ON THE ALGEBRAIC K- AND L-THEORY OF WORD HYPERBOLIC GROUPS

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Abstract. In this paper, the assembly maps in algebraic K- and L-theory for the family of finite subgroups are proven to be split injections for word hyperbolic groups. This is done by analyzing the compactification of the Rips complex by the boundary of a word hyperbolic group.

1. Introduction

In [10, 11], conditions were given for discrete groups Γ under which the assembly maps in algebraic K- and L-theory are split injective. For such groups, a portion of the K- and L-theory of a group ring $R\Gamma$ is then described by an appropriate equivariant homology theory evaluated on the universal space for proper Γ-actions. Tools such as spectral sequences and Chern characters can then be used to calculate the homology groups, so that a piece of the geometrically important K- and L-groups of $R\Gamma$ can be understood. In this note, we show that *word hyperbolic* groups satisfy the conditions of [10, 11], thus proving the following theorem:

Theorem 1.1. *Let Γ be a word hyperbolic group. Then,*

1. *the assembly map $H_*^\Gamma(\underline{E}\Gamma; \mathbb{K}^{-\infty}(R\Gamma_x)) \to K_*(R\Gamma)$, in algebraic K-theory, is a split injection for any ring with unit R;*
2. *the assembly map $H_*^\Gamma(\underline{E}\Gamma; \mathbb{L}^{-\infty}(R\Gamma_x)) \to L_*^{\langle -\infty \rangle}(R\Gamma)$, in algebraic L-theory, is a split injection for any ring with involution R such that for sufficiently large i, $K_{-i}(RH) = 0$ for every finite subgroup H of Γ.*

Theorem 1.1 implies the classical Novikov conjecture for word hyperbolic groups (see for example Lück and Reich [7]). This, however, also follows from the injectivity of the Baum-Connes assembly map, which was proved by Higson [6]. More recently, Mineyev and Yu [9] have shown that the Baum-Connes assembly map is in fact an isomorphism for these groups. It is also important to note that in the case of torsion-free word hyperbolic groups, Theorem 1.1 follows from Carlsson and Pedersen [3].

It is proved in [10, 11] that a discrete group Γ will satisfy statements (1) and (2) of Theorem 1.1 if there is a finite Γ-CW model for the universal space for proper Γ-actions that admits a compactification X such that

- the Γ-action extends to X;

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- X is metrizable;
- $\underline{E}\Gamma^H$ is dense in X^H for every finite subgroup H of Γ;
- X^H is contractible for every finite subgroup H of Γ;
- compact subsets of $\underline{E}\Gamma$ become small near $X - \underline{E}\Gamma$. That is, for every compact subset $K \subset \underline{E}\Gamma$ and for every neighborhood $U \subset X$ of $y \in X - \underline{E}\Gamma$, there exists a neighborhood $V \subset X$ of y such that $g \in \Gamma$ and $gK \cap V \neq \emptyset$ implies $gK \subset U$.

This result generalized work of Carlsson and Pedersen [3], who proved it for torsion-free groups. In this note, Theorem 1.1 is proved by showing that word hyperbolic groups satisfy the above conditions.

Meintrup and Schick [8] proved that for word hyperbolic groups Γ, the Rips complex, $P_d(\Gamma)$, with d sufficiently large, is a finite Γ-CW model for the universal space for proper Γ-actions. The desired boundary, $\partial\Gamma$, was introduced by Gromov [5] and is defined as follows. For $x, y \in \Gamma$, let $(x \cdot y) = \frac{1}{2}\left(d(x,1) + d(y,1) - d(x,y)\right)$. A sequence $\{x_i\}$ in Γ is *convergent at infinity* if $(x_i \cdot x_j) \to \infty$ as $i, j \to \infty$. Two such sequences $\{x_i\}$ and $\{y_i\}$ are *equivalent* if $(x_i \cdot y_j) \to \infty$ as $i, j \to \infty$. The boundary, $\partial\Gamma$, can then be defined as the set of equivalence classes of sequences that are convergent at infinity. We topologize $P_d(\Gamma) \cup \partial\Gamma$ by defining a typical neighborhood of $a \in \partial\Gamma$ to be the set of points $y \in \Gamma \cup \partial\Gamma$ with $(a \cdot y) \geq R$, along with the simplices of $P_d(\Gamma)$ that they span. By [4, 5], $P_d(\Gamma) \cup \partial\Gamma$ is a compact, metrizable, finite-dimensional space, so we choose it as our candidate for X. It turns out that the most delicate part of the proof is showing that the fixed sets X^H, for the finite subgroups H of Γ, are contractible. The case when H is the trivial group was done by Bestvina and Mess [1]. The general case can be handled similarly but requires a careful analysis of the contractibility of $P_d(\Gamma)^H$, given in Meintrup and Schick [8].

