Project III (MATH3382) 2014-15

Optimal Location Problems and the Hexagonal Tiling

Description. Where is the best place in a city to locate glass recycling points, or postboxes, or bank machines? Or where should Edinburgh locate its polling stations for the Scottish Referendum? In this project we will study a simple model for these optimal location problems and show that the best place to put these resources is at the centres of hexagons! See the figure below. This result is known as Fejes Tóth's Theorem on Sums of Moments [2] after the Hungarian geometer László Fejes Tóth. Modern analytical proofs were given in [3] & [5] and the result can be demonstrated numerically using tools from computational geometry such as Centroidal Voronoi Tessellations and Lloyd's Algorithm [1].

While hexagonal patterns are ubiquitous in nature - the beehive, Giant's Causeway, patterns on a giraffe's skin - this is a rare example of a problem where you can actually prove that the hexagonal pattern is the best.

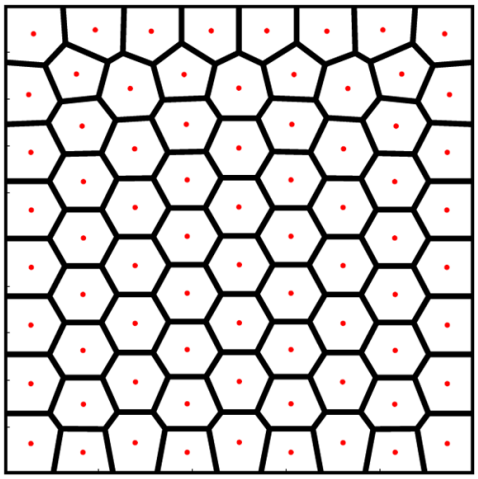

Figure 1: The red points represent the optimal locations of recycling points in a square city.

Prerequisites. This project could be steered in the direction of analysis (e.g., following the proof of the optimality of the hexagonal tiling from [3] & [5]) or numerical methods (e.g., implementing Lloyd's algorithm from [1] in MATLAB to numerically demonstrate the optimality of the hexagonal pattern), depending on the background and interests of the student. Alternatively, Voronoi diagrams could be studied in their own right, following [6], or students could focus on applications of Centroidal Voronoi diagrams (e.g., image compression) from [1], or learn more about discrete and convex geometry from [4].

References

[1] Q. Du, V. Faber, and M. Gunzburger. Centroidal Voronoi tessellations: Applications and algorithms. *SIAM Rev.*, 41:637–676, 1999.

[2] L. Fejes Tóth. *Lagerungen in der Ebene, auf der Kugel und im Raum.* Springer, 1972.

[3] P. M. Gruber. A short analytic proof of Fejes Tóth's theorem on sums of moments. *Aequationes Math.*, 58:291–295, 1999.

[4] P. M. Gruber. *Convex and Discrete Geometry.* Springer, 2007.

[5] F. Morgan and R. Bolton. Hexagonal economic regions solve the location problem. *Amer. Math. Monthly*, 109:165–172, 2002.

[6] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams.* Wiley, 2nd edition, 2000.