Geodesics in Non-Positively Curved Plane Tessellations

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April 27, 2005

Abstract

We introduce a natural combinatorial curvature function on the corners of plane tessellations and relate it to the global metric geometry of their corresponding edge and dual graphs. If the combinatorial curvature in the corners is non-positive then we prove that any geodesic path in such a graph may be extended to infinity. Moreover, if the combinatorial curvature is negative we show that every pair of geodesic segments with the same end points does not enclose any vertices. We apply these results to establish an estimate for the growth of distance balls, Gromov hyperbolicity, and four-colourability of certain classes of plane tessellations.

Introduction

A plane tessellation (sometimes also called a tiling of the plane, see §1 for definition) induces the structure of a combinatorial cell complex on the plane. The one-skeleton (edge-graph) of a plane tessellation is called a *plane tessellating graph*. A tessellating graph $\mathcal{G}$ is locally finite, and, with its combinatorial (word) metric $d_{\mathcal{G}}$ in which each edge has length one, $\mathcal{G}$ obtains the structure of a *geodesic metric space*. This article is about the basic question how global *geometric* properties of the metric space $(\mathcal{G}, d_{\mathcal{G}})$ are determined by local *combinatorial* properties of the tessellation.

Combinatorial curvature We introduce now the curvature function which we use to control the local combinatorial properties of tessellations. A *corner* of a tessellation is a pair (v, f), where v is a vertex of $\mathcal{G}$, and f is a face of the tessellation which satisfies $v \in \partial f$. We define the *combinatorial curvature* κ on the corners of a tessellation, as

$$\kappa(v, f) = \frac{1}{|v|} + \frac{1}{E_{\partial f}} - \frac{1}{2},$$

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where $|v|$ denotes the degree of the vertex v, and $E_{\partial f}$ the number of edges of the face f.

The curvature function κ, which is defined on the set $\mathcal{C}$ of corners of the tessellation, relates nicely to various combinatorial curvature conditions which have been considered in the literature. The most well known such conditions may be expressed using the characteristic functions ψ, χ, and ϕ (see for example [Wo2]) which are defined on the set of vertices, faces and edges of a tessellation respectively. These functions can be written in terms of the curvature κ as $\psi(v) = -\sum_{(v,f)\in\mathcal{C}} \kappa(v,f)$, $\chi(f) = -\sum_{(v,f)\in\mathcal{C}} \kappa(v,f)$ and $\phi(vv') = -\kappa(v,f) - \kappa(v',f')$, where vv' is an edge with adjacent faces f, f'.

The strong geometric significance of the vertex function ψ (and its dual function χ) is exhibited, for example, by the result of Stone [Sto1, Sto2] who proved a combinatorial analogue of Myers' theorem for tessellations using the positive curvature-condition $-\psi > 0$.

Corresponding negative curvature conditions have been applied mostly in the context of isoperimetric inequalities (compare [Do, Soa, Mo, Żu, Wo2, BP, LPZ]). In [Wo2] Woess proves a strong isoperimetric inequality for all graphs for which one of the functions ψ, χ, ψ is strictly positive. This result generalized former results of Dodziuk [Do] (who considered tessellations with $|v| > 7$ for all $v \in \mathcal{V}$), and Soardi [Soa] (who considered certain tessellations satisfying $\chi > 0$). Moreover, Żuk [Żu] proved Gromov hyperbolicity (see below for definition) of graphs with $|v| > 7$, for all $v \in \mathcal{V}$.

The notion of non-positive curvature in tessellations has been expressed by the (p,q)-*condition*, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, which requires that at least p edges emanate at each vertex, and that each face has at least q edges. This condition has been used by Lyndon [Ly2], and also by the authors [BP], to study the distance function in plane tessellations. The (p,q)-condition, however, is much stronger than the condition $\kappa \leq 0$ on the combinatorial curvature function on corners of the tessellation. The aim of this paper is to study the behaviour of the distance function in tessellations which satisfy $\kappa \leq 0$.

Main results A connected path in $\mathcal{G}$ whose length realizes the distance of its end points is called a *geodesic* of $\mathcal{G}$. To control the geometry of $\mathcal{G}$ it is important to understand the behaviour of geodesics. For a fixed vertex $v \in \mathcal{G}$ we consider the distance function $d_{\mathcal{G}}(v, \cdot\,)$ which defines a function on the set of vertices of $\mathcal{G}$. We say that $\mathcal{G}$ has *no cut locus* if all functions $d_{\mathcal{G}}(v,\cdot\,)$ have no local maxima, or equivalently if all geodesic paths in $\mathcal{G}$ can be extended to infinity. Our main result is the following combinatorial analogue of the Theorem of Hadamard-Cartan in differential geometry:

Theorem 1 *Let $\mathcal{G}$ be a plane tessellating graph. If $\kappa(v,f) \leq 0$, for all corners (v,f), then the metric space $(\mathcal{G}, d_{\mathcal{G}})$ has no cut locus.*

The proof of Theorem 1 is given in Section 2 and is based on certain refined "convexity"-considerations for distance balls. Note that the theorem fails if the condition on κ is replaced by a condition on a weaker curvature function. An example is given in Section 1. Theorem 1 has precursors in the literature. Lyndon [Ly2] showed that $\mathcal{G}$ has no cut locus if each face of the tessellation has at least 6 edges. More generally, the authors established in [BP] that $\mathcal{G}$ has no cut locus if $\mathcal{G}$ satisfies the (p,q)-condition, where $(p,q) \in \{(3,6),(4,4),(6,3)\}$.

Our second result is about bigons in negatively curved plane tessellations. A pair $(v_1,\ldots,v_k)$ and $(w_1,\ldots,w_k)$ of geodesic segments is called a *bigon*, if $v_1 = w_1$ and $v_k = w_k$. A bigon is *minimal* if $v_j \neq w_j$ for $j \neq 1,k$. The *interior* of a minimal bigon consists of all vertices enclosed by the geodesic segments but not belonging to them.

Theorem 2 *Let $\mathcal{G}$ be a plane tessellating graph. If $\kappa(v,f) < 0$, for all corners (v,f), then any minimal bigon in $\mathcal{G}$ has empty interior.*

Theorem 2 is proved in Section 3. It generalizes a result of Żuk [Żu] on tessellations satisfying $|v| > 7$, for all $v \in \mathcal{V}$.

In differential geometry, the distance function of two parametrized geodesics in a non-positively curved simply connected manifold is convex and, therefore, any two geodesics with the same end points agree. Theorem 2 therefore states a weak analogue in the discrete case, namely that the two geodesics of a bigon cannot open up to contain a vertex in their interior. However, the theorem fails to be true in the case $\kappa(v,f) \leq 0$ (see Figure 1).

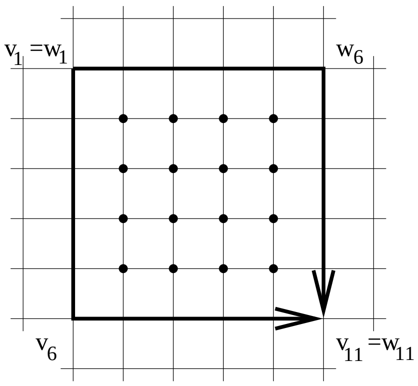

Figure 1: A bigon in the $(4,4)$-regular plane tessellation. The interior vertices are marked.

Applications Let $\mathcal{C}$ denote the set of all corners of the tessellation. Then we define the *curvature at a vertex* v by

$$\kappa(v) = \sum_{(v,f)\in\mathcal{C}} \kappa(v,f) = 1 - \frac{|v|}{2} + \sum_{(v,f)\in\mathcal{C}} \frac{1}{E_{\partial f}}. \tag{1}$$

Note that the curvature $\kappa(v)$ coincides up to factor 2 with Stone's vertex curvature [Sto1] and up to sign with the characteristic number $\psi(v)$ of [Wo2].