2. Basic definitions

Let Γ be a finitely generated group and $d(\cdot,\cdot)$ the word metric with respect to some finite symmetric set of generators.

Definition 2.1. Let d be a positive integer. The *Rips complex*, $P_d(\Gamma)$, is the simplicial complex whose k-simplices are $(k+1)$-tuples $(g_0, \ldots, g_k)$ of pairwise distinct elements of Γ with $\max\{d(g_i, g_j)\} \leq d$.

In particular, the 0-skeleton of $P_d(\Gamma)$ coincides with Γ. Because of the left invariance of the word metric, there is a simplicial action of Γ on $P_d(\Gamma)$ given by $g \cdot (g_0, \ldots, g_k) = (gg_0, \ldots, gg_k)$.

Following Meintrup and Schick [8], we write $d(K, L) = \max\{d(k,l) \,|\, k \in K, l \in L\}$ for the maximal distance between finite subsets K and L of Γ. We also call $d(K) = d(K,K)$ the *diameter* of the finite subset K of Γ.

Definition 2.2. Let Γ be a finitely generated group and $\delta \geq 0$. Then Γ is δ-*hyperbolic* if for any four points $x, y, z, w \in \Gamma$,

$$d(x,y) + d(z,w) \quad \leq \quad \max\{d(x,z) + d(y,w), d(x,w) + d(y,z)\} + 2\delta.$$

A group Γ is called *word hyperbolic* if there is a $\delta \geq 0$ such that Γ is δ-hyperbolic.

We remark that being δ-hyperbolic for a specific δ is a property of Γ and a chosen word metric, while being word hyperbolic does not depend on the word metric.

We want to define a boundary for a word hyperbolic group. For this, let

$$(x \cdot y) \quad = \quad \frac{1}{2}\left(d(x,1) + d(y,1) - d(x,y)\right)$$

be the *overlap function*, where $x, y \in \Gamma$. The following lemma is easy to see.

Lemma 2.3. *Let Γ be a finitely generated group and $\delta \geq 0$. Then Γ is δ-hyperbolic if and only if*

$$(x \cdot y) \quad \geq \quad \min\{(x \cdot z), (y \cdot z)\} - \delta$$

for all $x, y, z \in \Gamma$. □

From now on we assume that Γ is δ-hyperbolic for some $\delta \geq 0$. A sequence $\{x_i\}$ in Γ is *convergent at infinity* if $(x_i \cdot x_j) \to \infty$ as $i, j \to \infty$. Two such sequences $\{x_i\}$ and $\{y_i\}$ are *equivalent* if $(x_i \cdot y_j) \to \infty$ as $i, j \to \infty$. Define the boundary of Γ, $\partial\Gamma$, to be the set of equivalence classes of sequences that are convergent at infinity. We will denote the equivalence class of a sequence $\{x_i\}$ by $[\{x_i\}]$. If $a \in \partial\Gamma$ and $y \in \Gamma$, define

$$(a \cdot y) \quad = \quad \sup\left\{\liminf_{i\to\infty}(x_i \cdot y) \,\middle|\, [\{x_i\}] = a\right\}.$$

Notice that $(x_i \cdot y) \leq d(y,1)$ and that there is a sequence $\{z_i\}$ representing a with $(a \cdot y) = (z_i \cdot y)$ for large i. Furthermore, for any sequence $\{x_i\}$ representing a,

$$\liminf_{i\to\infty}(x_i \cdot y) \quad \geq \quad (a \cdot y) - \delta,$$

by Lemma 2.3. The overlap function extends to $a, b \in \partial\Gamma$ by setting

$$(a \cdot b) \quad = \quad \sup\left\{\liminf_{i,j\to\infty}(x_i \cdot y_j) \,\middle|\, [\{x_i\}] = a, [\{y_j\}] = b\right\}.$$

Because of the supremum in the definition, we get

$$(x \cdot y) \quad \geq \quad \min\{(x \cdot z), (y \cdot z)\} - 2\delta \tag{1}$$

for all $x, y, z \in \Gamma \cup \partial\Gamma$. (Compare Bridson and Haefliger [2, p.433].)

