I use † to indicate (what I consider to be) trickier problems.

Thursday 19th February should be handed in for marking at the lecture on Thursday 19th February.

There will be a problem class on this chapter on Monday 16th February.

16. Bernstein polynomial approximation. Compute the approximations using Bernstein polynomials of degree $n = 1$ and $n = 2$ to the function $f(x) = 1 - |x - \frac{1}{3}|$ on $[0, 1]$. Verify that the approximation is converging in the $\infty$-norm.

Solution: We have

\[ B_1(f, x) = f(0) \binom{1}{0} (1-x) + f(1) \binom{1}{1} x = f(0)(1-x) + f(1)x = \frac{2}{3}(1-x) + \frac{1}{3}x = \frac{2}{3} - \frac{1}{3}x, \]
\[ B_2(f, x) = f(0) \binom{2}{0} (1-x)^2 + f(\frac{1}{2}) \binom{2}{1} x(1-x) + f(1) \binom{2}{2} x^2 \]
\[ = f(0)(1-x)^2 + 2f(\frac{1}{2})x(1-x) + f(1)x^2 = \frac{2}{3}(1-x)^2 + \frac{2}{3}x(1-x) + \frac{1}{3}x^2 = -\frac{2}{3}x^2 + \frac{4}{3}x + \frac{2}{3}. \]

In pictures,

![Graph of Bernstein polynomials](image)

To verify convergence, we compute $\|f - B_1(f, x)\|_\infty$ and $\|f - B_2(f, x)\|_\infty$.

From the picture, we see that the maximum of $|f(x) - B_1(f, x)|$ occurs at $x = \frac{1}{3}$, so

\[ \|f - B_1(f, x)\|_\infty = 1 - (\frac{2}{3} - \frac{1}{3}) = \frac{2}{3}. \]

To find $\|f - B_2(f, x)\|_\infty$, we check each subinterval. In $[0, \frac{1}{3}]$, we have

\[ f(x) - B_2(f, x) = \frac{2}{3} + x - (-\frac{2}{3}x^2 + \frac{1}{3}x + \frac{2}{3}) = \frac{2}{3} + \frac{2}{3}x \quad \implies \quad \frac{d}{dx}(f(x) - B_2(f, x)) = \frac{2}{3} + \frac{4}{3}x. \]

Hence $|f(x) - B_2(f, x)|$ is largest at $x = \frac{1}{4}$, where $f(x) - B_2(f, x) = \frac{3}{27}$. On the other hand, in $[\frac{1}{3}, 1]$, we have

\[ f(x) - B_2(f, x) = \frac{2}{3} - x - (-\frac{2}{3}x^2 + \frac{1}{3}x + \frac{2}{3}) = \frac{2}{3} - \frac{4}{3}x + \frac{2}{3}x^2 \quad \implies \quad \frac{d}{dx}(f(x) - B_2(f, x)) = -\frac{4}{3} + \frac{4}{3}x. \]

We conclude that the largest value of $|f(x) - B_2(f, x)|$ in this subinterval is also at $x = \frac{1}{3}$. Hence $\|f - B_2(f, x)\|_\infty = \frac{3}{27}$. This is less than $\|f - B_1(f, x)\|_\infty = \frac{2}{3} = \frac{12}{27}$, so we are indeed seeing convergence as $n$ increases.

17. Recursive definition of Bernstein polynomials. Let $b_{n,k}$ for $k = 0, \ldots, n$ be the Bernstein basis functions, as defined in the lecture. Show that these basis functions satisfy the recursion relation

\[ b_{n,k}(x) = (1-x)b_{n-1,k}(x) + xb_{n-1,k-1}(x). \]

Remark: This is the basis of de Casteljau’s fast algorithm for drawing Bézier curves.
Cubic Bézier curves.

18. Derivatives of Bernstein polynomials. Show that the derivatives of the Bernstein basis functions \( b_{n,k}(x) \) for \( k = 0, \ldots, n \) satisfy

\[
\frac{d}{dx} b_{n,k}(x) = n \left( b_{n-1,k-1}(x) - b_{n-1,k}(x) \right).
\]

Solution: This can be shown by direct differentiation:

\[
\frac{d}{dx} b_{n,k}(x) = \binom{n}{k} \frac{d}{dx} \left( x^k (1-x)^{n-k} \right),
\]

\[
= \frac{k n!}{k!(n-k)!} x^{k-1} (1-x)^{n-k} + \frac{(n-k)!}{k!(n-k)!} x^k (1-x)^{n-k-1},
\]

\[
= \frac{n(n-1)!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} + \frac{n(n-1)!}{k!(n-k-1)!} x^k (1-x)^{n-k-1},
\]

\[
= n \left( b_{n-1,k-1}(x) - b_{n-1,k}(x) \right).
\]

19. Cubic Bézier curves. Verify that the cubic Bézier curve \( B_3(t) \) with control points \( x_0, x_1, x_2, x_3 \) is tangent (i) at \( x_0 \) to the line joining \( x_0 \) and \( x_1 \), and (ii) at \( x_3 \) to the line joining \( x_2 \) and \( x_3 \).