We use Theorem 1 to deduce an estimate for the growth of distance balls. Let B_r denote a metric ball of radius r in $\mathcal{G}$, and $|B_r|$ the number of vertices contained in B_r. By [BP, Corollary 5.2], Theorem 1 gives

Corollary 3 *Let $\mathcal{G}$ be a plane tessellating graph which satisfies $\kappa(v,f) \leq 0$ for all corners (v,f) and $\kappa(v) \leq -c < 0$. Then*

$$|B_r| \geq (1+2c)^r ,$$

for any distance ball B_r of radius r in $\mathcal{G}$.

Note that $\kappa(v,f) < 0$ implies $\kappa(v,f) \leq -\frac{1}{42}$ and, consequently, $\kappa(v) \leq -\frac{1}{14}$. Therefore, we have

Corollary 4 *Let $\mathcal{G}$ be a plane tessellating graph which satisfies $\kappa(v,f) < 0$ for all corners (v,f). Then*

$$|B_r| \geq \left(\frac{8}{7}\right)^r .$$

Corollary 3 and Corollary 4 provide *explicit* growth estimates of distance balls in terms of the curvature κ. This considerably strengthens the results of Woess [Wo2] in our (however less general) situation.

We are also interested in the geometric properties of tessellations with negative combinatorial curvature. We have:

Corollary 5 *Let $\mathcal{G}$ be a plane tessellating graph which satisfies $\kappa(v,f) < 0$ for all corners (v,f) and has a uniform upper bound on the degree of faces. Then the metric space $(\mathcal{G}, d_{\mathcal{G}})$ is Gromov hyperbolic and its Gromov boundary is homeomorphic to S^1.*

The proof of Corollary 5 is given in Section 3.3. The Gromov-hyperbolicity is an immediate application of Theorem 2. Recall that a bigon is *δ-thin* if $d_{\mathcal{G}}(v_j, w_j) \leq \delta$ for all j. It was shown in [Pa] that graphs with the property that all bigons are δ-thin, for some $\delta > 0$, are Gromov hyperbolic. (Note that this fact is only valid in discrete settings: the Euclidean plane $\mathbf{R}^2$ is not Gromov hyperbolic but all bigons are 0-thin.)

It is well known that a strong isoperimetric inequality in $\mathcal{G}$ implies that the simple random walk in $\mathcal{G}$ is transient. We remark that if $\mathcal{G}$ is Gromov hyperbolic, satisfies a strong isoperimetric inequality and has bounded vertex degree, then the simple random walk on $\mathcal{G}$ converges almost surely to its Gromov boundary (cf. [Wo1] or [An]).

Let us finally remark that if in Corollary 5 the condition on the degree of faces is dropped then $\mathcal{G}$ is in general not a Gromov hyperbolic space, although its geometry still exhibits hyperbolic properties. One may view $\mathcal{G}$ then as a *conformally hyperbolic space* and it is possible to show that the Floyd-boundary (cf. [Ka]) of $\mathcal{G}$ is homeomorphic to S^1.

We would also like to point out that the proof of Theorem 1 exhibits strong combinatorial information about the structure of distance balls in non-positively curved plane tessellations (see Section 2.4) which are of own interest.

This is used in [KLPS] to prove the absence of finitely supported eigenfunctions of elliptic operators on non-positively curved plane tessellations.

As another consequence of the combinatorial structure of distance balls in tessellations with non-positive curvature we can give a straightforward proof of the fact that such tessellations may be coloured using only four colours (see Corollary 2.9).

Acknowledgement The authors wish to thank Anders Karlsson, Urs Lang, Viktor Schroeder, Andrzej Żuk, and, in particular, the anonymous referees for many helpful comments related to the contents and presentation of this paper.

1 Preliminaries

All graphs $\mathcal{G}$ under consideration are assumed to be embedded in the (topological) plane $\mathbf{R}^2$. Let $\mathcal{V}$ and $\mathcal{E}$ denote the sets of vertices and edges of $\mathcal{G}$. The vertex set $\mathcal{V}$ is countable, and $\mathcal{G}$ is assumed to have no loops nor multiple edges. Let $\mathcal{F}$ denote the family of connected components of the complement of the image of $\mathcal{G}$ in $\mathbf{R}^2$. The elements of $\mathcal{F}$ are called *faces* for $\mathcal{G}$. A *path* of length $|w| = k$ in $\mathcal{G}$ is a finite sequence of vertices $w = (v_0, v_1, v_2 \ldots, v_k)$ so that subsequent vertices are connected by an edge. In particular, if $k \geq 1$, then $|w| = E_w$ is the number of edges contained in the path. We say that an egde e belongs to a path w, if the end points of e coincide with two subsequent vertices of w. Let $A \subset \mathcal{F}$ be a finite set of faces of $\mathcal{G}$. A is called a *polygon*, if

$$\mathcal{P}(A) = \bigcup_{f \in A} \bar{f} \subset \mathbf{R}^2$$

is homeomorphic to a closed disc and if its topological boundary $\partial\mathcal{P}(A)$ defines a simply closed path of vertices which is then called the *boundary cycle* ∂A of A. This definition also applies for A consisting of a single face f in which case we denote the boundary cycle by ∂f. The edges and vertices which are contained in $\partial\mathcal{P}(A)$ are called the *sides* and *vertices* of A. If a polygon A has k sides it is called a *k-gon*. The number of edges emanating from a vertex $v \in \mathcal{V}$ is called the *degree of* v, denoted by $|v|$. If v belongs to ∂A then the *inner degree* $|v|_A^i$ of v (relative to A) is the number of faces $f \in A$ meeting in v, and the *exterior degree* $|v|_A^e$ is $|v|_A^e := |v| - |v|_A^i$.

We now introduce the notion of a plane tessellation:

Definition 1.1 A graph $\mathcal{G}$ in $\mathbf{R}^2$ is called *tessellating*, if the following conditions are satisfied:

i) Any edge is a side of precisely two different faces.

ii) Any two distinct faces have at most one vertex or at most one side in common.

iii) Any face $f \in \mathcal{F}$ is a polygon with finitely many sides.

iv) Every vertex has finite degree.

We will only consider *locally finite* tessellations. A tessellation is called locally finite if, for every $p \in \mathbf{R}^2$, there is an open set containing p which meets only finitely many faces of the tessellation.

1.1 Duality of tessellations

If $\mathcal{G}$ is a tessellating graph we can define its dual $\mathcal{G}^*$ which is again a tessellating graph. To each face f of $\mathcal{G}$ there corresponds a vertex $v^*(f)$ of $\mathcal{G}^*$, and vertices $v^*(f)$ and $v^*(f')$ are connected by an edge e^* if and only f and f' share an edge e. Thus geometric properties of $\mathcal{G}$ are equivalent to corresponding dual notions in $\mathcal{G}^*$. This motivates the following definitions.

The *distance* $d(f, f')$ of two faces f, f' of a tessellating graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ is the combinatorial distance of their corresponding vertices in the dual graph $\mathcal{G}^*$. Two faces are called *neighbours*, if they have an edge in common. A sequence $(f_0, f_1, \ldots)$ of faces is a *connected path*, if any two subsequent faces f_j, f_{j+1} are neighbours. Such a path corresponds to a connected path of vertices in the dual $\mathcal{G}^*$. A *minimal geodesic path of faces* in $\mathcal{G}$ is then a path $(f_0, f_1, \ldots)$ which satisfies $d(f_i, f_j) = |i - j|$ for all i, j. For a given face f_0 we define the *distance balls* and *distance spheres* by

$$\begin{aligned} A_r(f_0) &= \{f \in \mathcal{F} \mid d(f, f_0) = r\} , \\ B_r(f_0) &= \{f \in \mathcal{F} \mid d(f, f_0) \leq r\} . \end{aligned}$$

Definition 1.2 Let $f_0 \in \mathcal{F}$ be a fixed face and $d_0(f) := d(f_0, f)$, for all $f \in \mathcal{F}$, be the distance function of f_0. The *cut locus* $C(f_0)$ of f_0 is the set

$$C(f_0) \; := \; \{ \; f' \in \mathcal{F} \mid d_0(f) \leq d_0(f') \text{ for all neighbours } f \text{ of } f' \; \}.$$

That is, $C(f_0) \subset \mathcal{F}$ is the set where the function d_0 attains its local maxima.