Now we can put a topology on $\Gamma \cup \partial\Gamma$ in which Γ is a discrete subset. A typical neighborhood of $a \in \partial\Gamma$ is defined to be $\{y \in \Gamma \cup \partial\Gamma \mid (a \cdot y) \geq R\}$, where $R > 0$. This gives a compactification of Γ by [4, 5]. Similarly, we can topologize $\overline{P_d(\Gamma)} = P_d(\Gamma) \cup \partial\Gamma$ by defining a typical neighborhood of $a \in \partial\Gamma$, $U_R(a)$, to be the set of points $y \in \Gamma \cup \partial\Gamma$ with $(a \cdot y) \geq R$, along with the simplices of $P_d(\Gamma)$ that they span. By [4, 5], $\overline{P_d(\Gamma)}$ is a compact, metrizable, finite-dimensional space.

3. Proof of the main theorem

Lemma 3.1. *Let $x_1, x_2, g \in \Gamma$. Then $|(x_1 \cdot x_2) - (gx_1 \cdot gx_2)| \leq d(g,1)$.*

Proof. Since the metric is left-invariant,

$$d(x_i, 1) \leq d(x_i, g^{-1}) + d(g^{-1}, 1) = d(gx_i, 1) + d(g^{-1}, 1).$$

Therefore,

$$\begin{aligned}(x_1 \cdot x_2) &= \frac{1}{2}\left(d(x_1,1)+d(x_2,1)-d(x_1,x_2)\right)\\ &\leq \frac{1}{2}\left(d(gx_1,1)+d(gx_2,1)-d(gx_1,gx_2)\right)+d(g^{-1},1)\\ &= (gx_1 \cdot gx_2)+d(g^{-1},1).\end{aligned}$$

The same argument gives $(gx_1 \cdot gx_2) \leq (x_1 \cdot x_2) + d(g,1)$, which proves the lemma. □

Lemma 3.1 allows us to define an action on $\partial\Gamma$ by setting $g \cdot a = [\{gx_i\}]$, where $\{x_i\}$ is a sequence representing $a \in \partial\Gamma$. This gives a well defined action of Γ on $\overline{P_d(\Gamma)}$.

The next lemma is an observation of Bestvina and Mess in the proof of [1, Theorem 1.2].

Lemma 3.2. *Let $a \in \partial\Gamma$, and let $x, y \in \Gamma$ with $(a \cdot x), (a \cdot y) \geq 2R + 6\delta$. If $z \in \Gamma$ is a point on a geodesic in $P_d(\Gamma)$ between x and y, then $(a \cdot z) \geq R$.*

Proof. Since $d(x,z) + d(z,y) = d(x,y)$, we have $(x \cdot z) + (y \cdot z) = (x \cdot y) + d(z,1) \geq (x \cdot y)$. Thus, $(x \cdot z) \geq \frac{1}{2}(x \cdot y)$ or $(y \cdot z) \geq \frac{1}{2}(x \cdot y)$. Assume $(x \cdot z) \geq \frac{1}{2}(x \cdot y)$. Using (1), $(x \cdot y) \geq \min\{(a \cdot x), (a \cdot y)\} - 2\delta \geq 2R + 4\delta$. Therefore $(x \cdot z) \geq R + 2\delta$. This implies $(a \cdot z) \geq \min\{(a \cdot x), (x \cdot z)\} - 2\delta \geq R$. □