Solution: The cubic Bézier curve is

\[
B_3(t) = (1-t)^3 x_0 + 3t(1-t)^2 x_1 + 3t^2(1-t) x_2 + t^3 x_3,
\]

so

\[
B_3'(t) = -3(1-t)^2 x_0 + 3(1-t)(1-3t)x_1 + 3t(2-3t)x_2 + t^2 x_3.
\]

The tangent direction at \( x_0 \) is \( B_3'(0) = -3(x_1 - x_0) \), while at \( x_3 \) it is \( B_3'(1) = -3(x_3 - x_2) \). This shows that the required lines are indeed the tangents at these two points.

20. A Bézier curve. Find the parametric equations of the Bézier curve with control points \( (0,1), \left( \frac{1}{3}, \frac{3}{2} \right), \left( \frac{3}{3}, 2 \right) \) and \( (1,0) \). Find the slope of the curve at each of its end-points and make a rough sketch of the curve.

As a check, you could try drawing the curve in Postscript.

Solution: We have

\[
x(t) = (1-t)^3(0) + 3t(1-t)^2 \left( \frac{1}{3} \right) + 3t^2(1-t) \left( \frac{2}{3} \right) + t^3(1) = \frac{1}{2} t(3 + 3t - t^2),
\]

\[
y(t) = (1-t)^3(1) + 3t(1-t)^2 \left( \frac{3}{2} \right) + 3t^2(1-t)(2) + t^3(0) = \frac{1}{2} (1-t)(2 + 5t + 5t^2).
\]
For the slope, note that \( x'(t) = \frac{3}{5} + \frac{6}{5} t - \frac{3}{5} t^2 \) and \( y'(t) = \frac{3}{7} - \frac{15}{7} t^2 \). So at the endpoints \( \frac{dy}{dx}(0) = \frac{y'(0)}{x'(0)} = \frac{3}{2} \) and \( \frac{dy}{dx}(1) = \frac{y'(1)}{x'(1)} = -5 \). Alternatively, you could get these from the slopes of the straight lines between the control points. Note that \( x'(t) \) is always positive for \( t \in [0,1] \), so that \( x \) is monotonically increasing. On the other hand, \( y'(t) \) changes sign, so there is a maximum in \( y \). The curve and its control points are shown below:

![Graph showing the curve and control points.](image)

21. **Minimax approximation.** Find the minimax linear approximation to \( f(x) = \sinh(x) \) on \([0,1]\).

**Solution:** We look for a straight line \( p_1^*(x) = a + bx \) such that \( f, p_1^* \) have an alternating set \( \{0, \theta, 1\} \). We require

\[
\begin{align*}
 f(0) - p_1^*(0) &= 0 - a = E, \\
 f(\theta) - p_1^*(\theta) &= \sinh(\theta) - a - b\theta = -E, \\
 f(1) - p_1^*(1) &= \sinh(1) - a - b = E.
\end{align*}
\]

There are four unknowns \( a, b, \theta, E \) but only three equations - we get a fourth equation by requiring that the error has a turning point at \( x = \theta \). This gives

\[
\cosh(\theta) - b = 0.
\]

Eliminating \( E \) from (3) gives \( b = \sinh(1) = 1.1752 \), and from (2) gives \( a = \frac{1}{b} (\sinh(\theta) - \sinh(1)\theta) \approx -0.0343 \), where \( \theta \) is given by \( \cosh(\theta) = \sinh(1) \) [from (4)]. The solution looks like this:
22. **Minimax approximation to a polynomial.** Find the minimax approximation of degree 4 to the polynomial \( f(x) = x^5 + 2x^2 - x \).

**Solution:** As shown in the lecture, 
\[ p_4^*(x) = f(x) - \frac{1}{2} T_5(x). \]

We use the recurrence relation to compute the Chebyshev polynomial \( T_5(x) \):
\[
\begin{align*}
T_0(x) &= 1, & T_1(x) &= x, \\
T_2(x) &= 2xT_1(x) - T_0(x) = 2x^2 - 1, \\
T_3(x) &= 2xT_2(x) - T_1(x) = 4x^3 - 3x, \\
T_4(x) &= 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1, \\
T_5(x) &= 2xT_4(x) - T_3(x) = 16x^5 - 20x^3 + 5x.
\end{align*}
\]
Therefore
\[ p_4^*(x) = x^5 + 2x^2 - x - x^5 + \frac{5}{4}x^3 - \frac{5}{16}x = \frac{5}{4}x^3 + 2x^2 - \frac{21}{16}x. \]

23. **Non-monic polynomials.** Prove that, if \( p_m^* \) is the minimax polynomial of degree \( m \) for a polynomial \( f \in P_{m+1} \), then \( \alpha p_m^* \) is the minimax approximation for \( \alpha f \).

