1. Let us look at the following triangulation of the Klein bottle.

Note that $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, generated by cycles $[0, 1] + [1, 2] - [0, 2]$ (this one generates the $\mathbb{Z}$) and $[0, 3] + [3, 4] - [0, 4]$ (this one has order 2 in homology). We denote the homology class of the first cycle by $x$ and the homology class of the second by $y$.

The red (vertical) circle represents a cocycle $A : C_1(K) \to \mathbb{Z}/2$. If we denote the cohomology class by $a$, we get $a(x) = A([1, 2]) = \bar{1}$, and $a(y) = 0$.

The blue (horizontal) circle represents a cocycle $B : C_1(K) \to \mathbb{Z}/2$, whose cohomology class $b$ satisfies $b(x) = 0$ and $b(y) = B([3, 4]) = \bar{1}$.

In particular, $a$ and $b$ are linearly independent, and so they generate $H^1(K; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

We only need to work out $a \cup a$, $b \cup b$ and $a \cup b = b \cup a$. This is done in the same way as in the lectures.

Note that $A \cup A$ will vanish on all 2-simplices which don’t intersect the
red line. The remaining evaluations are

\[
\begin{align*}
A \cup A([1, 2, 6]) &= A([1, 2])A([2, 6]) = \bar{1} \cdot \bar{0} = \bar{0} \\
A \cup A([1, 5, 6]) &= \bar{0} \\
A \cup A([5, 6, 8]) &= \bar{0} \\
A \cup A([5, 7, 8]) &= \bar{0} \\
A \cup A([2, 7, 8]) &= \bar{1} \\
A \cup A([1, 2, 7]) &= \bar{1}
\end{align*}
\]

Thus we get \( A \cup A = [1, 2, 7]^* + [2, 7, 8]^* \).

Similarly,

\[
B \cup B([3, 6, 8]) = \bar{1}
\]

while \( B \cup B \) vanishes on all other 2-simplices. In other words, \( B \cup B = [3, 6, 8]^* \).

Finally, \( A \cup B = [5, 6, 8]^* \) as one checks in the same way (note that there are only two potentially non-zero simplices anyway).

Now as in the case of \( \mathbb{R}P^2 \), we see that every cocyle \([l, m, n]^* \) generates \( H^2(K; \mathbb{Z}/2) \cong \mathbb{Z}/2 \). It follows that \( b \cup b = a \cup b \) are this generator, and \( a \cup a = 0 \).

2. We represent the Klein bottle as in the picture.

Here the skew-diagonal represents the circle of connection in \( \mathbb{R}P^2 \# \mathbb{R}P^2 \), which we can identify with the Klein bottle \( K \). Note that the picture is different from the one in Question 1. The map \( f : \mathbb{R}P^2 \# \mathbb{R}P^2 \to \mathbb{R}P^2 \vee \mathbb{R}P^2 \) is then obtained by collapsing the skew-diagonal circle to the wedge
point. As in the case of the surface of genus 2, \( f^* \) induces an isomorphism on \( H^k \) for \( k \leq 1 \), and \( f^* : H^2(\mathbb{RP}^2 \vee \mathbb{RP}^2; \mathbb{Z}/2) \to H^2(K; \mathbb{Z}/2) \) sends the two generators to the generator of \( H^2(K; \mathbb{Z}/2) \). As \( f^* \) is a ring homomorphism, we get that \( H^1(K; \mathbb{Z}/2) \) is generated by two elements \( r, s \) which come from a copy of \( H^1(\mathbb{RP}^2; \mathbb{Z}/2) \) each, and \( r^2 = s^2 \) is the generator of \( H^2(K; \mathbb{Z}/2) \), and \( r \cup s = 0 \) by the behaviour of cup products over different wedge copies.

Note that the element \( a \in H^1(K; \mathbb{Z}/2) \) which satisfied \( a \cup a = 0 \) is here given by \( r + s \).

3. (a) If the inclusion \( i : U \to X \) is homotopic to a constant map \( k : U \to X \) with \( k(u) = x_0 \) for all \( u \in U \), then \( i^* = k^* \). Also, \( k = j \circ c \), where \( c : U \to \{x_0\} \) is the constant map and \( j : \{x_0\} \to X \) is inclusion. Now \( k^* = c^* \circ j^* \), and so \( k^* \) factors through the group \( H^0(\{x_0\}; k) \), which is trivial for \( q > 0 \). Therefore \( k^* = i^* \) is the trivial map.

(b) This follows directly from part (a), using the long exact sequence of cohomology.

(c) If we remove the north pole from \( S^2 \), we get an open set \( U \) which is homeomorphic to \( \mathbb{R}^2 \). Removing the south pole from \( S^2 \) gives another open set \( V \) also homeomorphic to \( \mathbb{R}^2 \), and \( S^2 = U \cup V \).

Now \( U \) and \( V \) are both contractible in themselves, so composing this contraction with inclusion to \( S^2 \) gives the result.

(d) Assume that \( \mathbb{RP}^2 = U \cup V \) with \( U, V \) open and both contractible in \( \mathbb{RP}^2 \). Now let \( x \in H^1(\mathbb{RP}^2; \mathbb{Z}/2) \) be the non-zero element. By part (b), both \( j_1^* : H^1(\mathbb{RP}^2, U; \mathbb{Z}/2) \to H^1(\mathbb{RP}^2; \mathbb{Z}/2) \) and \( j_2^* : H^1(\mathbb{RP}^2, V; \mathbb{Z}/2) \to H^1(\mathbb{RP}^2; \mathbb{Z}/2) \) are surjective. therfore we can find \( x_1 \in H^1(\mathbb{RP}^2, U; \mathbb{Z}/2) \) and \( x_2 \in H^1(\mathbb{RP}^2, V; \mathbb{Z}/2) \) with \( j_1^*(x_1) = x \) and \( j_2^*(x_2) = x \). It follows that

\[
x \cup x = j_1^*(x_1) \cup j_2^*(x_2)
\]

By naturality and basic properties of cup products, we get that

\[
j_3^*(x_1) \cup j_3^*(x_2) = j_3^*(x_1 \cup x_2)
\]

where \( j_3^* : H^1(\mathbb{RP}^2, U \cup V; \mathbb{Z}/2) \to H^1(\mathbb{RP}^2; \mathbb{Z}/2) \) is induced by inclusion \( j_3 : (\mathbb{RP}^2, \emptyset) \to (\mathbb{RP}^2, U \cup V) \). But \( U \cup V = \mathbb{RP}^2 \) and \( H^*(\mathbb{RP}^2, \mathbb{RP}^2; \mathbb{Z}/2) = 0 \), so \( x \cup x = 0 \). This contradicts our calculation \( x \cup x \neq 0 \). Hence the sets \( U \) and \( V \) cannot exist.