Example: Let $\mathcal{G}$ be the dual of the Cayley graph of the group $\langle s, t \mid s^3 = t^3 = (st)^3 = 1\rangle$ (see [Ha, pp. 161]). Since we work in the dual setting the curvature (1) at a vertex translates into the curvature $\kappa(f)$ at a face as follows

$$\kappa(f) = 1 - \frac{E_{\partial f}}{2} + \sum_{(v,f) \in \mathcal{C}} \frac{1}{|v|}.$$

$\mathcal{G}$ is illustrated in Figure 2. One easily checks that $\kappa(f) = 0$ at all faces and that the cut locus $C(f_0)$ is not empty (even infinite) [1]. This example shows that the condition of Theorem 1 cannot be weakened to the condition $\kappa(f) \leq 0$ for all $f \in \mathcal{F}$.

Note that there is a natural one-to-one correspondence of corners in a graph $\mathcal{G}$ and its dual $\mathcal{G}^*$ and that the curvature function $\kappa(v, f)$ coincides on corners and their duals. In the subsequent sections of our paper we will study the distance function d on the set of faces of tessellations, and prove corresponding results on geodesic paths of faces. *By duality this will prove the results as presented in the introduction.*

[1] We thank R. Grigorchuck for bringing this particular example to our attention.

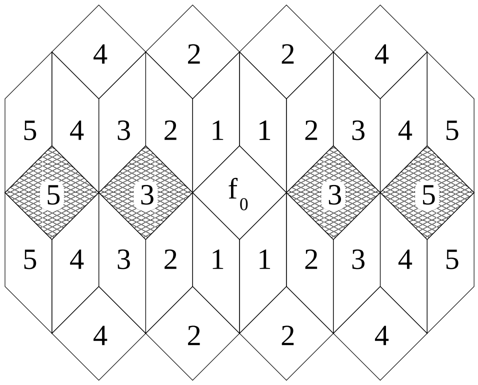

Figure 2: A plane tessellation with vanishing curvature at the faces. The (nonempty) cut locus $C(f_0)$ is hatched.

2 Distance balls in non-positively curved tessellations

2.1 The Gauß-Bonnet formula and Lyndon's inequality

An important tool in [BP] is Lyndon's formula which holds in a tessellating graph $\mathcal{G}$ of type (p,q). The purpose of this section is to provide a suitable generalization of Lyndon's formula which holds in all $\mathcal{G}$ with non-positive combinatorial curvature. To achieve this, and for a more conceptual understanding of the curvature formulas which play a role here, we briefly recall the notion of *conformal structure* on a tessellation (see [Co] for further reference).

Conformal structures and curvature We denote with $\mathcal{C}$ the set of all corners of $\mathcal{G}$. A *conformal structure* on $\mathcal{G}$ is a real function $\mu : \mathcal{C} \to \mathbf{R}$. We may think of the value $\mu(v,f)$ as an angle in the corner (v,f). In this sense, a conformal structure is a device to introduce geometry in a combinatorial setting.

Let $\mu : \mathcal{C} \to \mathbf{R}$ be a conformal structure on $\mathcal{G}$. We put $\mu(v) = \sum_{(v,f)\in\mathcal{C}} \mu(v,f)$, and $\mu(f) = \sum_{(v,f)\in\mathcal{C}} \mu(v,f)$. The expression

$$\kappa_\mu(v) = 1 - \mu(v)$$

is called the *curvature of* μ at the vertex v, and

$$\kappa_\mu(f) = \mu(f) - \left(\frac{E_{\partial f}}{2} - 1\right)$$

is called the *curvature (area defect)* of μ at the face f.

Let P be a polygon in $\mathcal{G}$. We put $C_P = \{(v,f) \in \mathcal{C} \mid f \in P\}$ for the set of corners contained in P. For $v \in \partial P$, we define the *interior curvature* (relative to P) as

$$\kappa^i_{\mu,P}(v) = \frac{1}{2} - \sum_{(v,f)\in C_P} \mu(v,f) \ .$$

Similarly, one can define the *exterior curvature*

$$\kappa^e_{\mu,P}(v) = \frac{1}{2} - \sum_{(v,f)\in\mathcal{C}, f\notin P} \mu(v,f).$$

Note that $\kappa_\mu(v) = \kappa^i_{\mu,P}(v) + \kappa^e_{\mu,P}(v)$ for all $v \in \partial P$. Also, we define the curvature of the boundary ∂P as

$$\kappa^i_\mu(\partial P) = \sum_{v\in\partial P} \kappa^i_{\mu,P}(v) \ .$$

With these notations, the *Gauß-Bonnet formula* for μ and P (see [Co] for a proof) reads as

$$1 = \kappa^i_\mu(\partial P) + \sum_{v\in P, v\notin\partial P} \kappa_\mu(v) + \sum_{f\in P} \kappa_\mu(f) \ . \tag{2}$$

Combinatorial curvature functions Let $\mathcal{G}$ be a tessellating graph. Natural choices for conformal structures on $\mathcal{G}$ which carry combinatorial information are, for example,

$$\mu_1(v,f) = \frac{1}{2} - \frac{1}{E_{\partial f}} \quad \text{and} \quad \mu_2(v,f) = \frac{1}{|v|} \ .$$

For our purposes, we will always consider $\mathcal{G}$ together with the conformal structure μ_1. *All curvature functions on $\mathcal{G}$ will therefore be with respect to μ_1. Consequently, from now on we drop the subscript μ_1 in the notation.*

With this convention, $\kappa(f) = 0$, for all $f \in \mathcal{F}$, and the vertex curvature for μ_1 recovers the curvature $\kappa(v)$ as defined in the introduction, that is, for all $v \in \mathcal{V}$,

$$\kappa_{\mu_1}(v) = \kappa(v) = 1 - \frac{|v|}{2} + \sum_{(v,f)\in\mathcal{C}} \frac{1}{E_{\partial f}} \ .$$

Let us also recall the definition of the combinatorial curvature $\kappa(v,f)$ in the corners of $\mathcal{G}$ as stated in the introduction, namely:

$$\kappa(v,f) = \frac{1}{|v|} + \frac{1}{E_{\partial f}} - \frac{1}{2} \ .$$

Consequently, we have

$$\kappa(v) = \sum_{(v,f)\in C} \kappa(v,f).$$

If P is a polygon, and $v \in \partial P$, then

$$\begin{aligned} \kappa^i_P(v) &= \frac{1}{2} + \sum_{(v,f)\in C_P} \left(\frac{1}{E_{\partial f}} - \frac{1}{2}\right) \\ &= \frac{1}{2} + \sum_{(v,f)\in C_P} (\kappa(v,f) - \frac{1}{|v|}) \ . \end{aligned} \tag{3}$$

In particular, if the curvature in the corners is non-positive then

$$\kappa^i_P(v) \le \frac{1}{2} - \frac{|v|^i_P}{|v|} \ . \tag{4}$$

Note that negative curvature in all corners implies strict inequality in (4).

From the Gauß-Bonnet formula (2) for μ_1, we deduce:

Proposition 2.1 (Lyndon's inequality) *Let $\mathcal{G}$ be a tessellating plane graph which satisfies $\kappa(v) \leq 0$, for all v in $\mathcal{V}$. Let P be a polygon with boundary curve ∂P. Then*

$$1 \leq \kappa^i(\partial P) = \sum_{v \in \partial P} \kappa_P^i(v) \tag{5}$$

Remark A tessellating graph $\mathcal{G}$ of type (p, q) satisfies $\kappa(v, f) \leq 0$. Using Proposition 2.1 and $1/p + 1/q = 1/2$ we conclude

$$1 \leq \sum_{v \in \partial P} \left(\frac{1}{2} + \sum_{(v,f) \in C_P} \left(\frac{1}{q} - \frac{1}{2} \right) \right) = \sum_{v \in \partial P} \left(\frac{1}{2} - \frac{|v|_P^i}{p} \right),$$

which is Lyndon's classical formula for graphs of type (p, q) (compare [Ly1], [BP, §2]).

2.2 Admissible Polygons

Let B be a polygon. We introduce a labeling of the boundary vertices of B with the letters a and b. This labeling reflects convexity properties of the polygon B. First, for any vertex v of $\mathcal{G}$, we define $N(v) = \min_{f, v \in \partial f} E_{\partial f}$. A vertex $v \in \partial B$ obtains label a if $|v|_B^i = 1$, or if both $N(v) = 3$ and $|v|_B^i \leq 3$ hold. All other vertices of ∂B obtain label b. Moreover, if v is an a-vertex satisfying $|v|_B^i = 1$ then we say that v is of type a^+. (Note that the labeling in this article is more refined than the one introduced in [BP]).

Definition 2.2 A connected path $w \subset \partial B$ is called *admissible* if for every vertex $v \in w$ with label b its neighbours in w carry the label a^+. We call the polygon B *admissible* iff ∂B is admissible.

We consider a^+-vertices as particularly convex vertices whereas b-vertices are particularly concave. With this interpretation, a polygon is admissible iff particularly concave boundary vertices are compensated by particularly convex neighbours along ∂B.

Admissible polygons play a prominent role in our considerations. We describe now their curvature properties. For any subpath w contained in ∂B, we define

$$\kappa_B^e(w) = \sum_{v \in w} \kappa_B^e(v) .$$

In tessellations with non-positively curved corners, the labeling of ∂B controls, to some extent, the exterior curvature on ∂B.

Lemma 2.3 *Let B be a polygon in $\mathcal{G}$, and $v \in \partial B$. Then*

a) $\kappa_B^i(v), \kappa_B^e(v) \leq \frac{1}{3}$.

If $\kappa(v, f) \leq 0$, for all $(v, f) \in \mathcal{C}$, then:

b) If v has label a^+ then $\kappa_B^e(v) < 0$.

c) If v has label a then $\kappa_B^e(v) \leq 0$.

d) If v has label a^+ and if $v_1 \in \partial B$ is any neighbour of v then $\kappa_B^e(v) + \kappa_B^e(v_1) \leq 0$.

e) If v satisfies $|v|_B^e = 1$ then $\kappa_B^i(v) < 0$, and for every neighbour $v_1 \in \partial B$: $\kappa_B^i(v) + \kappa_B^i(v_1) \leq 0$.

f) If w is an admissible subpath of ∂B then $\kappa_B^e(w) \leq \frac{1}{3}$.

(If $\kappa(v,f) < 0$, for all $(v,f) \in \mathcal{C}$, then the inequalities in c), d) and e) are strict.)

Proof. Since $E_{\partial f} \geq 3$, for all $f \in \mathcal{F}$, $\kappa_B^i(v) \leq \frac{1}{2} - \frac{|v|_B^i}{6} \leq \frac{1}{3}$. Analogously, $\kappa_B^e(v) \leq \frac{1}{3}$. Therefore a) holds.

Next, we show b.) By assumption, $v \in \partial B$ has label a^+, and $\kappa(v,f) \leq 0$, for all corners. We deduce

$$\kappa_B^e(v) \;\leq\; \frac{1}{2} - (|v|-1)\frac{1}{|v|} = -\frac{1}{2} + \frac{1}{|v|} < 0. \tag{6}$$

If v has label a^+ then b) implies c) a fortiori. We may assume therefore that v has label a and $N(v) = 3$. Since $N(v) = 3$, there exists a corner (v,f), so that $E_{\partial f} = 3$. From $\kappa(v,f) \leq 0$ it follows that $|v| \geq 6$. Therefore,

$$\kappa_B^e(v) \;\leq\; \frac{1}{2} - (|v|-3)\frac{1}{|v|} \;\leq\; 0\;.$$

For the proof of d), let f denote the face outside B with edge $vv_1 \in \partial B$. Using (6) we deduce

$$\begin{aligned}\kappa_B^e(v_1) + \kappa_B^e(v) \;\leq\; \frac{1}{2} + &\sum_{(v,f')\in\mathcal{C},f'\notin B} \left(\kappa(v_1,f') - \frac{1}{|v_1|}\right) + \left(-\frac{1}{2} + \frac{1}{|v|}\right)\\ &\leq \kappa(v_1,f) - \frac{1}{|v_1|} + \frac{1}{|v|} = \kappa(v,f) \leq 0.\end{aligned}$$

Hence, d) holds.

The proof of e) follows along the same lines as the proofs of b) and d).

It remains to prove f). Now c) implies that only b–vertices give a positive contribution to $\kappa_B^e(w)$. By d), such a contribution is compensated by any neighbour with label a^+. In particular, if w is an admissible subpath of ∂B then either $\kappa_B^e(w) \leq 0$, or $\kappa_B^e(w) \leq \kappa_B^e(u)$, where $u \in w$ has label b and is one of the endpoints of w. Hence, a) implies f).

For graphs with negatively curved corners the strict inequalities for c) d), and e) are immediate consequences of the above arguments. □

2.3 The Bridge Lemma

We establish now an important *convexity property* of admissible polygons in non-positively curved tessellations. To formulate the result we have to introduce the notion of a bridge over a polygon. The following definitions are slight modifications of the ones given in [BP, §4]: Let B and P be polygons of the graph $\mathcal{G}$, and $\partial B, \partial P \subset \mathcal{V}$ their boundary cycles. Assume that B, P satisfy the following properties:

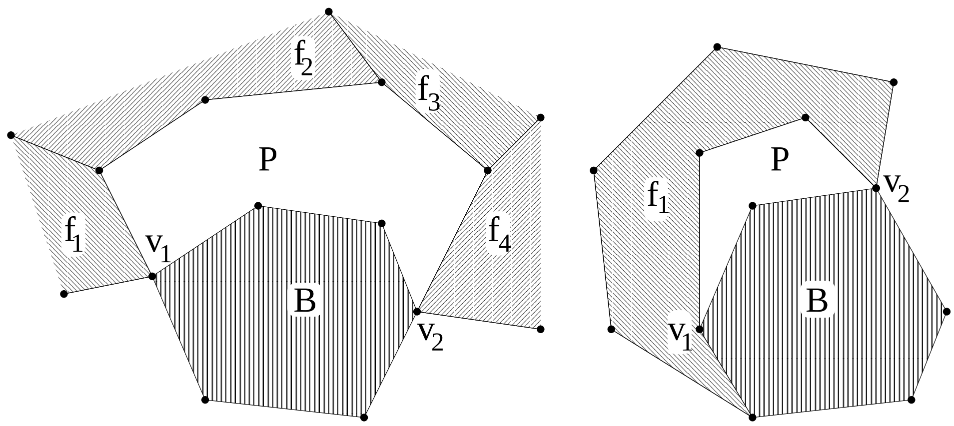

Figure 3: Examples of bridges of length 4 and 1. The second bridge is attached to B in the corner (v_1, f_1).

i) $\partial P \cap \partial B \neq \emptyset$,

ii) $\partial P \cap \partial B$ is a connected path,

iii) $P \cap B = \emptyset$.

Let v_1, v_2 denote the two end points of the path $\partial P \cap \partial B$. (Note that $v_1 = v_2$ if $\partial P \cap \partial B$ consists of a single vertex.) Let $w \subset \partial P$ be the path complementary to $\partial P \cap \partial B$ except for its end points v_1, v_2. Starting in v_1 let us follow the path w until v_2. Thereby we encounter, as shown in Figure 3, a sequence $\Gamma = \{f_1, f_2, \ldots, f_k\}$ of faces in $\mathcal{F} - P$ which share at least one edge with ∂P. We call the sequence Γ a *bridge over* B of length $L(\Gamma) = k$. We say that Γ *is attached to* B *in the corner* (v_1, f_1), if ∂B contains an edge of f_1 which emanates from v_1. Correspondingly, we say that Γ *is attached to* B *in the corner* (v_2, f_k), if $\partial f_k \cap \partial B$ contains an edge which emanates from v_2.

Lemma 2.4 (Bridge Lemma) *Let $\mathcal{G}$ be a tessellation which is non-positively curved in all corners, and let B,P be polygons in $\mathcal{G}$. Assume further that B is admissible, and that B,P satisfy the properties above. If Γ is attached to B in one corner then*

$$L(\Gamma) \geq 2 \; .$$

If Γ is attached to B in both corners then

$$L(\Gamma) \geq 3 \; .$$

Proof. Let w_1 denote the connected path in ∂P which is disjoint to ∂B. We conclude with Lemma 2.3 a), e) that $\kappa_P^i(w_1) \leq \; (L(\Gamma) - 1)\frac{1}{3}$. Let us assume first that $v_1 = v_2$. Then

$$\kappa_P^i(\partial P) \;\leq\; (L(\Gamma) - 1)\frac{1}{3} + \kappa_P^i(v_1) \;\leq L(\Gamma)\frac{1}{3} \; .$$

Lyndon's inequality (5) implies $L(\Gamma) \geq 3$ in this case. So we are left with the case $v_1 \neq v_2$. Let w_2 be the path along $\partial P \cap \partial B$, but without the endpoints v_1,

v_2. (Note that w_2 may be empty.) As above, we conclude:

$$\kappa_P^i(\partial P) \leq (L(\Gamma)-1)\frac{1}{3} + \kappa_P^i(v_1) + \kappa_P^i(v_2) + \kappa_P^i(w_2) .$$

Suppose now that Γ is attached to B in one corner. We assert that then

$$\kappa_P^i(v_1) + \kappa_P^i(v_2) + \kappa_P^i(w_2) \leq \frac{2}{3} . \tag{7}$$

It follows that, $\kappa_P^i(\partial P) \leq (L(\Gamma)+1)\frac{1}{3}$. Lyndon's inequality (5) implies $L(\Gamma) \geq 2$.

If Γ is attached to B in both corners then we assert that

$$\kappa_P^i(v_1) + \kappa_P^i(v_2) + \kappa_P^i(w_2) \leq \frac{3}{6} . \tag{8}$$

This implies that $1 < (L(\Gamma)+1)\frac{1}{3}$, and hence $L(\Gamma) \geq 3$. For the proof of the Bridge lemma it remains therefore to prove our assertions. To accomplish this, we start with three observations. (In the following all labels of vertices are meant with respect to the polygon B.)

Observation 1 *Assume that Γ is attached in (v_1, f_1). If v_1 is of type a then $\kappa_P^i(v_1) \leq \frac{1}{6}$.*

Proof. If v_1 is of type a^+ then we deduce from equation (4) that

$$\kappa_P^i(v_1) \leq \frac{1}{2} - \frac{(|v_1|-2)}{|v_1|} = -\frac{1}{2} + \frac{2}{|v_1|} \leq \frac{1}{6} . \tag{9}$$

If v_1 is of type a, but not a^+, then $N(v_1) = 3$. By the curvature condition, $|v_1| \geq 6$, so that $|v_1|_P^i \geq 6 - 4 = 2$, and therefore

$$\kappa_P^i(v_1) = \frac{1}{2} + \sum_{(v_1,f)\in C_P} \left(\frac{1}{E_{\partial f}} - \frac{1}{2}\right) \leq \frac{1}{2} + 2\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{6}.$$

Observation 2 *Assume that Γ is attached in (v_1, f_1). If both end points of w_2 carry label b then $\kappa_P^i(v_1) + \kappa_P^i(w_2) \leq \frac{1}{|v_1|}$.*

Proof. Since ∂B is admissible, v_1 has label a^+. Let v' be the end point of w_2 next to v_1. Let h denote the face of P with edge v_1v'. Using (9) and $\kappa(v_1, h) \leq 0$ we deduce that

$$\kappa_P^i(v_1) + \kappa_P^i(v') \leq \left(-\frac{1}{2} + \frac{2}{|v_1|}\right) + \frac{1}{E_{\partial h}} \leq \frac{1}{|v_1|} .$$

Note that the (possibly empty) path w' which is obtained from w_2 by removing v' satisfies $\kappa_P^i(w') \leq 0$. This implies Observation 2.

Observation 3 *If $\kappa_P^i(w_2) < 0$, and v_1,v_2 carry label b then $\kappa_P^i(v_1) + \kappa_P^i(w_2) \leq 0$.*

Proof. The neighbour $v' \in w_2$ of v_1 carries label a^+. It follows as in the proof of Lemma 2.3 d) that $\kappa^i_P(v_1) + \kappa^i_P(v') \leq 0$. Since the (possibly empty) path w' which is obtained from w_2 by removing v' satisfies $\kappa^i_P(w') \leq 0$, Observation 3 holds.

Now we can prove our first assertion. We may assume that Γ is attached in (v_1, f_1). If $\kappa^i_P(w_2) \leq 0$ then (7) is immediate from Lemma 2.3 a). Assume therefore $\kappa^i_P(w_2) > 0$. Since w_2 is admissible, Lemma 2.3 d) implies that both end points of w_2 have to carry label b. Then (7) is implied by Observation 2.

We are left to prove the second assertion. By assumption Γ is attached in both corners. We consider first the case that $\kappa^i_P(w_2) \leq 0$. We note that $\kappa^i_P(w_2) = 0$ is only possible if one of v_1,v_2 carries label a. But then, by Observation 1, $\kappa^i_P(v_1)+\kappa^i_P(v_2)+\kappa^i_P(w_2) \leq \frac{1}{6}+\frac{1}{3} = \frac{3}{6}$. So assume that $\kappa^i_P(w_2) < 0$, and that both v_1 and v_2 carry label b. By Observation 3, $\kappa^i_P(v_1)+\kappa^i_P(v_2)+\kappa^i_P(w_2) \leq \kappa^i_P(v_2) \leq \frac{1}{3}$, and hence (8) holds. It remains to deal with the case $\kappa^i_P(w_2) > 0$. In this situation, both v_1 and v_2 carry label a^+. Now, by Observation 2, $\kappa^i_P(v_1) + \kappa^i_P(w_2) \leq \frac{1}{3}$. Therefore, using Observation 1, we get (8).

□

2.4 Admissibility of distance balls

Let P be a polygon. Then the set $T = \{f \in \mathcal{F} \mid d(f,B) \leq 1\}$ is called the *distance one tube of P*. Under the assumption that $\mathcal{G}$ is non-positively curved in all corners, we establish here that the distance one tube of an admissible polygon in $\mathcal{G}$ is again an admissible polygon. This is the crucial step to prove that distance balls are polygons. In fact, we show here that distance balls are admissible polygons.

Our first result is a direct consequence of the Bridge Lemma.

Proposition 2.5 *Let $\mathcal{G}$ be non-positively curved in all corners, and let B be an admissible polygon. Then the distance one tube $T = \{f \in \mathcal{F} \mid d(f,B) \leq 1\}$ is a polygon, and every face $f \in T - B$ satisfies:*

(a) $\partial f \cap \partial B$ is a connected path of edges of length ≤ 2,

(b) $\partial f \cap \partial T$ is a connected path of length ≥ 1.

Proof. If $\partial f \cap \partial B$ were not connected, f would form a bridge of length one over B which is attached in (at least) one corner. By the Bridge Lemma this is not possible. Therefore $\partial f \cap \partial B$ is connected. Since interior vertices of the path $\partial f \cap \partial B$ have label b with respect to B, the admissibility of B implies that the length of $\partial f \cap \partial B$ is ≤ 2. This shows a).

Following the edges of the boundary cycle ∂B, we get an enumeration

$$T - B = \{f_1, f_2, \dots, f_k\},$$

of the faces in $T - B$. We remark that only faces with subsequent indices (mod k) intersect in their closures. This follows since any two intersecting faces which are not subsequent form a bridge of length 2 over B which is attached in both corners. We deduce that, for all j, $\partial f_j \cap \partial T$ is connected, and contains at least

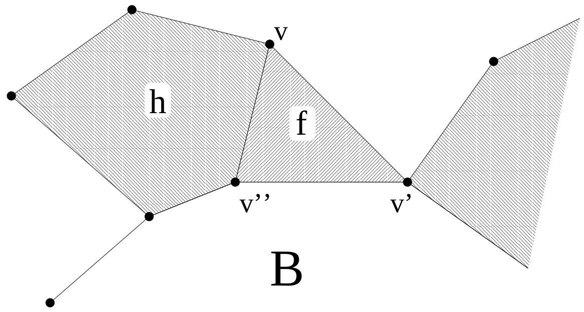

Figure 4: The case of a triangle f with $|\partial f \cap \partial B| = 1$

one edge. (For, otherwise f_{j-1} and f_{j+1} intersect in their closures.) This shows b). It follows also that ∂T is a closed path of edges without repeated vertices, and hence, T is a polygon. $\square$

Proposition 2.6 *Let $\mathcal{G}$ be non-positively curved in all corners, and let B be an admissible polygon. Then the distance one tube T is an admissible polygon.*

Proof. We have to show that the polygon T is admissible. Observe first that ∂T is admissible if and only if every length 1 subpath of ∂T is admissible. Since any such path is an edge of some $f \in T - B$ it suffices therefore to prove that, for all $f \in T - B$, the path $\partial f \cap \partial T$ is admissible.

First, we assert that every face $f \in T - B$ with $E_{\partial f} \geq 4$ satisfies $|\partial f \cap \partial T| \geq 2$, and hence contributes an admissible path to ∂T. (Note that all inner vertices of the path $\partial f \cap \partial T$ carry a^+ labels). If $E_{\partial f} \geq 6$ then it follows from Proposition 2.5 (a) that $|\partial f \cap \partial T| \geq 2$. Assume next that $4 \leq E_{\partial f} \leq 5$. Note that if one of the endpoints of $\partial f \cap \partial B$ is of type a with respect to B then f can not share an edge with its corresponding neighbour along ∂B. If $|\partial f \cap \partial B| = 2$ both endpoints of $\partial f \cap \partial B$ are of type a^+. In particular, since $E_{\partial f} \geq 4$, $|\partial f \cap \partial T| \geq 2$. If $|\partial f \cap \partial B| = 1$ then, by admissibility of ∂B, at least one of the two endpoints of $\partial f \cap \partial B$ is of type a. It follows that $|\partial f \cap \partial T| \geq 2$. Therefore our first assertion holds.

If a triangle contributes two edges to ∂T then the inner vertex of the path $\partial f \cap \partial T$ carries an a^+ label and the path is admissible.

Next we assert that if a triangle $f \in T - B$ contributes only one edge to ∂T then both of its endpoints v and v' carry label a with respect to T. If $|\partial f \cap \partial B| = 2$ then v and v' belong to ∂B, and are of type a^+ with respect to B. Since $N(v) = N(v') = 3$ they are of type a with respect to T. We assume now that $|\partial f \cap \partial B| = 1$. Then f shares an edge $v''v$ with precisely one neighbour $h \in T - B$ (see Figure 4). Since $N(v) = 3$, $v \in \partial T$ carries label a. Note that v'' has label b with respect to B. It follows that v' has label a^+ with respect to B, and hence label a with respect to T. Therefore we proved our last assertion. It follows that T is admissible. $\square$

Remark In the course of the proof we showed that any face $f \in T - B$ which contributes *only* a single edge to ∂T is a triangle.

Since every face $f \in \mathcal{F}$ constitutes an admissible polygon, we deduce from the two Propositions:

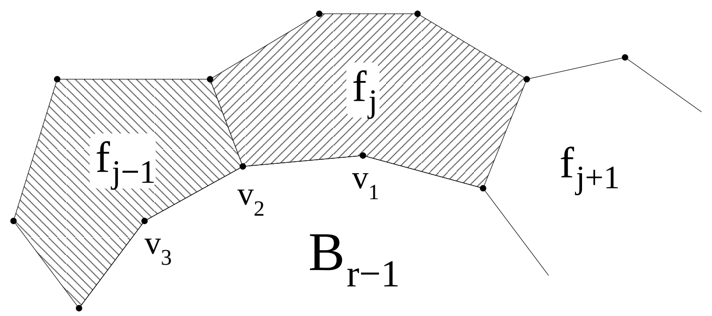

Figure 5: Analyzing the faces of A_{r-1}

Corollary 2.7 *Let $\mathcal{G}$ be a tessellating plane graph which is non-positively curved in all corners. Then every distance ball $B_r(f)$ in $\mathcal{G}$ is an admissible polygon, and every face of the distance sphere $A_r(f)$ contributes an edge to the boundary ∂B_r.*

Proof of Theorem 1. We prove the dual statement for the distance function d defined on faces. We show inductively that the distance balls $B_r(f_0)$ are polygons with the property that every face f with $d(f, f_0) = r$ contributes at least one edge to $\partial B_r(f_0)$. This obviously holds for $r = 0$. The induction step is carried out in Propositions 2.5 and 2.6. $\square$

2.5 Colouring of distance spheres

Let $\mathcal{G}$ be non-positively curved in all corners, and let

$$A_r = A_r(f) = \{f_1, f_2, \ldots, f_k\}$$

be a distance sphere in $\mathcal{G}$ with a cyclic enumeration of its faces. We remark:

Lemma 2.8 *There is at least one face $f_j \in A_r$ with the following property: either $|\partial f_j \cap \partial B_{r-1}(f)| = 1$ or f_j does not share a common edge with both f_{j-1} and f_{j+1} (mod k).*

Proof. Assume to the contrary that all faces f_j of A_r have two distance one neighbours in A_r and satisfy $|\partial f_j \cap \partial B_{r-1}| = 2$. Note that this is only possible if $r \geq 2$ by Definition 1.1 ii). Now it follows that all faces h of A_{r-1} are 6-gons which have two distance one neighbours in A_{r-1} and satisfy $|\partial h \cap \partial B_{r-2}| = 2$. (Compare Figure 5: The vertices v_1 and v_3 are b-vertices with respect to B_{r-1}. Since B_{r-1} is admissible, v_2 carries label a^+. Hence $|v_2| = 3$ and there exists $h \in A_{r-1}$ which satisfies $\partial h \cap \partial B_{r-1} = (v_3, v_2, v_1)$. By the curvature condition, $|E_{\partial h}| \geq 6$. By Proposition 2.5 h shares at most two edges with ∂B_{r-2}, and h is a 6-gon which has distance one to both neighbours in A_{r-1}.) Hence, our assumption must be satisfied for all $r \geq 1$, a contradiction. $\square$

The lemma implies

Corollary 2.9 *The faces of any non-positively curved tessellating plane graph $\mathcal{G}$ are colourable with four colours.*

Proof. We assign a colour to the fixed face of reference $f \in \mathcal{F}$ and colour the distance spheres A_r inductively: Assume that all faces of B_{r-1} are coloured. Let $A_r = \{f_1, \ldots, f_k\}$ and assume by Lemma 2.8 that f_k is the face which satisfies $|\partial f_k \cap \partial B_{r-1}| = 1$ or has at most one distance one neighbour in A_r, namely f_{k-1}. Since $|\partial f_j \cap \partial B_{r-1}| \leq 2$ (Proposition 2.5), we can colour $f_1, f_2, \ldots, f_k$ successively, since each face f_j ($1 \leq j \leq k$) has at most three neighbours which are already coloured. This extends the colouring to B_r and finishes the induction step. □

3 Bigons in negatively curved tessellations

In this section we study properties of geodesic bigons in a plane tessellating graph $\mathcal{G}$ with negative curvature in all corners. If $\mathcal{G}$ has a uniform upper bound δ on vertex degrees then our results imply that the bigons in $\mathcal{G}$ are δ-thin.

3.1 Bridges in negatively curved tessellations

To understand the properties of bridges in negatively curved tessellations we use the calculus which we applied in the proof of the Bridge Lemma.

We will need the following lemma:

Lemma 3.1 *Let $\mathcal{G}$ be negatively curved in all corners, let P be a polygon in $\mathcal{G}$, $v \in \partial P$, and let $v_1 \in \partial P$ be a neighbour of v.*

a) If $|v|_P^e \leq 2$ then $\kappa_P^i(v) \leq \frac{1}{7}$, and $\kappa_P^i(v) + \kappa_P^i(v_1) < \frac{1}{3}$.

b) If $|v|_P^e = 3$ then $\kappa_P^i(v) \leq \frac{1}{5}$, and $\kappa_P^i(v) + \kappa_P^i(v_1) < \frac{1}{2}$.

Proof. Assume that $|v|_P^e \leq 2$. Then we use (4) and $|v|_P^i \geq |v| - 2$ to deduce $\kappa_P^i(v) \leq -\frac{1}{2} + \frac{2}{|v|}$. If $|v| \geq 4$ this yields $\kappa_P^i(v) \leq 0$. If $|v| = 3$ then all faces f with $v \in \partial f$ satisfy $E_{\partial f} \geq 7$, hence $\kappa_P^i(v) \leq \frac{1}{E_{\partial f}} \leq \frac{1}{7}$. This shows the first part of a). Next we compute as in the proof of Observation 2 that $\kappa_P^i(v) + \kappa_P^i(v_1) < \frac{1}{|v|} \leq \frac{1}{3}$. This shows a).

Assume that $|v|_P^e = 3$. If $|v| \geq 5$ then $\kappa_P^i(v) \leq -\frac{1}{2} + \frac{3}{|v|} < \frac{1}{5}$. If $|v| = 4$ then all faces f meeting in v satisfy $E_{\partial f} \geq 5$, and hence $\kappa_P^i(v) \leq \frac{1}{5}$. This shows the first part of b). Next we compute that $\kappa_P^i(v) + \kappa_P^i(v_1) < \frac{2}{|v|}$. Since $|v| \geq 4$ this shows b). □

Let $\Gamma = \{f_1, \ldots, f_k\}$ be a bridge over a polygon B which encloses the polygon P. Let w_1 denote the connected path in ∂P disjoint to ∂B, and let $v_1, v_2 \in \partial B$ denote the endpoints of the path $\partial P \cap \partial B$. We define the *weight of* Γ, as $\mu(\Gamma) = \kappa_P^i(w_1)$. (Recall from Section 2.3 that if $\mathcal{G}$ is non-positively curved in all corners then $\mu(\Gamma) \leq (L(\Gamma) - 1)\,\frac{1}{3}$.) Finally, we say that Γ is *semi-attached* in the corner (v_1, f_1), if this corner is separated from B by at most one more corner outside P (when moving counterclockwise from (f_1, v_1) to B). Correspondingly, we define that Γ is semi-attached in (v_2, f_k).

Proposition 3.2 *Let $\mathcal{G}$ be a tessellation which is non-positively curved in all corners, and let Γ be a bridge over an admissible polygon B. Then $\mu(\Gamma) \geq 0$, and* *i) If Γ is attached to B in one corner then $\mu(\Gamma) \geq \frac{1}{3}$.*

ii) If Γ is attached to B in both corners then $\mu(\Gamma) \geq \frac{1}{2}$.

If $\mathcal{G}$ is negatively curved in all corners then also the following holds:

iii) If Γ is semi-attached to B in one corner then $\mu(\Gamma) > \frac{1}{6}$.

iv) If Γ is semi-attached to B in both corners then $\mu(\Gamma) > \frac{1}{3}$.

Proof. We continue with the notational conventions introduced just before the proposition. In particular, P denotes the polygon enclosed by the bridge Γ, and v_1, v_2 denote the endpoints of $\partial P \cap \partial B$. We first consider the special case $v_1 = v_2$. Then Lyndon's inequality (5) states that $1 \leq \mu(\Gamma) + \kappa^i_P(v_1)$. Since $\kappa^i_P(v_1) \leq \frac{1}{3}$ it follows that $\mu(\Gamma) \geq \frac{2}{3}$. In particular, i)-iv) hold, and we assume $v_1 \neq v_2$ for the rest of the proof.

Let w_2 denote the path along $\partial P \cap \partial B$ stripped of v_1, v_2. Then

$$1 \leq \mu(\Gamma) + \kappa^i_P(v_1) + \kappa^i_P(v_2) + \kappa^i_P(w_2)\ .$$

Since $\kappa^i_P(v_1) + \kappa^i_P(v_2) + \kappa^i_P(w_2) \leq 1$ it follows that $\mu(\Gamma) \geq 0$. Note that i) and ii), being equivalent to (7) and (8), were already established in the proof of the Bridge-Lemma.

Assume now that $\mathcal{G}$ is negatively curved. For the rest of the proof all labels are meant with respect to the polygon B. To show iii) we assume that Γ is semi-attached in (v_1, f_1). To show $\kappa^i_P(v_1) + \kappa^i_P(v_2) + \kappa^i_P(w_2) < \frac{5}{6}$ we need only to consider the case $\kappa^i_P(w_2) > 0$. Then v_1 carries label a^+ and hence $|v_1|^e_P \leq 3$. Moreover, the neighbour of v_1 in w_2 carries label b. Let us denote this neighbour by v' and let w_3 denote the subpath of w_2 obtained by dropping v'. Then Lemma 3.1 b) and Lemma 2.3 a) imply that

$$\begin{aligned} \kappa^i_P(v_1) + \kappa^i_P(w_2) + \kappa^i_P(v_2) &= \big(\kappa^i_P(v_1) + \kappa^i_P(v')\big) + \kappa^i_P(w_3) + \kappa^i_P(v_2) \\ &< \frac{1}{2} + 0 + \frac{1}{3} = \frac{5}{6}\ , \end{aligned}$$

and iii) holds.

To show iv), it is enough to consider the case $\kappa^i_P(w_2) \geq 0$. We assume first $\kappa^i_P(w_2) > 0$. Then v_1 and v_2 carry label a^+. In particular, Lemma 3.1 implies that $\kappa^i_P(v_1) + \kappa^i_P(w_2) < \frac{1}{2}$. If $|v_2| \geq 5$ then $\kappa^i_P(v_2) \leq -\frac{1}{2} + \frac{3}{|v_2|} < \frac{1}{6}$, so that $\kappa^i_P(v_1) + \kappa^i_P(w_2) + \kappa^i_P(v_2) < \frac{2}{3}$. If $|v_2| \leq 4$ then all faces f with $v_2 \in \partial f$ satisfy $E_{\partial f} \geq 5$. This implies $\kappa^i_P(v_2) \leq \frac{1}{5}$ and $\kappa^i_P(w_2) \leq \frac{1}{5}$. By Lemma 3.1, also $\kappa^i_P(v_1) \leq \frac{1}{5}$. Hence, $\kappa^i_P(v_1) + \kappa^i_P(v_2) + \kappa^i_P(w_2) \leq \frac{3}{5} < \frac{2}{3}$. Hence iv) holds if $\kappa^i_P(w_2) > 0$. We assume now that $\kappa^i_P(w_2) = 0$. Since the curvature is negative this implies that either both end points of w_2 carry label b (this case has already been considered) or that w_2 is empty. In particular v_1 and v_2 are neighbours along ∂P. If v_1 is of label a but not a^+ then $|v_1| \geq 7$, by negative curvature, and hence $\kappa^i_P(v_1) \leq -\frac{1}{2} + \frac{5}{|v_1|} \leq -\frac{1}{2} + \frac{5}{7}$. Therefore, if both v_1 and v_2 carry label a, $\kappa^i_P(v_1) + \kappa^i_P(v_2) \leq \frac{3}{7}$. If v_1 is of label a^+ then $|v_1|^e_P \leq 3$, and by Lemma 3.1, $\kappa^i_P(v_1) + \kappa^i_P(v_2) < \frac{1}{2}$. Hence iv) holds. $\square$

We can draw some conclusions about short bridges in tessellations which are negatively curved in all corners:

1) Bridges of length one are never attached, nor semi-attached. (In fact, any bridge of length one satisfies $\mu(\Gamma) = 0$, and $\kappa_P^i(v_1)+\kappa_P^i(v_2)+\kappa_P^i(w_2) = 1$. This implies that P consists of a single face h which is a triangle.)

2) Bridges of length two are never semi-attached in both corners. If a bridge $\Gamma = \{f_1, f_2\}$ of length two is attached in one corner then we must have $d(f_1, f_2) > 1$. This follows readily from Lemma 3.1 a) and Lemma 2.3 e).

3.2 The interior of bigons

Our principal result in this section is Theorem 2 of the introduction. As with Theorem 1, we are going to prove a dual statement. Therefore, we start with translating the properties of bigons (of geodesic path of vertices) in $\mathcal{G}$ into the properties of the corresponding bigons of geodesic paths of faces in $\mathcal{G}^*$.

Let $F = \{f_0, f_1, f_2, \dots, f_k\}$ and $G = \{g_0, g_1, \dots, g_k\}$ be geodesic segments (of faces) in $\mathcal{G}$. The pair (F, G) is called a *bigon* if F and G have the same endpoints, i.e., if $f_0 = g_0$ and $f_k = g_k$. We say that a bigon (F, G) is *minimal* if $f_j \neq g_j$, $1 \leq j \leq k-1$. The *interior* of (F, G) is defined to be the set of all faces $f \in \mathcal{F}$ which are enclosed by the geodesic segments F and G, but do not belong to F or G. Let us denote the interior of (F, G) by $\mathrm{int}(F, G)$. A bigon is called *δ-thin* if $d(f_j, g_j) \leq \delta$, for all $j \in \{0, \dots, k\}$, where d is the combinatorial distance between faces in $\mathcal{G}$. Our aim is to proof:

Theorem 3.3 *Let $\mathcal{G}$ be a plane tessellating graph. If $\kappa(v, f) < 0$, for all corners (v, f), then any minimal bigon (F, G) of geodesic paths of faces in $\mathcal{G}$ satisfies $\mathrm{int}(F, G) = \emptyset$.*

We begin now with preparations for the proof. If (F, G) is a bigon we denote the set of faces enclosed by F and G or belong to F or G with $\overline{\mathrm{int}}(F, G)$. Let us consider the intersection of $\overline{\mathrm{int}}(F, G)$ with the combinatorial distance spheres around the two endpoints f_0 and f_k:

$$A_j^0 = \{f \in \overline{\mathrm{int}}(F, G) \mid d(f, f_0) = j\} \text{ and } A_j^1 = \{f \in \overline{\mathrm{int}}(F, G) \mid d(f, f_k) = j\} .$$

If $\mathcal{G}$ has no cut locus then $\overline{\mathrm{int}}(F, G) = \bigcup_{j=0}^k A_j^0 = \bigcup_{j=0}^k A_j^1$. Furthermore we notice

Lemma 3.4 *If $\mathcal{G}$ has no cut locus then*

$$A_j^0 = A_{k-j}^1 .$$

Proof. Assume inductively that $A_i^0 = A_{k-i}^1$, $i = 0, \dots, j$. Every face $f \in A_{k-j-1}^1$ has a neighbour in A_{k-j}^1, and therefore in A_j^0. Since f does not belong to A_i^0, $i = 0, \dots, j$, it follows that $f \in A_{j+1}^0$. Therefore $A_{k-j-1}^1 \subset A_{j+1}^0$. Analogous arguments yield the opposite inclusion $A_{j+1}^0 \subset A_{k-j-1}^1$. □

Remark The arguments of the previous proof show also that every face $f \in \overline{\mathrm{int}}(F, G)$ lies on a geodesic segment which is contained in $\overline{\mathrm{int}}(F, G)$ and which connects f_0 and f_k.

Proof of Theorem 3.3. Since $\partial B_{k-1}(f_k)$ is admissible, Proposition 2.5 implies that $\partial f_0 \cap \partial B_{k-1}(f_k)$ is connected and of length ≤ 2. Therefore $A_{k-1}^1 =$

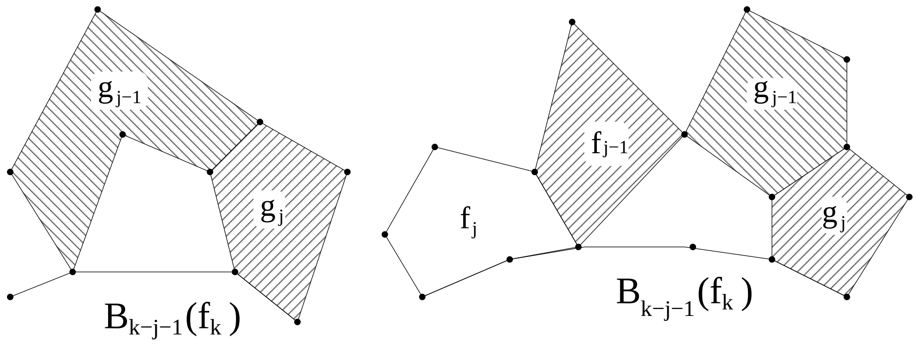

Figure 6: Bridges $\Gamma \subset S$

$A_1^0 = \{f_1, g_1\}$. We will prove now inductively that $A_j^0 = A_{k-j}^1 = \{f_j, g_j\}$, $j = 1, \ldots, k-1$.

Assume therefore that the induction hypothesis holds for $i = 1, \ldots, j-1$. Let $S = \{f_j, f_{j-1}, g_{j-1}, g_j\}$. If $\mathrm{int}(F,G) \cap A_{k-j}(f_k) \neq \emptyset$ then there exist a subset Γ of S which forms a bridge over $B_{k-j-1}(f_k)$ and Γ encloses (a part of) $\mathrm{int}(F,G) \cap A_{k-j}(f_k)$. Every subset of S which contains at least one of f_{j-1}, g_{j-1} could possibly occur as such a bridge Γ (see Figure 6). The bridge Γ has the additional property that it is always semi-attached in at least one corner, and it is attached (in at least one corner) if f_j or g_j appear. Moreover, the fact that f_j and f_{j-1} (or g_j and g_{j-1}) share an edge allows us to estimate $\mu(\Gamma)$ from above.

If $\Gamma = S$ then we use a) of Lemma 3.1, and also e) of Lemma 2.3, to deduce that $\mu(\Gamma) < \frac{1}{3} + \frac{1}{7} < \frac{1}{2}$. (Note that there is at most one vertex v with $|v|_P^e > 2$ in w_1, where w_1 denotes the connected path in ∂P which is disjoint to ∂B. If v_1 is any neighbour of v along w_1 then $\kappa_P^i(v) + \kappa_P^i(v_1) < \frac{1}{3}$ in the case $|v_1|_P^e = 2$, and $\kappa_P^i(v) + \kappa_P^i(v_1) \leq 0$ if $|v_1|_P^e = 1$.) Since the bridge Γ is then also attached in both corners we obtain a contradiction to part ii) of Proposition 3.2. If $L(\Gamma) = 3$ then we see, similarly, that $\mu(\Gamma) < \frac{1}{3}$. Since Γ is also attached in at least one corner this contradicts to i) of Proposition 3.2. If $L(\Gamma) = 2$ then Γ is either semi-attached in both corners (contradicting Proposition 3.2 iv)) or attached in one corner. In the latter case we have $\Gamma = \{f_j, f_{j-1}\}$ or $\Gamma = \{g_j, g_{j-1}\}$, contradicting the second conclusion after Propositon 3.2. Therefore, $\mathrm{int}(F,G) \cap A_{k-j}(f_k) = \emptyset$. $\square$

3.3 Hyperbolicity of tessellations

Proof of Corollary 5. Gromov hyperbolicity follows from Theorem 2 by a result of Papasoglu [Pa], as explained in the introduction.

To prove that the boundary is S^1 we follow the reasoning explained in [Bo, §3,4]. In fact, the Gromov hyperbolicity of the plane tessellation and the boundedness of the faces allows to define a hyperbolic metric on the plane $\mathbf{R}^2$ which is quasiisometric to the edge graph $\mathcal{G}$ of the tessellation. By [Bo, Theorem 3.1] to show that the boundary is homeomorphic to S^1 it is enough to show that the boundary is not a singleton. Assume that the boundary is a singleton. By

[Bo, Theorem 3.2] there exists a constant $L > 0$ such that any compact set K is separated from infinity by a simple closed curve in $\mathcal{G}$ of length at most L. But this is not possible since $\mathcal{G}$ is without cut locus. $\square$

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