**Solution:** We need to show that
\[ \|\alpha f - \alpha p_m^*\|_\infty \leq \|\alpha f - p_m\|_\infty \]
for all \( p_m \in P_m \). For a scalar \( \alpha \), all norms satisfy
\[ \|\alpha g\| = |\alpha| \|g\|, \]
so for any \( q_m \in P_m \) we have
\[ \|\alpha f - \alpha p_m^*\|_\infty = |\alpha| \|f - p_m^*\|_\infty \leq |\alpha| \|f - q_m\|_\infty = \|\alpha f - \alpha q_m\|_\infty. \]

Writing \( q_m = p_m/\alpha \) gives the result (unless \( \alpha = 0 \) in which case it is trivial).

\[ \star 24. \textbf{De la Vallée Poussin Theorem.} \text{ Let } f(x) = -\cos(x) \text{ and } q_1(x) = 0.5x - 1.1. \]

(a) Show that \( \{0, \frac{1}{2}, 1\} \) is a non-uniform alternating set for \( f \) and \( q_1 \) on \([0, 1]\).

(b) Use the De la Vallée Poussin Theorem with these points to find a lower bound for \( \|f - p_1^*\|_\infty \), where \( p_1^* \) is the minimax degree 1 polynomial for \( f \) on \([0, 1]\).

(c) Use \( q_1 \) to find an upper bound for \( \|f - p_1^*\|_\infty \).

(d) By postulating a suitable alternating set, or otherwise, find \( p_1^* \).

**Solution:** (a) We have
\[
\begin{align*}
f(0) - q_1(0) &= 0.1 := e_0, \\
f(\frac{1}{2}) - q_1(\frac{1}{2}) &= -0.0276 := e_1, \\
f(1) - q_1(1) &= 0.0597 := e_2.
\end{align*}
\]
The points are ordered and the successive \( e_i \) alternate in sign, so this is a non-uniform alternating set for \( f \) and \( q_1 \).

(b) By the DLVP Theorem, it follows from (a) that \( \|f - p_1^*\|_\infty > 0.0276 \).

(c) To find an upper bound, we can use \( \|f - q_1\|_\infty \). To find this, consider the derivative
\[ f'(x) - q_1'(x) = \sin(x) - 0.5. \]
Thus the error has a turning point at \( \sin(x) = 0.5 \), or \( x = \frac{\pi}{6} \). At this point \( f(\frac{\pi}{6}) - q_1(\frac{\pi}{6}) = -0.0278 \). Thus the maximum on \([0, 1]\) is \( \|f - q_1\|_\infty = 0.1 \) (at the left end). Hence our upper bound is
\[ \|f - p_1^*\|_\infty \leq 0.1. \]
(d) We look for a straight line \( p_1^*(x) = a + bx \) such that \( f, p_1^* \) have an alternating set \( \{0, \theta, 1\} \). We require

\[
\begin{align*}
    f(0) - p_1^*(0) &= -1 - a = E, \quad (5) \\
    f(\theta) - p_1^*(\theta) &= -\cos(\theta) - a - b\theta = -E, \quad (6) \\
    f(1) - p_1^*(1) &= -\cos(1) - a - b = E. \quad (7)
\end{align*}
\]

There are four unknowns \((a, b, \theta, E)\) but only three equations - we get a fourth equation by requiring that the error has a turning point at \( x = \theta \). This gives

\[
\sin(\theta) - b = 0. \quad (8)
\]

Eliminating \( E \) from (7) gives \( b = 1 - \cos(1) = 0.4597 \), and from (6) gives \( a = \frac{1}{2}(-1 - \cos(\theta) - [1 - \cos(1)]\theta) \approx -1.0538 \), where \( \theta \) is given by \( \sin(\theta) = 1 - \cos(1) \) [from (8)]. The solution looks like:

25. The Equioscillation Theorem. In light of the Chebyshev Equioscillation Theorem, explain why the function \( q_1(x) \) in Problem 24 could not possibly be the minimax degree 1 polynomial.