Lemma 3.3. *Let H be a finite subgroup of Γ, $a \in (\partial\Gamma)^H$, and $R > 0$. If $y \in \Gamma$ such that $(a \cdot y) \geq R + d(H)$, then $Hy \subset U_R(a)$. That is, $(a \cdot hy) \geq R$ for every $h \in H$.*

Proof. Choose a representative $\{x_i\}$ of a such that $(a \cdot y) = (x_i \cdot y)$ for large i. Let $h \in H$. Since $ha = a$, $\{hx_i\}$ is also a representative of a. Thus,

$$(a \cdot hy) \geq \liminf_{i\to\infty}(hx_i \cdot hy) \geq \liminf_{i\to\infty}(x_i \cdot y) - d(H) = (a \cdot y) - d(H) \geq R,$$

by Lemma 3.1. □

The next lemma is essentially taken from Meintrup and Schick [8, Lemma 6] but with a slight variation that will become important.

Lemma 3.4. *Let H be a finite subgroup of Γ, let y_0 be a vertex of $P_d(\Gamma)$ with $d(Hy_0) = R$, and let $h \in H$ such that $d(y_0, hy_0) = R$.*

(1) *Then there is an $x \in \Gamma$ on a geodesic between y_0 and hy_0 such that*

$$d(Hx, Hy_0) \leq \left\lceil \frac{R}{2} \right\rceil + 2\delta + 1, \text{ and } d(Hx) \leq 8\delta + 4.$$

(2) *If, in addition, $R \geq 8\delta + 2$ and x_0 is a vertex of $P_d(\Gamma)$, then*

$$d(Hx, x_0) \quad \leq \quad d(x_0, y_0) + d(Hx_0).$$

Proof. By Meintrup and Schick [8, Lemma 6(a)], there is an $x \in \Gamma$ satisfying the inequalities of (1). An inspection their proof verifies that x is indeed chosen on a geodesic between y_0 and hy_0. Assume $R \geq 8\delta + 2$. Choose $h' \in H$ such that $d(h'y_0, x_0) = d(Hy_0, x_0)$. By [8, Lemma 6(b)], $d(Hx, x_0) \leq d(x_0, h'y_0)$. Therefore,

$$d(Hx, x_0) \leq d(x_0, h'y_0) \leq d(x_0, h'x_0) + d(h'x_0, h'y_0) \leq d(x_0, y_0) + d(Hx_0),$$

which finishes the proof. □

Lemma 3.5. *Let H be a finite subgroup of Γ, and let $d \geq 8\delta + 4$. Then $P_d(\Gamma)^H$ is dense in $\overline{P_d(\Gamma)}^H$.*

Proof. Let $a \in (\partial\Gamma)^H$ and $R > 0$ be given. We must find a point in $U_R(a) - \partial\Gamma$ that is fixed by H. Choose a representative $\{x_i\}$ of a. Since $(x_i \cdot x_j) \to \infty$ as $i, j \to \infty$, there is an N such that $(x_i \cdot x_j) \geq (2R + 6\delta) + d(H)$ for all $i, j \geq N$. Then $(a \cdot x_N) \geq \liminf(x_i \cdot x_N) \geq (2R + 6\delta) + d(H)$. By Lemma 3.3, $Hx_N \subset U_{2R+6\delta}(a)$. By Lemma 3.2, any element of Γ on a geodesic between two points of Hx_N is contained in $U_R(a)$. By Lemma 3.4, there is an $x \in \Gamma$ on such a geodesic with $d(Hx) \leq 8\delta + 4$. Notice that Hx is contained in the union of all geodesics between points of Hx_N. Therefore $Hx \subset U_R(a)$. The elements of Hx form a simplex in the Rips complex since $d \geq 8\delta + 4 \geq d(Hx)$. Since this simplex is invariant under H, it has a fixed point. □

Proposition 3.6. *Let H be a finite subgroup of Γ, and let $d \geq 40\delta + 20$. Let $a \in (\partial\Gamma)^H$ and U a neighborhood of a in $\overline{P_d(\Gamma)}^H$ be given. Then there is a neighborhood V of a in $\overline{P_d(\Gamma)}^H$ such that every compact subset C of $V - (\partial\Gamma)^H$ is contractible in $U - (\partial\Gamma)^H$.*

Proof. We can assume that $U = U' \cap \overline{P_d(\Gamma)}^H$, where U' is a typical neighborhood of a in $\overline{P_d(\Gamma)}$. That is, there is an $R > 0$ such that U' contains the vertices x of $P_d(\Gamma)$ with $(a \cdot x) > R$ and the simplices in $P_d(\Gamma)$ that they span. By Lemma 3.3, there exists an H-equivariant neighborhood V' of a in $\overline{P_d(\Gamma)}$ such that for every vertex $x \in V'$, $(a \cdot x) > 2R + 6\delta$. Let $V = V' \cap \overline{P_d(\Gamma)}^H$.

Let F be the subcomplex of $P_d(\Gamma)$ consisting of all simplices of $P_d(\Gamma)$ that contain an H-fixed point and their faces. This subcomplex is the same complex as the one defined by Meintrup and Schick in the proof of [8, Proposition 7]. Note that if x is a vertex of F, then $d(Hx) \leq d$.

Now let C be a compact subset of $V - (\partial\Gamma)^H$. Define the subcomplex D of F by setting $D = F \cap U'$. Notice that $U - (\partial\Gamma)^H \subset D$. Let K' be a finite subcomplex of $D \cap V'$ containing C, and let $K = H \cdot K'$. Since V' is H-invariant, $K \subset V'$. Following Meintrup-Schick, we will show that the inclusion $K \hookrightarrow D$ is H-equivariantly homotopic to a constant map. By passing to fixed sets, this will imply the statement of the proposition. We do this by modifying the construction of the H-equivariant homotopy in the proof of [8, Proposition 7], making sure that it is in fact a map into D.

By Lemma 3.4, there is a vertex $x_0 \in \Gamma \cap V'$ with $d(Hx_0) \leq 8\delta + 4$. Without loss of generality, we can assume $x_0 \in K$. Let K^0 be the 0-skeleton of K. If $d(x_0, y) \leq \frac{d}{2}$ for all $y \in K^0$, then K is contained in a simplex of D. Thus, it can be contracted H-equivariantly.

Now suppose that there exists a $y \in K^0$ with $d(x_0, y) > \frac{d}{2}$. For every orbit $Hy \subset K^0$ with $d(x_0, Hy) > \frac{d}{2}$, choose a representative y and a geodesic segment c_y from x_0 to y. Note that $c_y \subset U'$ for all such y by Lemma 3.2. Pick $y_0 \in K^0$ to be a representative of an orbit Hy_0 with $d(x_0, Hy_0)$ maximal. Notice that we do not require $d(x_0, y_0) = d(x_0, Hy_0)$ as in the proof of [8, Proposition 7]. This will result in slightly different inequalities along the way. Let y_0' be the vertex on c_{y_0} with $d(y_0, y_0') = [\frac{d}{4}]$. We claim $y_0' \in D$. That is, $d(Hy_0') \leq d$. If $d(Hy_0) \leq \frac{d}{2}$ this follows from the triangle inequality. So assume $d(Hy_0) > \frac{d}{2}$. By Lemma 3.4, we can find a vertex x with $d(Hx) \leq 8\delta + 4$, $d(Hx, y_0) \leq \frac{d}{2} + 2\delta + 1$, and $d(Hx, x_0) \leq d(x_0, y_0) + 8\delta + 4$. Since we assumed $d \geq 40\delta + 20$, hyperbolicity yields the following.

$$\begin{aligned} d(hx, y_0') &\leq \max\{d(hx, y_0) + d(y_0', x_0), d(hx, x_0) + d(y_0', y_0)\} - d(y_0, x_0) + 2\delta \\ &\leq \max\left\{\frac{d}{2} + 4\delta + 1 - \left[\frac{d}{4}\right], \left[\frac{d}{4}\right] + 10\delta + 4\right\} \\ &\leq \frac{d}{2} \end{aligned}$$

The triangle inequality now gives $d(Hy_0') \leq d$.

Define $f_0 : (K^0, x_0) \to (D, x_0)$ by $f_0(hy_0) = hy_0'$, and $f_0(y) = y$ if $y \in K^0 - Hy_0$. To see that f_0 extends to a simplicial map, $f : (K, x_0) \to (D, x_0)$, we must show that $d(f_0(x), f_0(y)) \leq d$ whenever $d(x, y) \leq d$ and $x, y \in K^0$. As in the proof of [8, Proposition 7], it suffices to show that $d(y, y_0) \leq d$ implies $d(y, y_0') \leq d$ for $y \in K^0 - Hy_0$. Choose $h \in H$ so that $d(hy_0, x_0) = d(Hy_0, x_0)$. By maximality, $d(hy_0, x_0) \geq d(y, x_0)$. Thus, using hyperbolicity,

$$\begin{aligned} d(y, y_0') &\leq \max\{d(y, y_0) + d(y_0', x_0), d(y, x_0) + d(y_0', y_0)\} - d(y_0, x_0) + 2\delta \\ &\leq \max\left\{d - \left[\frac{d}{4}\right] + 2\delta, d(y, x_0) + \left[\frac{d}{4}\right] - d(hy_0, x_0) + d(hx_0, x_0) + 2\delta\right\} \\ &\leq \max\left\{d - \left[\frac{d}{4}\right] + 2\delta, \left[\frac{d}{4}\right] + 10\delta + 4\right\} \\ &\leq d. \end{aligned}$$

Note that by the definition of U', the image of f is contained in D. Next, Meintrup and Schick [8, p.6] observe that for every simplex σ of K, the set $f(\sigma) \cup \sigma$ is contained in a simplex of D. Thus, there is an H-equivariant homotopy between f and the inclusion $K \hookrightarrow D$. Notice that $f(K)$ is a finite subcomplex of D and that $f(K^0) = Hy_0' \cup (K^0 - Hy_0)$. We claim that $d(Hy_0', x_0) < d(Hy_0, x_0)$. For every $h \in H$,

$$\begin{aligned} d(hy_0', x_0) &\leq d(hy_0', hx_0) + d(hx_0, x_0) \leq d(x_0, y_0) - \left[\frac{d}{4}\right] + 8\delta + 4 \\ &\leq d(x_0, Hy_0) - \left[\frac{d}{4}\right] + 8\delta + 4 < d(x_0, Hy_0), \end{aligned}$$

since $d \geq 40\delta + 20$.

We wish to repeat this process with the finite subcomplex $f(K)$. If $d(Hy_0', x_0) > \frac{d}{2}$ and if y_0' was not in the original K^0, choose y_0' as the representative of its orbit, and choose the geodesic $c_{y_0'}$ to be a subset of c_{y_0}. This ensures that the subsequent homotopies will remain in U'. After finitely many steps every vertex will have distance from x_0 less than or equal to $\frac{d}{2}$. Then they will span a simplex of D that can be equivariantly contracted. □

Recall that a separable metric space X is an *absolute retract* (AR) if whenever it is embedded in a separable metric space Y as a closed subset, it is a retract of Y. It is called an *absolute neighborhood retract* (ANR) if whenever such a closed embedding is given, it is a retract of a neighborhood in Y. A closed subset W of a compact ANR X, is called a *Z-set* if for every open set U in X, the inclusion $U - W \to U$ is a homotopy equivalence. In particular, the inclusion $X - W \to X$ is a homotopy equivalence.

Theorem 3.7. *Let Γ be a δ-hyperbolic group and let $d \geq 40\delta + 20$. For every finite subgroup H of Γ, $\overline{P_d(\Gamma)}^H$ is an absolute retract, and $(\partial\Gamma)^H \subset \overline{P_d(\Gamma)}^H$ is a Z-set.*

This theorem generalizes Theorem 1.2 of Bestvina and Mess [1], which gives the result for the trivial group $H = \{1\}$. Because of Proposition 3.6, we can follow their line of proof which rests on the following proposition proven in [1].

Proposition 3.8. ([1, Proposition 2.1]) *Suppose that X is a compact metric space, and $W \subset X$ is a closed subset such that*

(1) $\operatorname{int} W = \emptyset$;
(2) $\dim X = n < \infty$;
(3) *for every $k = 0, \ldots, n$, every point $z \in W$, and every neighborhood U of z, there is a neighborhood V of z such that every map $\alpha : S^k \to V - W$ extends to $\tilde{\alpha} : B^{k+1} \to U - W$;*
(4) *$X - W$ is an ANR.*

Then X is an ANR, and $W \subset X$ is a Z-set. □

Proof of Theorem 3.7. We want to apply Proposition 3.8 with $X = \overline{P_d(\Gamma)}^H$ and $W = (\partial\Gamma)^H$. By Lemma 3.5, condition (1) is satisfied. Both X and W are closed subsets of $\overline{P_d(\Gamma)}$ and $\partial\Gamma$ respectively, so condition (2) follows from Gromov [5]. Condition (3) follows from Proposition 3.6. Finally, condition (4) is satisfied since $P_d(\Gamma)^H$ is a subcomplex of the second barycentric subdivision of $P_d(\Gamma)$. Thus, $\overline{P_d(\Gamma)}^H$ is an ANR, and $(\partial\Gamma)^H$ is a Z-set in $\overline{P_d(\Gamma)}^H$. Since $P_d(\Gamma)^H = \overline{P_d(\Gamma)}^H - (\partial\Gamma)^H$ is contractible by Meintrup and Schick [8, Proposition 7], $\overline{P_d(\Gamma)}^H$ is contractible. It follows that $\overline{P_d(\Gamma)}^H$ is an AR. □

Proof of Theorem 1.1. Let $\delta \geq 0$ be given so that Γ is δ-hyperbolic. Choose $\underline{E}\Gamma$ to be the second barycentric subdivision of the Rips complex, $P_d(\Gamma)$, with $d \geq 40\delta+20$. Meintrup and Schick [8] have shown that this is a finite Γ-CW model for the universal space for proper Γ-actions. We proceed by showing that $X = P_d(\Gamma) \cup \partial\Gamma$, with the Γ-action defined above, satisfies the following properties.

1. X is metrizable.
2. $\underline{E}\Gamma^H$ is dense in X^H for every finite subgroup H.
3. X^H is contractible for every finite subgroup H.
4. Compact subsets of $\underline{E}\Gamma$ become small near $\partial\Gamma$. That is, for every compact subset $K \subset \underline{E}\Gamma$ and for every neighborhood $U \subset X$ of $a \in \partial\Gamma$, there exists a neighborhood $V \subset X$ of a such that $g \in \Gamma$ and $gK \cap V \neq \emptyset$ implies $gK \subset U$.

By [10, 11], this implies the theorem.

Property 1 is proved in [4] and can also be found in [2]. Property 2 is satisfied by Lemma 3.5, and property 3 follows immediately from Theorem 3.7.

Let $\{y_1, \ldots, y_n\} \subset \Gamma$, $a \in \partial\Gamma$, and $R > 0$ be given. We must find an $R' > 0$ such that if $gy_j \in U_{R'}(a)$ for some $g \in \Gamma$ and some $j \in \{1, \ldots, n\}$, then $\{gy_1, \ldots, gy_n\} \subset U_R(a)$. This will imply property 4. Let $R' = R + A$, where $A = \max\{d(y_k, y_l)\}$. Without loss of generality we can assume $j = 1$. That is, $(a \cdot gy_1) \geq R'$. Choose a representative $\{x_i\}$ of a such that $(a \cdot gy_1) = (x_i \cdot gy_1)$ for large i. For each i and k we have

$$d(x_i, gy_k) \quad \leq \quad d(x_i, gy_1) + d(gy_1, gy_k) \leq d(x_i, gy_1) + A,$$

and

$$d(gy_1, 1) \leq d(gy_1, gy_k) + d(gy_k, 1) \leq d(gy_k, 1) + A.$$

Therefore,

$$(x_i \cdot gy_k) = \frac{1}{2}\left(d(x_i, 1) + d(gy_k, 1) - d(x_i, gy_k)\right) \geq (x_i \cdot gy_1) - A.$$

Thus, for every k,

$$(a \cdot gy_k) \geq \liminf_{i\to\infty}(x_i \cdot gy_k) \geq \liminf_{i\to\infty}(x_i \cdot gy_1) - A = (a \cdot gy_1) - A \geq R.$$

This completes the proof of the theorem. □

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