Solution: In the solution to Problem 24(c), we found that the local extrema of the error \( f - q_1 \) were 0.1, -0.0278, 0.0597. Therefore it is impossible to find an alternating set of length 3 for \( f \) and \( q_1 \) (remember that alternating sets must attain \( \pm \|f - q_1\|_\infty \) at each point), meaning that \( q_1 \) cannot possibly be the minimax polynomial (by the Equioscillation Theorem).

26. Every minimax polynomial is an interpolant. Let \( p_n^* \in P_n \) be a minimax approximation to \( f \in C[a,b] \). Show that there exist \( n + 1 \) distinct points \( a < x_0 < x_1 < \ldots < x_n < b \) such that \( p_n^* \) is the polynomial interpolant in \( P_n \) to \( f \) at these \( n + 1 \) points.

Solution: We know from the Equioscillation Theorem that \( f \) and \( p_n^* \) have an alternating set of length \( n + 2 \). Therefore, \( f - p_n^* \) changes sign at \( n + 1 \) distinct points, which are the required interpolation points.

†27. Minimax polynomials of even functions. Let \( f \in C[-1,1] \) be even, i.e. \( f(-x) = f(x) \).

(a) Use the Equioscillation Theorem to prove that the minimax polynomial \( p_n^* \) is even for any \( n \geq 0 \).

(b) Prove that for any \( n \geq 0 \), \( p_{2n}^* = p_{2n+1}^* \).

(c) Find the minimax polynomial of degree 1 for \( f(x) = |x| \) on \([-1,1]\).

Solution: (a) Since \( p_n^* \) is the minimax polynomial for \( f \), these two functions have an alternating set \( \{x_i\} \) of length \( n + 2 \) such that

\[
f(x_i) - p_n^*(x_i) = (-1)^i E, \quad \text{for } i = 0, \ldots, n + 1, \text{where } E = \|f - p_n^*\|_\infty.
\]
28. **Remez algorithm.** Use the Remez Exchange algorithm to compute the linear minimax approximation to \( f(x) = x^2 \) on \([0, 3]\), using the initial reference set \( \{0, 1, 3\} \). Comment on the convergence of the algorithm.

**Solution:** Let \( p_1 = a_0 + a_1 x \).

**Step 1:** solve the linear system

\[
\begin{align*}
a_0 + E &= 0^2 = 0, \\
a_0 + a_1 - E &= 1^2 = 1, \\
a_0 + 3a_1 + E &= 3^2 = 9.
\end{align*}
\]

Solving this system gives \( a_0 = -1, a_1 = 3, E = 1 \), i.e. \( p_1^{(1)} = -1 + 3x \).

**Step 2:** to update the reference set, we look for the point of maximum \( |f - p_1^{(1)}| \). We have

\[
f - p_1^{(1)} = x^2 - 3x + 1.
\]

This has a turning point at \( x = \frac{3}{2} \), where \( f(\frac{3}{2}) - p_1^{(1)}(\frac{3}{2}) = -\frac{5}{4} \). At the end-points, \( f(0) - p_1^{(1)}(0) = -1 \) and \( f(3) - p_1^{(1)}(3) = 1 \), so \( \|f - p_1^{(1)}\|_{\infty} = \frac{5}{4} \). At the middle point of the old reference set, \( f(1) - p_1^{(1)}(1) = -1 \). So we form the new reference set \( \{0, \frac{3}{2}, 3\} \).

**Step 1:** now solve the linear system

\[
\begin{align*}
a_0 + E &= 0^2 = 0, \\
a_0 + \frac{3}{2}a_1 - E &= (\frac{3}{2})^2 = \frac{9}{4}, \\
a_0 + 3a_1 + E &= 3^2 = 9.
\end{align*}
\]

Solving this system gives \( a_0 = -\frac{9}{8}, a_1 = 3, E = \frac{9}{8} \), i.e. \( p_1^{(2)} = -\frac{9}{8} + 3x \).

**Step 2:** Now we have

\[
f - p_1^{(2)} = x^2 - 3x + \frac{9}{8}.
\]

Again this has a turning point at \( x = \frac{3}{2} \), but now \( f(\frac{3}{2}) - p_1^{(2)}(\frac{3}{2}) = -\frac{9}{8} \). The end point values are now \( f(0) - p_1^{(2)}(0) = \frac{9}{8} \) and \( f(3) - p_1^{(2)}(3) = \frac{9}{8} \). Now the maximum \( |f - p_1^{(2)}| \) is achieved with alternating signs at \( \{0, \frac{3}{2}, 3\} \), so this is an alternating set. Hence (by Equioscillation Theorem) the minimax polynomial is

\[
p_1(x) = p_1^{(2)} = 3x - \frac{9}{8}.
\]

The algorithm has converged to the exact solution after two steps. See the illustration below: