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**A guide to**

# Quantum field theory

**Kasper Peeters and Marija Zamaklar**

Lecture notes for the module "Advanced Quantum Theory" at Durham University.

Department of Mathematical Sciences
University of Durham
South Road
Durham DH1 3LE
United Kingdom

kasper.peeters@durham.ac.uk
marija.zamaklar@durham.ac.uk

# 1

# Overview and guide to the literature

## 1.1. Introductory remarks

Quantum Field Theory (QFT henceforth) was born out of the struggle to combine special relativity with quantum mechanics. Doing this consistently has led to an understanding of five deep facts about Nature:

1. The existence of different, yet totally indistinguishable, copies of elementary particles.
2. The relation of the statistics of particles (behaviour under exchange) to their spin.
3. The existence of anti-particles.
4. The ubiquity of particle creation & destruction.
5. The association of forces with particle exchange.

It is the goal of these lectures to explain these points, along with the required mathematics. Moreover, we will discuss some aspects of string theory, not only because it is an arena in which one can explain quantum field theory in simple terms, but also because it is, by many, considered to be an important step beyond quantum field theory (how this all fits together will become clear as we go along).

The crucial aspect of QFT is that it describes the quantum mechanics of an arbitrary number of relativistic particles. Quantum mechanics itself already suggests, by virtue of the uncertainty relation $\Delta E\,\Delta t \geq \hbar$, that particles can be created out of the vacuum, and hence that a theory based on a fixed number of particles is not going to be consistent. An inconsistency also shows up if you try to generalise the propagation amplitude for a particle to go from point $\vec{x}$ to $\vec{y}$,

$$U(t) = \langle \vec{y}|\, e^{-i\hat{H}t} |\vec{x}\rangle\,. \tag{1.1}$$

This amplitude is non-zero for all positions and times, indicating that particles can propagate with any speed. If you generalise this naively to the relativistic formula, $\hat{H} = \sqrt{\hat{p}^2 + m^2}$, you still encounter problems with causality. Only the introduction of anti-particles, one of the predictions of QFT, resolves the issue.

QFT is a hard topic. First of all, there are many new concepts to learn. Several of them look highly confusing at first sight. In particular, the fact that we are forced to work with an infinite number of degrees of freedom makes life conceptually and technically difficult. Another reason why QFT is hard to learn is that there are various mathematical ways to describe the same thing, each with their own technical

complications and interpretational issues. Lack of time prevents us from discussing each of the available formalisms (the operator formalism, the Schrödinger formalism and the path integral formalism) in equal detail. If you get confused, the book by Hatfield [1] is a good guide, as it does many computations in all three formalisms, so you can compare them.

These notes are essentially a collection of bits from various books which we considered well-written. For more details, all sections contain references to the original material. We have tried to return, in every chapter, to the "complex scalar four-point function" as the main example on which to explain the theory. With the limited time available, and the lack of a discussion of fermions, there was unfortunately not much room to discuss other physical processes.

## 1.2. Recommended literature

You are *strongly* encouraged to read some other literature apart from these notes, if only to put things in perspective and give you a better feeling for *why* things are done the way in which we present them. Short texts with overviews of the field or particular sub-topics include:

- R. P. Feynman, "QED: The strange theory of light and matter", Princeton, 1985.
  A masterpiece to whet your appetite; no formulas.
- F. Wilczek, "Quantum field theory", *Rev. Mod. Phys.* **71** (1999) S85–S95.
  A short overview of the general principles of quantum field theory and their consequences.

Books purely about quantum field theory, with varying levels of particle physics content:

- M. Srednicki, "Quantum field theory", Cambridge, 2007.
  A very modern text, with emphasis on a logical structure rather than a historical exposition (as in many other books). A pre-publication draft version (very similar to the printed book) can be obtained for free from
  http://www.physics.ucsb.edu/~mark/qft.html.
- B. F. Hatfield, "Quantum field theory of point particles and strings", Addison-Wesley (Frontiers in Physics), 1992.
  Nice pedagogical text, often with an original way of explaining things. Also contains an introduction to string theory in quite some detail. Does not have too many particle-physics examples.
- L. H. Ryder, "Quantum Field Theory", Cambridge, 1985.
  A big book which takes a lot of time to develop the material (e.g. contains a large amount of material on classical field theory), but as a result contains a lot of worked out calculations.
- M. Peskin and D. Schroeder, "An introduction to quantum field theory", Perseus, 1995.
  By many considered to be one of the most readable and most complete books on quantum field theory: it is rigorous and comprehensive, both for the mathematics as well as the physics.
- A. Zee, "Quantum field theory in a nutshell", 2003.
  A good pedagogical book to get an overview of the field, explains many concepts in simple terms. A bit thin on actual computations.

- S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations", 1995; S. Weinberg, "The quantum theory of fields. Vol. 2: Modern applications", 1995; S. Weinberg, "The quantum theory of fields. Vol. 3: Supersymmetry", 1995.

  A massive three-volume text with many topics not found elsewhere. The level of detail and explanation changes a lot throughout the book, which perhaps makes it harder to read as a first introduction (but some parts are extremely well written).

- K. Huang, "Quarks, leptons and gauge fields", 1982.

  A book with main emphasis on particle physics, relatively compact and to the point.

- M. Veltman, "Diagrammatica", Cambridge, 1994.

  A recommended, compact and highly original book, written from the particle physics perspective.

- C. Itzykson and J. B. Zuber, "Quantum field theory", McGraw-Hill, 1980.

  A thorough reference book with many real-world examples, but not so good as a first introduction to the field.

- R. J. Rivers, "Path integral methods in quantum field theory", Cambridge, 1987.

  Only uses path integrals.

- N. Bogoljubov and D. V. Shirkov, "Quantum fields", Benjamin/Cummings, 1983.

  Classic masterpiece, Russian accessibility. Famous for its appendix with explicit expressions of propagators in coordinate space (which few other books care to list).

Quantum field theory methods are strongly related to those used in statistical field theory, and some books discuss these in one go:

- G. Parisi, "Statistical field theory", Addison-Wesley, 1988.

  A useful and compact book for those who want to understand the connection between methods used in quantum field theory and statistical mechanics (perhaps as an alternative to Zinn-Justin's book, which is much more elaborate).

- J. Zinn-Justin, "Quantum field theory and critical phenomena", *Int. Ser. Monogr. Phys.* **113** (2002) 1–1054.

  A massive 1000+ page book with everything you always wanted to know (and more) about both quantum field theory and statistical field theory.

In the second half of the course we will discuss some string theory. Good books for this topic are

- D. Lüst and S. Theisen, "Lectures on string theory", Springer, 1989.

  Goes much further than these notes, but the first few chapters will be useful. The book is unfortunately out of print, but the authors have given permission to use a scanned version; ask one of the lecturers for a link.

- M. B. Green, J. H. Schwarz, and E. Witten, "Superstring theory", Cambridge University Press, 1987.
  Two-volume bible of the field (for the present course the first volume is more than sufficient).

- J. Polchinski, "String theory", Cambridge University Press, 1998.
  A more recent two-volume book. Covers several modern topics in string theory which are not discussed in Green/Schwarz/Witten (e.g. D-branes).

- B. Zwiebach, "A first course in string theory", 2004.
  New pedagogical text book which also contains a discussion of many of the field theory concepts necessary to understand string theory. Get the 2nd edition.

## 1.3. Software

There is a lot of software for quantum field theory freely available. Some of the packages most useful in the context of these lectures are

- GRC. This is a Feynman diagram generator for arbitrary Lagrangians. Available from
  http://minami-home.kek.jp/.

- Feyncalc. A large Mathematica package for various computations related to Feynman diagrams, including generating diagrams and computing one-loop integrals. http://www.feyncalc.org/.

## 1.4. Conventions

- The metric signature throughout is "mostly plus", i.e. $g_{\mu\nu} = (-,+,+,\ldots,+)$ in any dimension (many field theory books use "mostly minus" convention, so beware). A massive particle thus has $p_\mu p^\mu = -E^2 + |\vec{p}|^2 = -m^2$.
- Greek indices $\mu, \nu\rho, \ldots$ run over both space-like and time-like directions.
- Roman indices $r, s, t, u, \ldots$ run only over the space-like directions.
- Operators are denoted with a hat accent: $\hat{\alpha}_n$ is the operator corresponding to the classical variable $\alpha_n$.
- Light-cone coordinates are defined by $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$ so that the metric in the $X^\pm$ directions is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.
- Lorentzian and Euclidean time are related by $t_L = -i\,\tau_E$.

# 2

# Classical field theory reminder

## 2.1. Relativistic invariance

Special relativity says that physical formulas are the same in all inertial frames. Those are frames related by a Poincaré transformation, i.e. transformations which act on the space-time coordinates as[1]

$$x^\mu \to \Lambda^\mu{}_\nu x^\mu + a^\mu\,, \qquad \mu = 0, \ldots 3\,. \tag{2.1}$$

Here $a^\mu$ is a constant vector and $\Lambda^{\mu\nu}$ is such that it leaves the Minkowski metric of flat space-time unchanged (invariant). The flat metric is

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}\,. \tag{2.2}$$

and the invariance condition is simply

$$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\,. \tag{2.3}$$

The transformations generated by $\Lambda^\mu{}_\nu$ are called the Lorentz group, while the full group including translations $a^\mu$ is the Poincaré group.

Let us have a closer look at the explicit form of the matrices $\Lambda^\mu{}_\nu$. For small (infinitesimal) transformations, we can write them as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \tilde{\Lambda}^\mu{}_\nu\,, \tag{2.4}$$

where we are assuming that all components of $\tilde{\Lambda}^\mu{}_\nu$ are small and higher order expressions in these components can be ignored. Plugging this into (2.3) we find that the $\tilde{\Lambda}_{\mu\nu}$ (with two lower indices) are anti-symmetric, $\tilde{\Lambda}_{\mu\nu} = -\tilde{\Lambda}_{\nu\mu}$. Such matrices have six independent components. We can thus write these matrices as $\tilde{\Lambda}^\mu{}_\nu = \omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu$, where the six independent components in $\omega_{\rho\sigma}$ are the transformation parameters and $(J^{\rho\sigma})^\mu{}_\nu$ are the constant components of the matrices that

[1] We will use a notation in which the 0-th component of a position vector $x^\mu$ denotes the moment in time and the 2-nd, 3-rd and 4-th components denote the position in space, i.e. $x^\mu = (t, \vec{x})^\mu$. Latin indices only denote space components, $x^i = (\vec{x})^i$. Contravariant (upper) indices can be made covariant (lower) by acting with the metric, $x_\mu = \eta_{\mu\nu}x^\nu = (-t, \vec{x})_\mu$. Note that derivatives have covariant indices, $\frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i})$.

generate the transformation,

$$(J^{12})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad (J^{13})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad (J^{23})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{\mu\nu}, \tag{2.5}$$

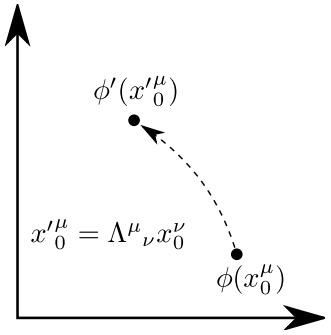
for rotations in the $(1,2)$, $(1,3)$ and $(2,3)$ planes respectively (or if you wish, around the $z$, $y$ and $x$ axes). The other three correspond to relativistic boosts in the three spatial directions,

$$(J^{01})_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad (J^{02})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad (J^{03})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}. \tag{2.6}$$

Furthermore $\omega_{12} = \theta_z$, $\omega_{13} = \theta_y$ and $\omega_{23} = \theta_x$ are the rotation parameters and $\omega_{01} = \gamma_x$, $\omega_{02} = \gamma_y$ and $\omega_{03} = \gamma_z$ are the boost parameters. A somewhat more convenient way to express these generators is

$$(J^{\mu\nu})_{\rho\sigma} = \delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} - \delta^{\nu}{}_{\rho}\delta^{\mu}{}_{\sigma}\,, \tag{2.7}$$

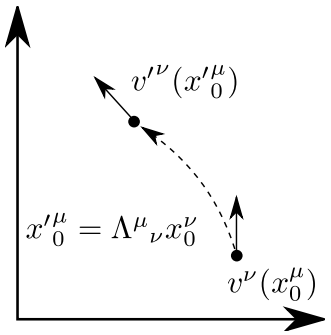
where the indices inside the brackets label the generators, while the indices outside the brackets label the components of the corresponding matrix. If you instead consider the matrices with one upper and one lower index (as they appear in the transformation (2.1)) you find that the rotation generators are anti-symmetric while the boost generators are symmetric matrices. By exponentiating these generators of infinitesimal transformations, the corresponding finite transformations can be obtained (see the exercises).




If we rotate points by a transformation $\Lambda^{\mu}{}_{\nu}$, the value of the transformed scalar field at the new point is the same as the value of the original field at the old point.

When we deal with classical fields, we also need to know how the transformations act on the fields. The simplest case is a scalar field, i.e. a function $\phi(x^{\mu})$ which associates a scalar value to each point in space-time. Let us think of transformations like (2.1) in the *active* way, i.e. moving physical points around in space-time but keeping the coordinate system fixed (the alternative is to consider them as *passive* transformations, in which the coordinate system changes but all the physical objects are kept at their original place). In that case, the value of the scalar field at the transformed point is equal to the value of the scalar field at the original point,

$$\phi'(x'^{\mu}) = \phi(x^{\mu})\,. \tag{2.8}$$




If we rotate points by a transformation $\Lambda^{\mu}{}_{\nu}$, the components of the transformed vector field are rotated versions of the components of the original vector field at the old point.

This is the simplest way in which a field can transform. When we discuss symmetries later on, it is often more useful to write this transformation behaviour in the form

$$\phi(x^{\mu}) \to \phi((\Lambda^{-1})^{\mu}{}_{\nu} x^{\nu})\,, \tag{2.9}$$

i.e. as a replacement rule. This simply says that the value of the scalar field at a given point after the transformation is obtained by looking at where that point was before the transformation.

For a vector field, things are a bit more complicated. By rotating all points around the origin, the vector directions also change. We thus have the transformation

$$v'^{\mu}(x'^{\nu}) = \Lambda^{\mu}{}_{\rho} v^{\rho}(x^{\nu})\,. \tag{2.10}$$

For tensors, there will be a factor of the $\Lambda$ matrix for every upper (covariant) index and a factor of $(\Lambda^{-1})^{T}$ for every lower (covariant) index. Again, this can also be expressed as a replacement rule,

$$v^{\rho}(x^{\mu}) \to \Lambda^{\rho}{}_{\sigma} v^{\sigma}((\Lambda^{-1})^{\mu}{}_{\nu} x^{\nu})\,. \tag{2.11}$$

We can also write down explicit forms for the infinitesimal generators of the Poincaré transformations on fields. For translations of scalar fields, we have the four generators (one for every value of $\mu$)

$$P_\mu = \frac{\partial}{\partial x^\mu} \,. \tag{2.12}$$

If we introduce an infinitesimal parameter $\epsilon^\nu$ the action of the infinitesimal translation generator is

$$\phi(x^\nu) \to \phi(x^\nu) + \epsilon^\rho \partial_\rho \phi(x^\nu) \,. \tag{2.13}$$

We could also have written this as $\phi'(x^\nu + \epsilon^\nu) = \phi(x^\nu)$. Similarly, we can write down differential operators that generate rotations,

$$J_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \,. \tag{2.14}$$

By making use of these explicit representations, it is possible to compute the algebra of the generators. One finds

The *Poincaré algebra* generates translations and rotations in Minkowski space-time.

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \,, \\ [P_\mu, J_{\nu\rho}] &= \eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu \,, \\ [J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho} \,. \end{aligned} \tag{2.15}$$

This algebra is called the *Poincaré* algebra. The subalgebra generated by the $J_{\mu\nu}$ is the *Lorentz* algebra. The commutator on the last line can of course also be verified directly at the level of the matrix representation (2.7).

Finally, there are also some discrete symmetries which leave the Minkowski metric invariant, i.e. symmetries which are not obtained by exponentiating generators. These are *parity*, which flips the sign of all space-like coordinates, and *time reversal*, which flips the sign of time. In nature, not all interactions are invariant under these reflections.

(end of lecture 1)

▸ ***See also:*** B. F. Schutz, "A first course in general relativity", Cambridge, 1985 chapter 1, 2 & 3 contain many examples and exercises in case you need to refresh your memory about special relativity.

## 2.2. Lagrangian and Hamiltonian methods

A convenient way to formulate classical field theories is to make use of the Lagrangian formalism. This formalism will play an important role especially in the path integral formulation of quantum field theory. The Hamiltonian formalism is important for the operator formalism of interacting field theories. Let us therefore briefly recall some of the basic concepts.

In the Lagrangian formalism, the basic object is a scalar quantity called the *action*, which is a functional of the fields $\phi$ in the model, as well as their space and time derivatives. The action is the time-integral of a Lagrangian, or a space-time integral of a Lagrangian density

$$S = \int \mathrm{d}t \, L(t) = \int \mathrm{d}^4 x \, \mathcal{L}\big(\phi(x), \partial_\mu \phi(x)\big) \,. \tag{2.16}$$

The principle of least action states that classically, the fields evolve from one configuration at $t = t_1$ to another configuration at $t = t_2$ according to a "path" $\phi(t, \vec{x})$ for

which the action is an extremum. That is, small fluctuations around this path (the first *variational derivative*) have to vanish. This condition reads

$$0 = \delta S = \int \mathrm{d}^4x \left[\frac{\delta \mathcal{L}(x)}{\delta\phi(y)}\delta\phi(y)\right] \tag{2.17}$$

All fields are assumed to fall off to zero sufficiently fast in the limit $x^\mu \to \pm\infty$ so that boundary terms which occur in the process of partial integration can be ignored.[2] In addition, the variation $\delta\phi(y)$ is of course zero at the initial and final time slice.

Classical equation of motion for a scalar field

Let us do a simple example, namely that of a real scalar field with a Lagrangian density given by $\mathcal{L}(x) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x)$. The variation of the action becomes[3]

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x \left[-\frac{\partial\delta^4(x-y)}{\partial x^\mu}\frac{\partial\phi(x)}{\partial x^\nu}\eta^{\mu\nu} - m^2\phi(x)\delta^4(x-y)\right]\delta\phi(y) \\ &= \int \mathrm{d}^4x \left[\partial_\mu\partial^\mu\phi(x) - m^2\phi(x)\right]\delta^4(x-y)\delta\phi(y) \\ &= \left[\partial_\mu\partial^\mu\phi(y) - m^2\phi(y)\right]\delta\phi(y)\,.\end{aligned} \tag{2.19}$$

Since this has to hold for any small variation $\delta\phi(y)$, the equation of motion is given by the factor in square brackets. In (quantum) field theory, we often say that the fields are *on shell* when they satisfy the equations of motion, and *off shell* when they do not.

On-shell versus off-shell.

The connection to the Hamiltonian formalism is made through the definition of the *conjugate momenta* to the fields. This is done in complete analogy with the definition of momenta for single-particle systems, where $p = \partial L/\partial\dot{q}$, using the notation $\dot{q} = \partial_t q$. We now have

$$\pi(t,\vec{x}) := \frac{\partial\mathcal{L}(t,\vec{x})}{\partial\dot{\phi}(t,\vec{x})}\,. \tag{2.20}$$

The Hamiltonian is then obtained as the Legendre transform of the Lagrangian,

$$H(t) = \int \mathrm{d}^3x\, \left[\pi(t,\vec{x})\dot{\phi}(t,\vec{x}) - \mathcal{L}(t,\vec{x})\right]\,. \tag{2.21}$$

Note that the Hamiltonian is an integral over a space-like, three-dimensional slice of space-time, in contrast to the action which is an integral over all of space-time. Also remember that time derivatives of fields should always be eliminated in favour of momenta.

Again, let us illustrate the above on the real scalar field example. For the momentum density we find

$$\pi(t,\vec{x}) = \partial_t\phi(t,\vec{x})\,. \tag{2.22}$$

The Hamiltonian is thus

$$H = \int \mathrm{d}^3x\, \frac{1}{2}\left[\pi^2 + (\partial_i\phi)^2 + m^2\phi^2\right]\,. \tag{2.23}$$

[2] There certainly do exist systems for which this is not true (general relativity for instance) and more care has to be taken, but we will not encounter them in these notes.

[3] Remember that for variational derivatives we have

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta^4(x-y)\,, \qquad \text{and thus} \qquad \frac{\delta(\partial_\mu\phi(x))}{\delta\phi(y)} = \frac{\partial}{\partial x^\mu}\delta^4(x-y)\,. \tag{2.18}$$

Inside a space-time integral, partial derivatives on Dirac delta functions can always be integrated by parts.

There are thus three contributions to the energy: kinetic energy because of the field changing in time, "shear" potential energy because of the gradient of the field, and potential energy simply because of the presence of the field.

All of the manipulations in this section can of course be extended to fields which carry extra space-time or internal indices, and we will see examples of those cases in the exercises.

## 2.3. Noether's theorem

Noether's theorem, roughly speaking, says that for any continuous transformation which leaves the action invariant up to a surface term, there exists a conserved current. That is, for any such transformation, we can find a vector field $\mathcal{J}^\mu(x)$ which satisfies

$$\partial_\mu \mathcal{J}^\mu = 0\,. \tag{2.24}$$

A charge is conserved when its time derivative $\partial_0 Q$ vanishes; this holds when the current satisfies the conservation equation $\partial_\mu \mathcal{J}^\mu = 0$.

The space integral of the zeroth component of this current is then a conserved charge, $Q = \int \mathrm{d}^3x\, \mathcal{J}^0$, because

$$\partial_0 Q = \int \mathrm{d}^3x\, \partial_0 \mathcal{J}^0 = -\int \mathrm{d}^3x\, \partial_i \mathcal{J}^i\,, \tag{2.25}$$

which is a total derivative and hence vanishing if the fields fall of at infinity fast enough.

Let us derive the form of this current. What is needed is only the infinitesimal form of the field transformation, which we will write as

$$\phi(x) \to \phi(x) + \epsilon \Delta\phi(x)\,. \tag{2.26}$$

Here $\Delta\phi(x)$ is a fixed expression of the field $\phi(x)$ (it can be equal to the field or some more complicated function of it, or alternatively it can be a constant; the details depend on the symmetry at hand). We call such transformations *global*, since they are parameterised by a space-time independent parameter $\epsilon$.[4] If this transformation leaves the action invariant, i.e. if

$$\delta S = \epsilon \int \mathrm{d}^4x \frac{\delta S}{\delta\phi(x)} \Delta\phi(x) = 0\,, \qquad \text{(off shell)} \tag{2.27}$$

then this transformation is a *global symmetry* . Note that this is only a non-trivial statement if it holds independent of the equations of motion: when the fields do satisfy the equations of motion, (2.27) is of course always zero.

If we would temporarily make $\epsilon$ dependent on $x$, then the same transformation would generically not leave the action invariant. However, we know that in this case the action has to transform as

$$\delta S = \int \mathrm{d}^4x\, \mathcal{J}^\mu(x)\, \frac{\partial\epsilon(x)}{\partial x^\mu}\,, \qquad \text{(off shell)} \tag{2.28}$$

otherwise it would not vanish when $\epsilon$ *is* a constant. Again, if the fields satisfy the equations of motion, we always have $\delta S = 0$, since even a transformation with $\epsilon = \epsilon(x)$ non-constant is a generic field variation. So on-shell, we have (using a partial integration)

$$0 = \delta S = -\int \mathrm{d}^4x\, \partial_\mu \mathcal{J}^\mu(x)\, \epsilon(x)\,. \qquad \text{(on shell)} \tag{2.29}$$

[4] A global symmetry may certainly change the field in a different way at every space-time point $x^\mu$ (as the transformation (2.26) in fact does); what counts is whether or not there is a finite number of free parameters in the transformation rule.

(any boundary terms which arise during partial integration will vanish because we put the fields on shell; recall that the derivatation of the equations of motion requires that the boundary terms vanish). Because (2.29) has to be true for any $\epsilon(x)$, the integrand rather than the integral has to vanish, and we have found a conserved current $\mathcal{J}^\mu$.

The procedure described above can be used in general. However, it often happens that not only the action is invariant, but also the Lagrangian density itself (in other words, that there are no boundary terms in the variation of the action). In this case we can give a somewhat more explicit expression for the current $\mathcal{J}$. Let us first write the generic variation as

$$\delta S = \int \mathrm{d}^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi(x)} \epsilon \Delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\mu \left( \epsilon \Delta\phi(x) \right) \right] . \tag{2.30}$$

Invariance of the Lagrangian itself, for constant $\epsilon$, requires that

$$0 = \frac{\partial \mathcal{L}}{\partial \phi(x)} \Delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\mu \left( \Delta\phi(x) \right) , \tag{2.31}$$

which are the equations of motion. On-shell only the term in which $\partial_\mu$ hits $\epsilon$ survives, and comparing with (2.28) thus gives us the conserved current

$$\mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \, . \tag{2.32}$$

It is of course not necessary to use this formula; just following the general logic described in the previous paragraph is equally good.

Let us illustrate this again at the level of an example. Consider the action of a complex scalar field,

$$S = \int \mathrm{d}^4x \left[ -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right] . \tag{2.33}$$

This action is unchanged under complex rotations

$$\phi(x) \to e^{i\alpha} \phi(x) \, , \qquad \phi^*(x) \to e^{-i\alpha} \phi^*(x) \, . \tag{2.34}$$

For infinitesimal $\alpha = \epsilon$ these take the form

$$\phi(x) \to \phi(x) + i\epsilon\phi(x) \, , \qquad \phi^*(x) \to \phi^*(x) - i\epsilon\phi^*(x) \, , \tag{2.35}$$

i.e. $\Delta\phi = i\phi$ and $\Delta\phi^* = -i\phi^*$. Independent of the equations of motion, the variation $\delta S / \delta\alpha$ always vanishes provided $\alpha$ is a constant. Now temporarily make $\alpha = \alpha(x)$ to obtain the variation

$$\delta S = \int \mathrm{d}^4x \, (\partial_\mu \alpha) \left[ i(\partial^\mu \phi)\phi^* - i\phi(\partial^\mu \phi^*) \right] . \tag{2.36}$$

The expression in brackets is the conserved Noether current associated to the symmetry of the action (2.34).

(end of lecture 2)

▸ ***Summary:*** Global symmetries of the action (i.e. field transformations which are parameterised by a space-time independent parameter and which leave the action invariant even when the fields do not satisfy the equations of motion) lead to conserved currents. These currents can be found by using a local parameter, rewriting the variation in the form (2.28) and putting the fields on shell.

▸ ***See also:*** S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations", 1995 section 7.3.

# 3
# Free quantum fields

## 3.1. Multi-particle non-relativistic quantum mechanics

In order to introduce the language in which we will describe relativistic quantum field theory, let us start from a well-known point: the Schrödinger equation of a fixed number $n$ of non-relativistic particles. When they interact with each other through a pair-wise interaction potential $V$, it reads

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}_1,\ldots,\vec{x}_n) = \left[-\sum_{j=1}^{n}\frac{\hbar^2}{2m}\nabla_j^2 + \sum_{j=1}^{n}\sum_{k=j+1}^{n}V(\vec{x}_j-\vec{x}_k)\right]\psi(t,\vec{x}_1,\ldots,\vec{x}_n)\,. \quad (3.1)$$

In order to generalise this equation, it is useful to write it in a more abstract form, using operators acting on the vectors in Hilbert space. That is, we want to write this in the form

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = \hat{H}|\psi,t\rangle\,, \quad (3.2)$$

where $\hat{H}$ is the Hamiltonian operator and $|\psi,t\rangle$ a vector in Hilbert space. In order to do so, we introduce a set of operators $\hat{a}(\vec{x})$ and $\hat{a}^\dagger(\vec{x})$, i.e. one operator for every point in space. Right now they come out of the blue, but their usefulness will become clear shortly. We take these operators to satisfy the commutation relations

$$\begin{aligned}[\hat{a}(\vec{x}),\hat{a}(\vec{x}')] &= 0\,,\\ [\hat{a}^\dagger(\vec{x}),\hat{a}^\dagger(\vec{x}')] &= 0\,,\\ [\hat{a}(\vec{x}),\hat{a}^\dagger(\vec{x}')] &= \delta^3(\vec{x}-\vec{x}')\,.\end{aligned} \quad (3.3)$$

Observe that these are much like the commutation relations of a continuous set of independent harmonic oscillators, labelled by a continuous parameter $\vec{x}$ (see also the next section). In terms of these operators, the ingredients of (3.2) can be written as

$$\hat{H} = \int \mathrm{d}^3x\,\hat{a}^\dagger(\vec{x})\left[-\frac{\hbar^2}{2m}\nabla^2\right]\hat{a}(\vec{x}) + \int \mathrm{d}^3x\mathrm{d}^3y\,V(\vec{x}-\vec{y})\,\hat{a}^\dagger(\vec{x})\hat{a}^\dagger(\vec{y})\hat{a}(\vec{y})\hat{a}(\vec{x})\,. \quad (3.4)$$

and

$$|\psi,t\rangle = \int \mathrm{d}^3x_1\cdots\mathrm{d}^3x_n\,\psi(t,\vec{x}_1,\ldots,\vec{x}_n)\,\hat{a}^\dagger(\vec{x}_1)\ldots\hat{a}^\dagger(\vec{x}_n)|0\rangle\,. \quad (3.5)$$

Here we have introduced a special state $|0\rangle$, which we will call the *vacuum*. We declare it to have the property that

$$\hat{a}(\vec{x})\,|0\rangle = 0\,. \quad (3.6)$$

One can now verify that the equation (3.2) holds if and only if the wave function $\psi(\vec{x}_1, \ldots, \vec{x}_n)$ satisfies (3.1).

The physical interpretation of the rewriting procedure we just did is simple. We interpret the state $|0\rangle$ as the state which contains no particles, $\hat{a}^\dagger(\vec{x}_1)|0\rangle$ as a state which contains a single particle at position $\vec{x}_1$, the state $\hat{a}^\dagger(\vec{x}_1)\hat{a}^\dagger(\vec{x}_2)|0\rangle$ as one which contains two particles, and so on. Note that important fact that all of the information about the number of particles $n$ has been moved into the state $|\psi, t\rangle$. The Hamiltonian (3.4) no longer refers to $n$ at all. This is what will re-appear also in quantum field theory.

Let us now set $V = 0$, i.e. consider free particles. We then perform a Fourier transformation to momentum space, transforming the operators as

$$\hat{a}(\vec{p}) = \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} e^{-i\vec{p}\cdot\vec{x}} \hat{a}(\vec{x}) \,. \tag{3.7}$$

The Hamiltonian operator then takes the simple form

$$\hat{H} = \int \mathrm{d}^3 p \, \frac{\hbar^2}{2m} \vec{p}^2 \, \hat{a}^\dagger(\vec{p}) \, \hat{a}(\vec{p}) \,. \tag{3.8}$$

This form suggests a simple generalisation to the relativistic case: the expression $\vec{p}^2/2m$ is nothing else but the single-particle energy of a particle with momentum $\vec{p}$. If we replace this with the relativistic expression for the energy, we get

$$\hat{H} = \int \mathrm{d}^3 p \, \sqrt{\vec{p}^2 + m^2} \, \hat{a}^\dagger(\vec{p}) \, \hat{a}(\vec{p}) \,. \tag{3.9}$$

The goal is now to understand this from a wider perspective, and in particular understand why this is a Lorentz covariant expression. Note that the Hamiltonian expression (3.9) is reminiscent of the Hamiltonian of a classical field theory, as the $\hat{a}$ operators depend on three-space or three-momentum.

▸ ***Summary:*** It is possible to rewrite the quantum mechanics of multiple particles in such a way that all the dependence on the number of particles sits in the state vector $|\psi, t\rangle$. This requires introducing a "field" of ladder operators.

▸ ***See also:*** The argument presented here can also be found in e.g. M. Srednicki, "Quantum field theory", Cambridge, 2007.

## 3.2. Harmonic oscillators and canonical quantisation

What we will do now is to follow a more bottom-up procedure to construct a relativistic multi-particle Hamiltonian, which starts from a classical field theory and ends up with (3.9). In order to do this, we first need to remind ourselves of the procedure of canonical quantisation and of ladder operators.

So let us go back to a non-relativistic harmonic oscillator, and recall its quantum mechanics. The action reads

$$S = \frac{1}{2} \int \mathrm{d}t \left( \dot{q}(t)^2 - \omega^2 q(t)^2 \right) . \tag{3.10}$$

For the canonical momentum and Hamiltonian one finds

$$p(t) = \dot{q}(t) \,, \qquad H = \frac{1}{2} \Big( p(t)^2 + \omega^2 q(t)^2 \Big) \,. \tag{3.11}$$

When we quantise this system, we replace the classical variables $q(t)$ and $p(t)$ with operators $\hat{q}(t)$ and $\hat{p}(t)$, satisfying the equal-time commutation relations[1]

$$\big[\hat{q}(t),\, \hat{p}(t')\big]\Big|_{t=t'} = i\hbar\,. \tag{3.12}$$

An analogous relation in fact also already exists in classical physics, in the form of the Poisson bracket. The Poisson bracket between two functions of $q$ and $p$ is defined as

$$\{f,\, g\}_{\text{Poisson}} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\,. \tag{3.13}$$

For $f = q$ and $g = p$ we then get

$$\{q,\, p\}_{\text{Poisson}} = 1\,. \tag{3.14}$$

Dirac hence postulated that a way to go from a classical theory to its quantised version is to replace all Poisson brackets by $(i\hbar)^{-1}$ times commutators.[2] This procedure is now known under the name of *canonical quantisation*, in contrast to e.g. path integral quantisation which will appear in later chapters.

A second ingredient of harmonic oscillators which we will encounter is that of ladder operators. Let us recall how those work. Starting from the action (3.10), the Hamiltonian (3.11) and the equal-time commutation relations (3.12) one introduces new operators $\hat{a}$ and $\hat{a}^\dagger$ according to

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}\Big(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\Big)\,, \quad \hat{p}(t) = (-i)\sqrt{\frac{\hbar\omega}{2}}\Big(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\Big)\,. \tag{3.15}$$

These operators satisfy the classical equations of motion

$$\dot{\hat{q}} = \hat{p}\,, \qquad \dot{\hat{p}} = -\omega^2\hat{q}\,, \tag{3.16}$$

or if you want $\ddot{\hat{q}} = -\omega^2\hat{q}$, which is the Lagrangian equation of motion for (3.10). The inverse formulas read

$$\hat{a} = \Big(\sqrt{\frac{\omega}{2\hbar}}\hat{q}(t) + \frac{i}{\sqrt{2\omega\hbar}}\hat{p}(t)\Big)e^{i\omega t}\,, \quad \hat{a}^\dagger = \Big(\sqrt{\frac{\omega}{2\hbar}}\hat{q}(t) - \frac{i}{\sqrt{2\omega\hbar}}\hat{p}(t)\Big)e^{-i\omega t}\,. \tag{3.17}$$

The equal-time commutation relation (3.12) now requires us to fix

$$[\hat{a}, \hat{a}^\dagger] = 1\,. \tag{3.18}$$

We have essentially traded the time-dependent Heisenberg operators $\hat{q}(t)$ and $\hat{p}(t)$ for the time-independent Schrödinger operators $\hat{a}$ and $\hat{a}^\dagger$. The Hamiltonian becomes

$$\hat{H} = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big)\,. \tag{3.19}$$

Furthermore, one finds the commutation relations

$$\big[\hat{H}, \hat{a}^\dagger\big] = \omega\hbar\hat{a}^\dagger\,, \quad \big[\hat{H}, \hat{a}\big] = -\omega\hbar\hat{a}\,. \tag{3.20}$$

These relations can then be used to verify that the eigenstates of the Hamiltonian are given by

$$|n\rangle = (\hat{a}^\dagger)^n|0\rangle\,, \quad \text{with} \quad \hat{a}|0\rangle = 0\,. \tag{3.21}$$

These Schrödinger eigenstates, built from the Schrödinger ladder operators, have energy eigenvalue $(n + \frac{1}{2})\hbar\omega$, are time-dependent and their time evolution is given by $|n, t\rangle = \exp(-i\hat{H}t)|n\rangle$.

[1] In this section we are making a slow transition from the Heisenberg picture, in which operators are time dependent and states are not, to the Schrödinger picture, in which the roles are reversed. There are various other ways to write the things discussed here; see e.g. [1] or [6] for more on this.

[2] This caricature 'explanation' does not do any justice to the beautiful theory of Dirac brackets and constrained systems which is often necessary in order to arrive at a consistent quantised model. For lack of time, we will not discuss any of these issues here.

## 3.3. Relativistic quantum fields

After this short reminder of the harmonic oscillator, let us now return to our main goal, namely to derive (3.9) from a systematic quantisation procedure. We will follow a logic which is essentially canonical quantisation as discussed in the previous section. That is, given e.g. a classical theory for a scalar field $\phi(t,\vec{x})$, we will compute its conjugate momentum $\pi(t,\vec{x})$ as explained in section 2.2, promote the fields to operators, expand them in a set of solutions to the classical equations of motion as in section 3.2, and then impose the equal-time commutation relations

$$\left[\hat{\phi}(t,\vec{x}),\, \hat{\pi}(t',\vec{x}')\right]\Big|_{t=t'} = i\hbar\,\delta^{(3)}(\vec{x}-\vec{x}')\,. \tag{3.22}$$

As with quantum mechanics, there is a large number of ways to motivate this, but in the end it is just Dirac's prescription for the transition from classical to quantum, so we will not dwell on this any further. Again, we will set $\hbar = 1$ from now.

We will start, as an example, with the quantisation of a theory for a complex scalar field $\phi$. This is also known as a *complex Klein-Gordon field*. We have seen the action for this system before, it is

$$S = \int \mathrm{d}^4x \left[ -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\right]. \tag{3.23}$$

What we will do first is to write the field $\phi$ in terms of a full set of modes which satisfy the equations of motion. The equation of motion is

$$\partial_\mu\partial^\mu\phi - m^2\phi = 0\,. \tag{3.24}$$

One particular solution is given by a plane wave,

$$\exp\left(i\vec{k}\cdot\vec{x} \pm i\omega_k t\right) \quad \text{with} \quad \omega_k = \sqrt{\vec{k}^2 + m^2}\,. \tag{3.25}$$

(This is the definition of $\omega_k$; it will always denote the positive root of $\omega^2 - \vec{k}^2 = m^2$). A generic solution to the equations of motion is then given by summing an arbitrary number of these plane wave solutions with arbitrary coefficients,

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + \tilde{b}^*(\vec{k})e^{i\vec{k}\cdot\vec{x}+i\omega_k t}\right]. \tag{3.26}$$

The names of the coefficients are of course arbitrary; the important fact is that this field satisfies the classical equations of motion for any values of the coefficients $a(\vec{k})$ and $\tilde{b}(\vec{k})$. The expansion above has, unfortunately, a slightly inconvenient form to continue, because the exponential in the second term does not look like a Lorentz covariant inner product. But we can define $b(\vec{k}) = -\tilde{b}(-\vec{k})$ and write instead

$$\begin{aligned}
\phi(t,\vec{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + b^*(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right], \\
\phi^*(t,\vec{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left[b(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + a^*(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right].
\end{aligned} \tag{3.27}$$

This is the analog of the decomposition of $\hat{q}(t)$ of the harmonic oscillator.

Let us now continue with our quantisation procedure. From the definition of the canonical momentum,

$$\pi(t,\vec{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(t,\vec{x})} = \dot{\phi}^*(t,\vec{x})\,, \tag{3.28}$$

we can find the expansion of the momentum density in terms of $a(\vec{k})$ and $b(\vec{k})$. It reads

$$\begin{aligned}\pi(t,\vec{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_k}{2}}\left[b(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega_k t} - a^*(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right],\\ \pi^*(t,\vec{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_k}{2}}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega_k t} - b^*(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right].\end{aligned} \tag{3.29}$$

These expansions are the analogs of the decomposition of $\hat{p}(t)$ of the harmonic oscillator.

We are now ready to promote the classical field and its momentum density to quantum operators. We implement this at the level of the decompositions given above by promoting the coefficients $a(\vec{k})$ and $b(\vec{k})$ to operators. The expansions are thus as above, but with all $\phi, \pi, a, a^*, b$ and $b^*$ equipped with hats (and complex conjugation replaced by hermitean conjugation). For the inverse relations one finds

$$\begin{aligned}\hat{a}(\vec{k}) &= \int \mathrm{d}^3x\,\left(\sqrt{\frac{\omega_k}{2}}\hat{\phi}(t,\vec{x}) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}^\dagger(t,\vec{x})\right)e^{-i\vec{k}\cdot\vec{x}+i\omega_k t},\\ \hat{b}(\vec{k}) &= \int \mathrm{d}^3x\,\left(\sqrt{\frac{\omega_k}{2}}\hat{\phi}^\dagger(t,\vec{x}) + \frac{i}{\sqrt{2\omega_k}}\hat{\pi}(t,\vec{x})\right)e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}.\end{aligned} \tag{3.30}$$

These, and their hermitean conjugates, are similar again to the harmonic oscillator relations (3.17).

As we mentioned above, the idea is now to impose (3.22). Clearly, that condition will only hold if the new operators $\hat{a}(\vec{k})$ and $\hat{b}(\vec{k})$ satisfy certain commutation relations as well. This is a bit of a messy computation that you should do a couple of times to get the hang of it (see the exercises). The result is that one needs

$$\begin{aligned}\left[\hat{a}(\vec{k}),\,\hat{a}^\dagger(\vec{k}')\right] &= (2\pi)^3\hbar\delta^3(\vec{k}-\vec{k}'),\\ \left[\hat{b}(\vec{k}),\,\hat{b}^\dagger(\vec{k}')\right] &= (2\pi)^3\hbar\delta^3(\vec{k}-\vec{k}'),\end{aligned} \tag{3.31}$$

while all other commutators that you can write down should vanish. Comparing these with (3.18), we see that what the field theory quantisation has produced for us is essentially an infinite number of harmonic oscillator raising/lowering operators, one for each value of the spacelike components of the momentum $\vec{k}$. In fact, the complex scalar field has in fact given us two such sets, $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$.

Let us now finally compute the Hamiltonian of the system in terms of these new raising/lowering operators. A somewhat tedious computation gives

$$\hat{H} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\omega_k\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) + \text{const.}\right]. \tag{3.32}$$

The constant is a formally infinite number, which arises from the non-zero commutators of the ladder operators. One now observes that (3.32) is the same as two copies of the relativistic multi-particle Hamiltonian (3.9) which we postulated before. There is one set of particles associated to $\hat{a}$ and one to $\hat{b}$.

(end of lecture 3)

▸ ***Summary:*** The Hamiltonian of free relativistic multi-particle quantum mechanics (3.9) can be obtained by the quantisation of a scalar field.

## 3.4. Interpretation of field quantisation: anti-particles

In order to give a physical interpretation of the complex scalar field quantisation we have just carried out, it is useful to remember that this system has a conserved charge. We discussed this charge in section 2.3, and found that it is given by

$$Q = \int \mathrm{d}^3x\, \mathcal{J}^0\,, \quad \text{with} \quad \mathcal{J}^0 = (\partial^0\phi)\phi^* - \phi(\partial^0\phi^*)\,. \tag{3.33}$$

Let us see what this expression looks like after quantisation. Using the quantised versions of the expansions (3.27) we find

$$\hat{Q} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Big[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) + \text{const.}\Big]\,. \tag{3.34}$$

Notice the subtle minus sign difference with respect to the Hamiltonian. The constant is again related to commutators, and one usually defines it away by taking $\hat{Q}$ to be *normal ordered* (we will return to this issue in the next chapter).

Consider now a state obtained by acting with one $\hat{a}^\dagger(\vec{k})$ or $\hat{b}^\dagger(\vec{k})$ operator on the vacuum $|0\rangle$. For these states we find[3]

$$\begin{aligned}
\hat{H}\Big(\hat{a}^\dagger(\vec{k})|0\rangle\Big) &= \omega_k\hat{a}^\dagger(\vec{k})|0\rangle\,, \qquad \hat{Q}\Big(\hat{a}^\dagger(\vec{k})|0\rangle\Big) = +\hat{a}^\dagger(\vec{k})|0\rangle\,,\\
\hat{H}\Big(\hat{b}^\dagger(\vec{k})|0\rangle\Big) &= \omega_k\hat{b}^\dagger(\vec{k})|0\rangle\,, \qquad \hat{Q}\Big(\hat{b}^\dagger(\vec{k})|0\rangle\Big) = -\hat{b}^\dagger(\vec{k})|0\rangle\,.
\end{aligned} \tag{3.35}$$

These two states thus have the same energy, but opposite charge. There are thus *two* types of particles contained in the field $\hat{\phi}$. We will say that the state $\hat{a}^\dagger(\vec{k})|0\rangle$ contains "one particle" and the state $\hat{b}^\dagger(\vec{k})|0\rangle$ contains "one *anti-particle*".[4] Note once more that the masses (or equivalently, the energies at given three-momentum) of particles and anti-particles are the same.

With this information, let us return to the operator-valued field $\hat{\phi}$ itself. Note from (3.27) that $\phi$ contains one term which destroys particles and one term which creates anti-particles. Both terms thus lower the charge of a state by one unit. Note that, *if* we had interpreted the field (3.27) as a wave function, then it would have contained both positive and negative energy contributions, which would quickly lead to trouble.

Let us also comment on *real* scalar fields. If we impose that the field $\phi$ is real (i.e. that $\phi = \phi^*$ classically or $\hat{\phi} = \hat{\phi}^\dagger$ after quantisation) we would get a relation between the $a$ and $b$ coefficients. From (3.27) we immediately see that this relation is $a(\vec{k}) = b(\vec{k})$. For a real field a particle is thus identified with its anti-particle (and indeed, consistent with the fact that a real scalar does not have a phase rotation symmetry, the charge $\hat{Q}$ is then identically zero).

▸ ***See also:*** A clear exposition of the meaning of anti-particles can be found in B. F. Hatfield, "Quantum field theory of point particles and strings", Addison-Wesley (Frontiers in Physics), 1992 around page 51.

## 3.5. Propagators and causality

Having discussed the interpretation of the two sets of creation and annihilation operators in terms of particles and anti-particles, let us now look at their propagation.

[3] The fact that there are no zero-point energies here is that we have ignored the infinite constant in the Hamiltonian.

[4] Whether we call the $a$ species particles or anti-particles is totally irrelevant; what matters is that both species are present and have the same properties except for their charge.

We will look at the process in which a unit positive charge is transported from position $x = (t, \vec{x})$ to $x' = (t', \vec{x}')$.

This can be done in two ways. Either one starts with $+1$ charge particle at $x$ and propagates it to $x'$. Or one starts with a $-1$ charge anti-particle at $x'$ and transports it to $x$. However, we have to be careful, since $x$ and $x'$ refer to four-dimensional coordinates. Since we cannot destroy a particle before it is created, the first process only makes sense when $t' > t$, while the second one can only happen when $t > t'$.

Now let us write this down in formulas. The states containing a particle of charge $+1$ at $x$ or an antiparticle of charge $-1$ at $x'$ are[5]

$$\begin{aligned} |\text{particle at } x\rangle &= \hat{\phi}^\dagger(x)|0\rangle\,, \\ |\text{anti-particle at } x'\rangle &= \hat{\phi}(x')|0\rangle\,, \end{aligned} \tag{3.36}$$

which can be seen by simply using (3.27). The propagation of a positive charge from $x$ to $x'$ is thus given by the sum of two terms,

$$G(x', x) = \theta(t' - t)\,\langle 0|\,\hat{\phi}(x')\hat{\phi}^\dagger(x)\,|0\rangle \;+\; \theta(t - t')\,\langle 0|\,\hat{\phi}^\dagger(x)\hat{\phi}(x')\,|0\rangle\,. \tag{3.37}$$

The first term corresponds to the propagation of the particle from $x$ to $x'$, while the second one corresponds to the propagation of the anti-particle from $x'$ to $x$. The Heaviside step functions implement the condition that you cannot destroy before you create.

The expression above can be written in a more compact form by using the so-called *time-ordering* symbol. It is defined by

$$T\Big(\hat{\phi}(x')\hat{\phi}^\dagger(x)\Big) = \theta(t' - t)\hat{\phi}(x')\hat{\phi}^\dagger(x) + \theta(t - t')\hat{\phi}^\dagger(x)\hat{\phi}(x')\,. \tag{3.38}$$

The time-ordering operator thus orders the fields on which it acts with respect to their time, with the earliest ones being put furthest to the right. We can then write

$$G(x', x) = \langle 0|\,T\Big(\hat{\phi}(x')\hat{\phi}^\dagger(x)\Big)\,|0\rangle\,. \tag{3.39}$$

This is the form in which one usually finds it in the literature.

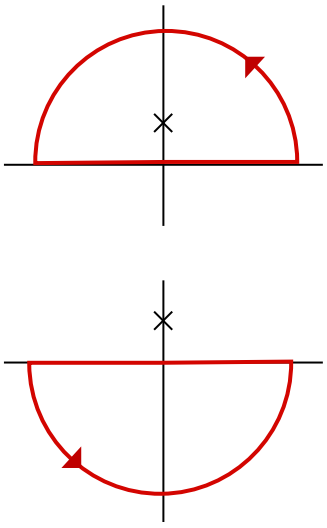
Having set on a notation, let us now *compute* this function $G(x', x)$. To do that, one has to insert the operator expansions (3.27) into (3.37) or (3.39) and use the creation/annihilation operator commutation relations (3.31) as well as the normalisation condition $\langle 0|0\rangle = 1$. A bit of algebra yields

$$G(x', x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\hbar}{2\omega_k}\Big[\theta(t' - t)e^{-i\omega_k(t'-t)+i\vec{k}(\vec{x}'-\vec{x})} + \theta(t - t')e^{i\omega_k(t'-t)+i\vec{k}(\vec{x}'-\vec{x})^\mu}\Big]\,. \tag{3.40}$$

In order to eliminate the Heaviside step functions from this expression, we will use the following integral representation.

$$\theta(t) = \lim_{\epsilon\downarrow 0}\int \frac{\mathrm{d}\omega}{2\pi i}\frac{e^{i\omega t}}{\omega - i\epsilon}\,. \tag{3.41}$$

Because $\epsilon > 0$, the pole has been moved above the real $\omega$ axis. The integral can be performed by noting that the exponent is suppressing for $t > 0, \mathrm{Im}\,\omega > 0$ and for $t < 0, \mathrm{Im}\,\omega < 0$. In the first case we pick up the pole, while in the second case we miss it.



Representation of the Heaviside step function (3.41). Top figure: contour when $t > 0$; bottom figure: contour when $t < 0$.

[5] These notations can be confusing, since they suggest that a vector in Hilbert space corresponds to a state of the system at one particular time only. This is not true. The state $\phi^\dagger(x)|0\rangle$ describes the entire time-evolution (history) of a particle. This history is *labelled* by the particle being at position $\vec{x}$ at the time $t$. If you have difficulty understanding this, read page 38 of [12].

With the help of (3.41) the propagator now becomes

$$G(x',x) = \lim_{\epsilon\downarrow 0} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int \frac{\mathrm{d}\omega}{2\pi i} \frac{\hbar}{2\omega_k} \frac{1}{\omega - i\epsilon} \Big[ e^{i(\omega-\omega_k)(t'-t)} e^{i\vec{k}\cdot(\vec{x}'-\vec{x})} + e^{-i(\omega-\omega_k)(t'-t)} e^{-i\vec{k}\cdot(\vec{x}'-\vec{x})} \Big] . \quad (3.42)$$

(pay attention to the difference between $\omega$ and $\omega_k$). We can rewrite this in a simpler form by introducing $k_0 = \omega_k - \omega$ in the first term and $k_0 = \omega - \omega_k$ in the second. Also substitute $\vec{k} \to -\vec{k}$ in the second term (in total this gives two sign flips for the first and one flip for the second term because of the integration measures). We then get

$$G(x',x) = \lim_{\epsilon\downarrow 0}(-i\hbar) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{ik_\mu(x'-x)^\mu}}{2\omega_k} \left( \frac{1}{\omega_k - k_0 - i\epsilon} + \frac{1}{\omega_k + k_0 - i\epsilon} \right) . \quad (3.43)$$

Now make use of the fact that we take the limit of $\epsilon$ going to zero to ignore all terms of order $\epsilon^2$. This finally gives

$$G(x',x) = \lim_{\epsilon\downarrow 0}(-i\hbar) \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{ik_\mu(x'-x)^\mu}}{k_\nu k^\nu + m^2 - i\epsilon} \equiv G_F(x'-x) . \quad (3.44)$$

The object $G_F(x'-x)$ is the *Feynman propagator* for the scalar field. The particular way in which the $\epsilon$ occurs here is important, as it implements the boundary condition that particle annihilation cannot occur before particle creation.

Note that the Feynman propagator as given above is obviously a Fourier transform of the propagator in momentum space,

$$G_F(k) = \frac{-i\hbar}{k_\mu k^\mu + m^2 - i\epsilon} . \quad (3.45)$$

In this form it is easy to see that the propagator is indeed a Green's function of the equation of motion (3.24). Using a Fourier transform we easily find

$$(-\partial_\mu\partial^\mu + m^2) G_F(x'-x) = -i\hbar\delta^4(x'-x) . \quad (3.46)$$

(end of lecture 4)

In practise we will mostly use the momentum space representation, because the integral in (3.44) leads to complicated Bessel functions in terms of $x'-x$.

However, there is one aspect of the coordinate space Feynman propagator which is important: it does *not* vanish outside the light cone. What does this mean? Does this not violate causality? The answer is that it does not. Literally, what the propagator measures is a correlation function for two fields at different points in space-time. The non-vanishing correlation between fields at two space-like separated points is in a sense similar to EPR correlations of entangled photons. While the correlation is non-zero, it cannot be used to send information faster than light.

So how *do* we figure out whether causality is violated? What is necessary is to establish whether space-like separated measurements can influence each other. In order to turn this into a mathematical statement, note that it requires that two operators commute when they are space-like separated. To be precise, we want

$$\big[\phi(x), \phi^\dagger(x')\big] = 0 \qquad \text{when} \quad (x-x')_\mu (x-x')^\mu > 0 , \quad (3.47)$$

i.e. when $x$ and $x'$ are space-like separated. This expression can be computed easily,

$$\begin{aligned}\left[\phi(x), \phi^\dagger(x')\right] &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int \frac{\mathrm{d}^3k'}{(2\pi)^3} \frac{1}{2\sqrt{\omega_k \omega_{k'}}} \\ &\qquad \left[\hat{a}(\vec{k}) e^{ik_\mu x^\mu} + \hat{b}^\dagger(\vec{k}) e^{-ik_\mu x^\mu},\ \ \hat{b}(\vec{k}') e^{ik'_\mu x'^\mu} + \hat{a}^\dagger(\vec{k}') e^{-ik'_\mu x'^\mu}\right] \qquad (3.48) \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\hbar}{2\omega_k} \left(e^{ik_\mu (x-y)^\mu} - e^{-ik_\mu (x-y)^\mu}\right).\end{aligned}$$

When the separation is space-like, we can use a Lorentz transformation to change $(x-y)$ to $-(x-y)$, and hence the two terms cancel. For time-like separation there is no continuous Lorentz transformation which flips the sign. Note that the two terms come from the $a$ and $b$ particles respectively, hence causality relies *crucially* on the presence of anti-particles!

▸ ***See also:*** The discussion here follows B. F. Hatfield, "Quantum field theory of point particles and strings", Addison-Wesley (Frontiers in Physics), 1992 and M. Peskin and D. Schroeder, "An introduction to quantum field theory", Perseus, 1995.

# 4
# Interacting quantum fields

## 4.1. Evolution operators and the perturbative expansion

In the previous chapter we have concerned ourselves with an analysis of free systems, that is, systems for which the action is quadratic in the fields. The key property of such actions is that the associated Hamiltonian eigenstates can be found exactly. This is especially manifest in section 3.3, where we saw that the operator field $\hat{\phi}(t,\vec{x})$ is a solution to the classical equations of motion: the expansion (3.27) makes essential use of the plane wave solutions to the equation of motion (3.25).

For more general actions, with higher powers of the fields, it is no longer possible to find a full set of solutions to the corresponding equations of motion. A simple example which will come back over and over again is the action[1]

$$S = \int \mathrm{d}^4x \left[ -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right]. \tag{4.1}$$

The equation of motion reads

$$\partial_\mu\partial^\mu\phi - m^2\phi - \frac{\lambda}{3!}\phi^3 = 0\,. \tag{4.2}$$

There is no known complete set of solutions to this equation, and hence we cannot hope to treat this without some sort of approximation.

In order to make progress, we have to go back again to the Schrödinger picture, in which operators are time-independent. We start by taking the Schrödinger operator $\hat{\phi}(\vec{x})$ on a given time-slice $t = t_0$. It reads

$$\hat{\phi}_{t_0}(\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left[\hat{a}(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right]. \tag{4.3}$$

We have labelled the field with a subscript $t_0$ to indicate the time slice at which the Schrödinger quantisation is performed; we can quantise at any fixed slice but once we pick it we should not change it anymore. For the free system, the time evolution of the operator field $\hat{\phi}(\vec{x})$, or in other words, the Heisenberg picture operator $\hat{\phi}(t,\vec{x})$, is given by

$$\hat{\phi}_0(t,\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left[\hat{a}(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega_k(t-t_0)} + \hat{a}^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega_k(t-t_0)}\right], \tag{4.4}$$

[1] We will in this chapter restrict ourselves to a real scalar field, so that particles and anti-particles are identified and the expressions become a bit more compact.

(a subscript zero has been attached to the field to emphasise that this expression is a solution of (4.2) only when $\lambda = 0$). But there is a more systematic way to understand this time evolution, namely by using the Hamiltonian operator. The expression (4.4) is more generically expressed by using the relation between the two pictures,

$$\hat{\phi}_0(t,\vec{x}) = e^{(i/\hbar)\hat{H}_0\,(t-t_0)}\;\hat{\phi}_{t_0}(\vec{x})\;e^{-(i/\hbar)\hat{H}_0\,(t-t_0)}\,. \tag{4.5}$$

This simply is the standard expression for the time evolution of operators known from quantum mechanics. It relates the Schrödinger picture operator $\hat{\phi}(\vec{x})$ to the Heisenberg picture operator $\hat{\phi}(t,\vec{x})$. Verifying that (4.4) indeed is the same as (4.5) requires only the form of the free Hamiltonian,

$$\hat{H}_0 = \int \mathrm{d}^3x \left[\frac{1}{2}\pi(\vec{x})^2 + \frac{1}{2}\partial_i\phi(\vec{x})\partial^i\phi(\vec{x}) + \frac{1}{2}m^2\phi(\vec{x})^2\right] = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\omega_k\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\,, \tag{4.6}$$

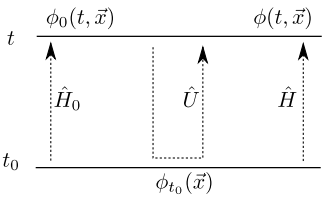
as well as the by now familiar commutation relation $[\hat{a}(\vec{k}),\hat{a}^\dagger(\vec{k}')] = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$. The idea is now to use an expression similar to (4.5) in order to write down the time dependence of the solutions to the equation (4.2), and hence obtain the Heisenberg operators $\hat{\phi}(t,\vec{x})$ for the interacting theory.

For this general case, when $\lambda \neq 0$, time evolution will be generated by a more complicated Hamiltonian. For the system (4.1) we have

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} = \hat{H}_0 + \frac{\lambda}{4!}\int \mathrm{d}^3x\,\hat{\phi}(\vec{x})^4\,. \tag{4.7}$$

The time evolution, however, works in exactly the same way as for the free theory. We thus have

$$\begin{aligned}\hat{\phi}(t,\vec{x}) &= e^{(i/\hbar)\hat{H}(t-t_0)}\,\hat{\phi}_{t_0}(\vec{x})\,e^{-(i/\hbar)\hat{H}(t-t_0)} \\ &= e^{(i/\hbar)\hat{H}(t-t_0)}\;e^{-(i/\hbar)\hat{H}_0\,(t-t_0)}\,\hat{\phi}_0(t,\vec{x})\,e^{(i/\hbar)\hat{H}_0\,(t-t_0)}\;e^{-(i/\hbar)\hat{H}(t-t_0)}\,.\end{aligned} \tag{4.8}$$




The evolution operator is used to write the interacting field $\hat{\phi}(t,\vec{x})$ purely in terms of the free field $\hat{\phi}_0(t,\vec{x})$; see (4.10).

In the second step we have simply inserted the free field time evolution (4.5). The particular combination of the Hamiltonian expressions entering here is given a special name,

$$\hat{U}(t,t_0) = e^{(i/\hbar)\hat{H}_0\,(t-t_0)}e^{-(i/\hbar)\hat{H}\,(t-t_0)}\,, \tag{4.9}$$

also called the *evolution operator*. When it acts on a state, it evolves it backward in time from $t$ to $t_0$ with the free Hamiltonian, and then forward in time again with the full Hamiltonian. Using this operator we have

$$\hat{\phi}(t,\vec{x}) = \hat{U}^\dagger(t,t_0)\hat{\phi}_0(t,\vec{x})\hat{U}(t,t_0)\,. \tag{4.10}$$

When $\hat{H}_0 = \hat{H}$, i.e. when $\lambda = 0$, the evolution is according to the equation of motion of the free field, and (4.8) reduces to a triviality. Note that $\hat{U}^\dagger(t,t_0) = \hat{U}(t_0,t)$.

(end of lecture 5)

We can find a more convenient and explicit expression for $U(t,t_0)$ by noting that this operator satisfies the equation

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) &= e^{(i/\hbar)\hat{H}_0\,(t-t_0)}(\hat{H}-\hat{H}_0)e^{-(i/\hbar)\hat{H}\,(t-t_0)} \\ &= e^{(i/\hbar)\hat{H}_0(t-t_0)}\,\hat{H}_{\text{int}}\,e^{-(i/\hbar)\hat{H}(t-t_0)} \\ &= e^{(i/\hbar)\hat{H}_0(t-t_0)}\,\hat{H}_{\text{int}}\,e^{-(i/\hbar)\hat{H}_0(t-t_0)}e^{(i/\hbar)\hat{H}_0(t-t_0)}e^{-(i/\hbar)\hat{H}(t-t_0)} \\ &= \hat{H}_I(t)\,\hat{U}(t,t_0)\,.\end{aligned} \tag{4.11}$$

To arrive at the third line we have simply inserted 'one'. Here we have defined $\hat{H}_I(t)$ which is the interaction part of the Hamiltonian, $\hat{H}_{\text{int}}$, evolved using the free Hamiltonian $\hat{H}_0$,

$$\hat{H}_I(t) = e^{(i/\hbar)\hat{H}_0(t-t_0)}\,\hat{H}_{\text{int}}\,e^{-(i/\hbar)\hat{H}_0(t-t_0)} = \frac{\lambda}{4!}\int \mathrm{d}^3x\,\hat{\phi}_0(t,\vec{x})^4\,. \tag{4.12}$$

This shows that it should be possible to find an expression for $\hat{U}(t,t_0)$ purely in terms of the time-dependent free fields $\hat{\phi}_0(t,\vec{x})$.

The solution to this problem is given by *Dyson's formula* (actually first figured out by Dirac, but not written in this compact form),

$$\hat{U}(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right]\,. \tag{4.13}$$

Here $T$ is the time-ordering operator already encountered in (3.38). We can verify that (4.13) solves (4.11) quite easily, by remembering that we can write the operators inside a time ordering operator in any order, since the time ordering operator will put them in a specific order already. We thus have

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right] &= T\left(\hat{H}_I(t)\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right]\right) \\ &= \hat{H}_I(t)T\left(\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right]\right)\,, \end{aligned} \tag{4.14}$$

where the last step is valid because $t$ is the largest time occurring in all operators inside $T$, and we can thus pull out $\hat{H}_I(t)$ to the left. Dyson's expression for the evolution operator thus satisfies (4.11).

So we have now expressed the time evolution of the full field, $\hat{\phi}(t,\vec{x})$, entirely in terms of the free field $\hat{\phi}_0(t,\vec{x})$, for which we know the time evolution. The next step is to use this knowledge to compute correlation functions in the interacting theory.

▸ ***Summary:*** Systems with actions which are higher than quadratic in the fields cannot be solved exactly, but need a perturbative treatment, with the Heisenberg picture field $\hat{\phi}_0$ of the unperturbed system as starting point. The crucial ingredient is the evolution operator, in the form of Dyson's formula (4.13).

▸ ***See also:*** This section follows section 4.2 of M. Peskin and D. Schroeder, "An introduction to quantum field theory", Perseus, 1995.

## 4.2. Correlation functions and Wick's theorem

Now that we have expressed the time evolution of the field $\hat{\phi}(t,\vec{x})$ entirely in terms of the field $\hat{\phi}_0(t,\vec{x})$, we can go and compute correlation functions in the interacting theory at $\lambda \neq 0$. Let us start by computing the correlator of a time-ordered product of fields, such as we have encountered in the computation of the Feynman propagator (3.39), but now with interactions. We thus want to compute

$$\langle 0|\,T\Big(\hat{\phi}(t,\vec{x})\,\hat{\phi}(t,\vec{y})\Big)\,|0\rangle\,, \tag{4.15}$$

and interpret this as the Feynman propagator in the presence of interactions. We have to be a bit careful with what we mean with $|0\rangle$. When there are no interactions, we know that there is a zero-energy eigenstate of the free Hamiltonian $\hat{H}_0$, let us call it $|0\rangle_0$, which represents a state without any particle excitations. We would like to

use such an empty state as a starting point in (4.15) as well. However, an empty state in general does not stay empty under time evolution when interactions are turned on. What we will thus do is take a state which is empty in the far past, and evolve it with the evolution operator to finite time,

$$|0\rangle := U(t_0, -\infty)|0\rangle_0\,, \tag{4.16}$$

which, to be precise, is the vacuum state of the free theory in the far past, evolved with the interacting Hamiltonian to $t_0$. A similar expression will be used for the outgoing state. The inner product of these states is then

$${}_0\langle 0|\, U(\infty, t_0)U(t_0, -\infty)\,|0\rangle_0 = {}_0\langle 0|\, U(\infty, -\infty)\,|0\rangle_0\,. \tag{4.17}$$

Unsurprisingly, this is called the *vacuum persistence amplitude*. It make sense to normalise this to one, or equivalently, divide all correlators by this amplitude.

We now just have to insert the expression for $\hat\phi(t, \vec x)$ and re-organise terms a bit. Using (4.10) to express the interacting fields in terms of the free fields we get

$$\begin{aligned}
&\langle 0|T\Big(\hat\phi(x)\hat\phi(y)\Big)|0\rangle \\
&= \frac{{}_0\langle 0|U(\infty, t_0)\, T\Big(U(t_0, x^0)\hat\phi_0(x)U(x^0, t_0)\;U(t_0, y^0)\hat\phi_0(y)U(y^0, t_0)\Big)U(t_0, -\infty)|0\rangle_0}{{}_0\langle 0|\, U(\infty, -\infty)\,|0\rangle_0} \\
&= \frac{{}_0\langle 0|T\Big(U(\infty, -\infty)\hat\phi_0(x)\hat\phi_0(y)\Big)|0\rangle_0}{{}_0\langle 0|\, U(\infty, -\infty)\,|0\rangle_0}\,.
\end{aligned} \tag{4.18}$$

The last step was possible because the time-ordering operator takes care of splitting the evolution operator up in factors and putting them at the right position in the correlator. Using Dyson's formula, the general expression for the time-ordered product of fields in the interacting theory is thus

$$\langle 0|T\Big(\hat\phi(x_1)\cdots\hat\phi(x_n)\Big)|0\rangle = \frac{{}_0\langle 0|\, T\Big(\hat\phi_0(x_0)\cdots\hat\phi_0(x_n)\exp\Big[-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\,\hat H_I(t)\Big]\Big)\,|0\rangle_0}{{}_0\langle 0|T\Big(\exp\Big[-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\,\hat H_I(t)\Big]\Big)|0\rangle_0}\,, \tag{4.19}$$

This expression only involves creation and annihilation operators of the free theory, and can hence in principle be worked out with the knowledge we have from the previous chapter.

Now we have to remember that $\hat H_{\text{int}}$ is proportional to the parameter $\lambda$, which we assume to be small. By expanding the exponents in (4.19) in powers of $\lambda$ and inserting the free field expansion, we will thus have to compute various time-ordered expectation values of free field operators,

$$\langle 0|\, T\Big(\hat\phi_0(x_1)\hat\phi_0(x_2)\cdots\hat\phi_0(x_n)\Big)\,|0\rangle\,. \tag{4.20}$$

We have done this for $n = 2$ by simply inserting the expansion of the field and using the algebra of the creation and annihilation operators. For larger $n$ such manipulations can still be done, but they become very cumbersome. It is much more useful to emply a clever trick, known as *Wick's theorem*, which allows us to write down these time-ordered products directly.

(end of lecture 6)

Wick's theorem provides a quick way to reduce time-ordered products of fields in terms of normal-ordered products of fields and propagators. We have encountered normal ordering briefly in the previous chapter. More explicitly, the normal

ordered product of two operators is denoted by $:\hat{A}\hat{B}:$ and defined in such a way that all creation operators are moved to the left of all annihilation operators. That is,

$$:\hat{a}(\vec{p})\hat{a}^\dagger(\vec{k})\hat{a}(\vec{q}):=\hat{a}^\dagger(\vec{k})\hat{a}(\vec{p})\hat{a}(\vec{q})\,. \tag{4.21}$$

This ensures that the vacuum expectation value of any normal ordered product vanishes,

$$\langle 0|\ :\hat{\phi}_0(x_1)\cdots\hat{\phi}_0(x_n):\ |0\rangle=0\,. \tag{4.22}$$

Time ordering and normal ordering differ at most by commutators of the fields, which are ordinary functions. This difference is called the *contraction* of two fields, and denoted with a square bracket under it. So we have

$$T\Big(\hat{\phi}_0(x)\hat{\phi}_0(y)\Big)=:\hat{\phi}_0(x)\hat{\phi}_0(y):+\underbrace{\hat{\phi}_0(x)\hat{\phi}_0(y)}\,, \tag{4.23}$$

in which the second term is some function of $x_1$ and $x_2$. By sandwiching this expression between vacuum states of the free Hamiltonian, the first term on the right-hand side vanishes and the term on the left-hand side becomes the free field Feynman propagator. We thus have

$$\underbrace{\hat{\phi}_0(x)\hat{\phi}_0(y)}=G_F(x-y)\,. \tag{4.24}$$

It is of course possible to derive this also by expanding the fields into creation and annihilation parts explicitly. If we write

$$\hat{\phi}_0(x)=\hat{\phi}_0^{\rm an}(x)+\hat{\phi}_0^{\rm cr}(x)\,, \tag{4.25}$$

where the first term contains the annihilation operators, which annihilate $|0\rangle_0$, and the second one the creation operators, which annihilate ${}_0\langle 0|$. We then have

$$\begin{aligned}T\Big(\hat{\phi}_0(x)\hat{\phi}_0(y)\Big)&=\theta(x^0-y^0)\Big(:\hat{\phi}_0(x)\hat{\phi}_0(y):+\big[\phi_0^{\rm an}(x),\hat{\phi}_0^{\rm cr}(y)\big]\Big)\\&\quad+\theta(y^0-x^0)\Big(:\hat{\phi}_0(x)\hat{\phi}_0(y):+\big[\phi_0^{\rm an}(y),\hat{\phi}_0^{\rm cr}(x)\big]\Big)\,.\end{aligned} \tag{4.26}$$

The two commutator terms add up to be precisely the Feynman propagator (to see this, use the expansion of the free field in terms of $\hat{a}$ and $\hat{a}^\dagger$ and compare with (3.40)).

The idea is now to relations similar to (4.23) also for products of more than two fields. The generic expression is called *Wick's theorem* and reads

$$T\Big(\hat{\phi}_0(x_1)\cdots\hat{\phi}_0(x_n)\Big)=:\hat{\phi}_0(x_1)\cdots\hat{\phi}_0(x_n)+\text{all possible contractions}:\,. \tag{4.27}$$

In order to explain what this means, let us consider the case of four fields. Wick's theorem then implies

$$\begin{aligned}T\Big(\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0(x_4)\Big)&=:\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0(x_4)\\&\quad+\underbrace{\hat{\phi}_0(x_1)\hat{\phi}_0}(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0(x_4)+\underbrace{\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0}(x_3)\hat{\phi}_0(x_4)\\&\quad+\underbrace{\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0}(x_4)+\hat{\phi}_0(x_1)\underbrace{\hat{\phi}_0(x_2)\hat{\phi}_0}(x_3)\hat{\phi}_0(x_4)\\&\quad+\hat{\phi}_0(x_1)\underbrace{\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)}\hat{\phi}_0(x_4)+\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\underbrace{\hat{\phi}_0(x_3)\hat{\phi}_0}(x_4)\\&\quad+\underbrace{\hat{\phi}_0(x_1)\hat{\phi}_0}(x_2)\underbrace{\hat{\phi}_0(x_3)\hat{\phi}_0}(x_4)+\underbrace{\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0}(x_4)\\&\quad+\underbrace{\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0(x_4)}:\,,\end{aligned} \tag{4.28}$$

where the entire right-hand side is inside the normal ordering symbols. In this notation, it is still understood that any contraction that involves non-neighbouring fields yields a propagator,

$$:\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0(x_4): = G_F(x_1-x_3)\ :\hat{\phi}_0(x_2)\hat{\phi}_0(x_4): \ . \tag{4.29}$$

The expression above is quite long (though was obtained with quite a bit less work than doing the creation/annihilation operator algebra). However, it reduces dramatically if we evaluate its vacuum expectation value, since all the terms involving normal ordered products will then disappear,

$$\begin{aligned} &{}_0\langle 0|\, T\Big(\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\hat{\phi}_0(x_3)\hat{\phi}_0(x_4)\Big)\,|0\rangle_0 \\ &= G_F(x_1-x_2)\,G_F(x_3-x_4) + G_F(x_1-x_3)\,G_F(x_2-x_4) + G_F(x_1-x_4)\,G_F(x_2-x_3)\,. \end{aligned} \tag{4.30}$$

We have thus reduced the four-point correlator to a sum of products of two-point correlators.

Proving Wick's theorem (4.27) is done by induction, i.e. by assuming that it holds for $m$ fields and then proving it for $m+1$ fields, using the known case $m=2$ as the starting point. We will come back to this in the exercises.

## 4.3. Feynman diagrams

Now remember that we did all this because we want to be able to work out a perturbative expression (valid for small $\lambda$) of (4.19). Let us consider a two-point correlator evaluated to first order in $\lambda$. In order to compute that, we expand both the numerator and denominator,

$$\begin{aligned} {}_0\langle 0|\, T\Big(\hat{\phi}_0(x)\hat{\phi}_0(y)\exp\Big[-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\,\hat{H}_I(t)\Big]\Big)\,|0\rangle_0 &= N_0+\lambda N_1+\mathcal{O}(\lambda^2)\,, \\ \Big({}_0\langle 0|\, T\Big(\exp\Big[-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\,\hat{H}_I(t)\Big]\Big)\,|0\rangle_0\Big)^{-1} &= \Big(D_0+\lambda D_1+\mathcal{O}(\lambda^2)\Big)^{-1} \\ &= \frac{1}{D_0}-\lambda\frac{D_1}{D_0^2}+\mathcal{O}(\lambda^2)\,. \end{aligned} \tag{4.31}$$

The full correlator to order $\lambda$ is then

$$\langle 0|\, T\Big(\hat{\phi}(x)\hat{\phi}(y)\Big)\,|0\rangle = \frac{N_0}{D_0}+\lambda\Big(\frac{N_1}{D_0}-\frac{N_0D_1}{D_0^2}\Big)+\mathcal{O}(\lambda^2)\,. \tag{4.32}$$

We can use Wick's theorem to compute all the $N_i$ and $D_1$ expansion coefficients. For instance, we have

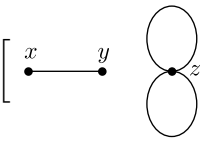
$$\begin{aligned} N_1 &= -\frac{i}{\hbar}\frac{\lambda}{4!}\,{}_0\langle 0|\, T\Big(\hat{\phi}_0(x)\hat{\phi}_0(y)\int\mathrm{d}^4z\,\big[\hat{\phi}_0(z)\big]^4\Big)\,|0\rangle_0 \\ &= -\frac{i}{\hbar}\frac{\lambda}{4!}\int\mathrm{d}^4z\,\Big(\text{all full contractions of }\hat{\phi}_0(x)\hat{\phi}_0(y)\big[\hat{\phi}_0(z)\big]^4\Big)\,. \end{aligned} \tag{4.33}$$

Computing such time-ordered products of a large number of free fields is essentially an exercise in combinatorics. It is convenient to use a graphical notation for the various contractions, suppressing the $\hat{\phi}_0$ and other irrelevant parts of the notation.

The idea behind *Feynman graphs* is to write every field $\hat{\phi}_0(x)$ as a dot, and every contraction $G_F(x-y)$ as a line connecting two dots. Let us do the expression (4.33)

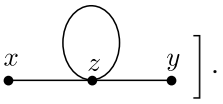
as an example. There are two essentially different types of contractions. One type involves contracting the $\hat{\phi}_0(x)$ and $\hat{\phi}_0(y)$ fields, and then contracting all the $\hat{\phi}_0(z)$ among each other. We can do that in 3 ways, but they all give the same result. The corresponding graph is drawn as

$$N_1^{\text{bubble}} = 3 \times \left(-\frac{i}{\hbar}\frac{\lambda}{4!}\right) \int \mathrm{d}^4 z \left[ \right. \quad \left. \right] . \tag{4.34}$$



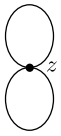

We call this a *bubble* contribution because the graph factorises into two disconnected pieces, one of which does not have *any* connections to the external points. The other option is to contract both $\hat{\phi}_0(x)$ and $\hat{\phi}_0(y)$ with the $\hat{\phi}_0(z)$, and then contract the remaining two $\hat{\phi}_0(z)$. This can be done in 12 ways, which are again all equivalent. So the second type of term gives

$$N_1^{\text{connected}} = 12 \times \left(-\frac{i}{\hbar}\frac{\lambda}{4!}\right) \int \mathrm{d}^4 z \left[ \right. \quad \left. \right] . \tag{4.35}$$



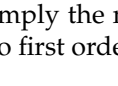

This is the *connected* contribution. The two expressions (4.34) and (4.35) added together represent the first order term in the expansion of the numerator, $N_1 = N_1^{\text{connected}} + N_1^{\text{bubble}}$. In the denominator, the term at order $\lambda$ is given by the simple graph

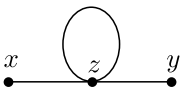
$$D_1 = 3 \times \left(-\frac{i}{\hbar}\frac{\lambda}{4!}\right) \int \mathrm{d}^4 z \left[ \right. \quad \left. \right] . \tag{4.36}$$




The $D_0$ contribution is simply the norm of the vacuum, and equal to one. For the full correlation function to first order in $\lambda$ we now find, using (4.32),

$$\langle 0| T\left(\hat{\phi}(x)\hat{\phi}(y)\right) |0\rangle \;=\; \quad + \; 12 \times \left(-\frac{i}{\hbar}\frac{\lambda}{4!}\right) \int \mathrm{d}^4 z \left[ \right. \quad \left. \right] . \tag{4.37}$$




Notice how the contribution of $N_1^{\text{bubble}}$ has been cancelled by the denominator. This is a general pattern: the expression in the denominator of (4.19) takes care of removing any bubble graphs (which might occur in the numerator) from the total expression. This is true at any order in $\lambda$.

The integrals corresponding to diagrams with loops are very often singular, and require special care. For this reason, we will not deal with them in any detail until chapter 7. However, there are some aspects which we can already discuss here. The most tricky part, when constructing the graphs, is to figure out the right overall factor (the 3 and 12 times $1/4!$ in the example above). For every vertex which comes from a term in the interaction Hamiltonian, we have in principle a factor

$$-\frac{i}{\hbar}\frac{\lambda}{4!} \int \mathrm{d}^4 z \,. \tag{4.38}$$

But whenever we connect a line to the $z$ dot, we can connect it to any of the four fields, so there are 4! ways of constructing the graph. A similar story holds true for the $1/n!$ from the Taylor series expansion. All the vertices are equivalent, so when we draw the graphs we will get $n!$ identical terms coming from re-ordering the vertices. In practise, we should thus expect that each vertex will come with a factor $-(i/\hbar)\lambda \int \mathrm{d}^4 z$.

However, in (4.34) we see a total factor $1/8$ and in (4.35) we see $1/2$. These additional factors are the so-called *symmetry* factors. They state in how many different ways we can exchange lines or vertices and end up with exactly the same diagram. For (4.34), we can flip each of the loops of the "figure eight" around the vertical axis, and we can also flip the eight upside down. This gives $2 \cdot 2 \cdot 2 = 8$ for the symmetry factor. In (4.35) we have only the symmetry which mirrors the loop, so a factor of 2. To obtain the correct expression for the correlator, we have to *divide* by these symmetry factors, and give each vertex the weight $-i\lambda \int \mathrm{d}^4 z$, without the 4!, as stated above.

(end of lecture 7)

We have so far discussed correlators in position space, i.e. with fields depending on space-time points. But we have already seen that the propagator in momentum space (3.45) is much simpler than the one in coordinate space (3.44). To figure out how to construct the Feynman diagrams in momentum space, consider once more the coordinate space Feynman propagator (3.44), repeated here for convenience,

$$G_F(x-y) = (-i\hbar) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik_\mu (x-y)^\mu}}{k_\nu k^\nu + m^2 - i\epsilon} \tag{4.39}$$

This expression shows that each line in a Feynman diagram has a momentum variable associated to it, which carries a direction. Moreover, a vertex, which involves an integration over a space-time point (like the $z$ in the example above), leads to a Dirac delta function. For instance, the exponentials of the various Fourier transforms in a $\hat{\phi}_0^4$ interaction term lead to

$$\int \mathrm{d}^4 z \, e^{ip_1 x} e^{ip_2 x} e^{ip_3 x} e^{ip_4 x} = (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \, . \tag{4.40}$$

With these rules we can write down the momentum-space two-point function in the interacting theory. We will come back to this in the next section.

▸ ***Summary:*** The correlator of $n$ time-ordered fields in the interacting theory is given by the sum of all connected graphs with $n$ external points. When computing the individual graphs, one should use a $-(i/\hbar)\lambda \int \mathrm{d}^4 z$ for each vertex, and divide by the symmetry factor of the graph.

▸ ***See also:*** A more complete discussion of the way in which bubble graphs get factored out can be found in [6] from (4.50) onwards. The simple way of computing symmetry factors discussed here was proven in G. Goldberg, "A rule for the combinatoric factors of Feynman diagrams", *Phys. Rev.* **D32** (1985) 3331.

## 4.4. Scattering matrix

Having shown how to compute correlators of fields in terms of Feynman diagrams, let us now go back to the problem of extracting physical information from such correlators. We have already mentioned, when we discussed the Feynman propagator for the free field in section 3.5, that a two-point function is related to the amplitude for the propagation of a particle from one point to the other. We would now like to make this a bit more explicit and also extend the logic to correlators of more than two fields.

We know that the creation operators $\hat{a}^\dagger(\vec{k})$ create on-shell particles with three-momentum $\vec{k}$. What we would like to do is to prepare such free particle states, make the particles scatter, and then express the result again in terms of free particle states. Because we have just seen how to compute correlators of free fields $\hat{\phi}_0(x)$, let us

use (3.30) to write the $\hat{a}(\vec{k})$ operators in terms of the free field. It is useful to write this as

$$\hat{a}(\vec{k}) = i \int \mathrm{d}^3x \, \frac{1}{\sqrt{2\omega_k}} e^{-ik_\mu x^\mu} \overset{\leftrightarrow}{\partial}_0 \hat{\phi}_0(t, \vec{x}) \,. \tag{4.41}$$

Here $\overset{\leftrightarrow}{\partial}_0$ is defined for any two functions $f$ and $g$ as

$$f \overset{\leftrightarrow}{\partial}_0 g = -(\partial_0 f) g + f(\partial_0 g) \,, \tag{4.42}$$

and we used $\partial_0 \hat{\phi}_0 = \hat{\pi}_0$. We would now like to use a similar expression also for the interacting field. That is, we want to define a new creation operator $\hat{a}_I^\dagger(\vec{k})$ which is the analogue of (4.41), but now using the interacting field $\hat{\phi}(x)$ on the right-hand side, instead of the free field $\hat{\phi}_0(x)$. We can then define a creation operator for a particle in the far past and an annihilation operator for a particle in the far future as

$$\begin{aligned} \hat{a}_{\text{in}}^\dagger(\vec{k}) &= \lim_{t \to -\infty} (-i) \int \mathrm{d}^3x \, \frac{1}{\sqrt{2\omega_k}} e^{ik_\mu x^\mu} \overset{\leftrightarrow}{\partial}_0 \hat{\phi}(t, \vec{x}) \,, \\ \hat{a}_{\text{out}}(\vec{k}) &= \lim_{t \to +\infty} (+i) \int \mathrm{d}^3x \, \frac{1}{\sqrt{2\omega_k}} e^{-ik_\mu x^\mu} \overset{\leftrightarrow}{\partial}_0 \hat{\phi}(t, \vec{x}) \,. \end{aligned} \tag{4.43}$$

For a complex field such as in the previous chapter, there would of course also be anti-particle operators. From the above definitions we find

$$\begin{aligned} \hat{a}_{\text{out}}^\dagger(\vec{k}) - \hat{a}_{\text{in}}^\dagger(\vec{k}) &= \int_{-\infty}^{\infty} \mathrm{d}t \, \partial_0 \hat{a}^\dagger(\vec{k}) \\ &= -i \int \mathrm{d}^4x \frac{1}{\sqrt{2\omega_k}} \partial_0 \Big( e^{ik_\mu x^\mu} \overset{\leftrightarrow}{\partial}_0 \hat{\phi}(x) \Big) \\ &= -i \int \mathrm{d}^4x \frac{1}{\sqrt{2\omega_k}} e^{ik_\mu x^\mu} \Big( \partial_0^2 + \omega^2 \Big) \hat{\phi}(x) \\ &= -i \int \mathrm{d}^4x \frac{1}{\sqrt{2\omega_k}} e^{ik_\mu x^\mu} \Big( -\partial_\mu \partial^\mu + m^2 \Big) \hat{\phi}(x) \,. \end{aligned} \tag{4.44}$$

In a similar way we can derive

$$\hat{a}_{\text{out}}(\vec{k}) - \hat{a}_{\text{in}}(\vec{k}) = i \int \mathrm{d}^4x \frac{1}{\sqrt{2\omega_k}} e^{-ik_\mu x^\mu} \Big( -\partial_\mu \partial^\mu + m^2 \Big) \hat{\phi}(x) \,. \tag{4.45}$$

Without interacctions, i.e. when the field $\hat{\phi}(x)$ equals $\hat{\phi}_0(x)$, the integrand equals the equation of motion and the right-hand side is zero.[2]

We now want to use the expression (4.45) above to compute the probability that a state of two incoming particles in the far past, with momenta $\vec{k}_1$ and $\vec{k}_2$, evolves to a state with two particles in the far future, with momenta $\vec{k}_3$ and $\vec{k}_4$. This probability is given by

$$\langle 0 | \, T \Big( \hat{a}_{\text{out}}(\vec{k_3}) \hat{a}_{\text{out}}(\vec{k_4}) \, \hat{a}_{\text{in}}^\dagger(\vec{k_1}) \hat{a}_{\text{in}}^\dagger(\vec{k_2}) \Big) \, | 0 \rangle \,, \tag{4.46}$$

(the time ordering operator does not really do anything here because the operators are already in the right order). Now we can use (4.45) to change the in-operators to out-operators and vice-versa, plus integral terms. When an in-operator is changed to an out-operator, it is moved to the left by the time ordering operator, and then hits the vacuum state, so that the result vanishes. What is left is just the integral terms, which read

The LSZ reduction formula relates time-ordered correlators to transition amplitudes for states with fixed external momenta.

[2] In deriving this formula we are cheating a bit with boundary terms when doing partial integrations. As usual, these arguments can be made more precise by properly smearing operators with wave packets (see e.g. [4]) and the result is that we can indeed just ignore the boundary terms.

$$\begin{aligned}
&\langle 0|\, T\Big(\hat{a}_{\text{out}}(\vec{k_3})\hat{a}_{\text{out}}(\vec{k_4})\,\hat{a}^\dagger_{\text{in}}(\vec{k_1})\hat{a}^\dagger_{\text{in}}(\vec{k_2})\Big)\,|0\rangle \\
&\quad = i^4 \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_4 e^{ik_1\cdot x_1 + ik_2\cdot x_2} e^{-ik_3\cdot x_3 - ik_4\cdot x_4}\Big(-\partial_1^2 + m^2\Big)\cdots\Big(-\partial_4^2 + m^2\Big) \\
&\qquad\qquad \times \langle 0|\, T\Big(\hat{\phi}(x_3)\hat{\phi}(x_4)\hat{\phi}(x_1)\hat{\phi}(x_2)\Big)|0\rangle \times \frac{1}{\sqrt{2\omega_{k_1}}\cdots\sqrt{2\omega_{k_4}}}\,. \qquad (4.47)
\end{aligned}$$

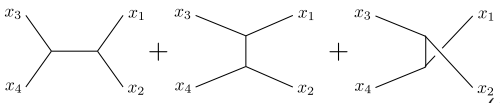
This expression, which relates a transition amplitude for particles of fixed momenta to a correlator of fields, is the *Lehmann-Symanzik-Zimmermann reduction formula*, or *LSZ formula* for short. The appearance of free field equation of motion operators (the $-\partial^2 + m^2$ factors) leads to what is called *amputation* of the external legs of a Feynman diagram; what that means will become clear in a minute.

(end of lecture 8)

Instead of using the action (4.1) for the $\phi^4$ model, let us use a slightly different one with a $\phi^3$ interaction, which will make the computations simpler. So we will use

$$S = \int \mathrm{d}^4x \Big[-\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3\Big]\,. \qquad (4.48)$$

In particular, because we have a three-point interaction instead of a four-point one, we will have a larger number of diagrams without loops at low order in $\lambda$ (we will not discuss loop diagrams in detail until chapter 7). The four-point correlator has a trivial contribution at $\lambda^0$, which just expresses non-interacting particles, so we will move on to $\lambda^2$. The four-point correlator at this order is expressed using Feynman graphs as

$\langle 0|\, T\Big(\hat{\phi}(x_3)\hat{\phi}(x_4)\hat{\phi}(x_1)\hat{\phi}(x_2)\Big)|0\rangle =$ 

(4.49)

(these three graphs are called the $s$, $t$ and $u$-channel graphs respectively; we will come back to that shortly). The mathematical expression associated to it is obtained by multiplying the appropriate Feynman propagators together, following the logic of the previous section. This gives

$$\begin{aligned}
= \Big(\frac{ig}{\hbar}\Big)^2 \int \mathrm{d}z_1\mathrm{d}z_2\, G_F(z_1 - z_2)\Big[\;& G_F(x_1 - z_2)G_F(x_2 - z_2)G_F(x_3 - z_1)G_F(x_4 - z_1) \\
&+ G_F(x_1 - z_2)G_F(x_3 - z_2)G_F(x_2 - z_1)G_F(x_4 - z_1) \\
&+ G_F(x_1 - z_2)G_F(x_4 - z_2)G_F(x_2 - z_1)G_F(x_3 - z_1)\Big]\,. \qquad (4.50)
\end{aligned}$$

We now have to insert this into the LSZ formula (4.47). This simplifies things a lot, because whenever a "$-\partial^2 + m^2$" operators hits a propagator with the corresponding coordinate, it produces a Dirac delta because of (3.46). For instance,

$$(-\partial_1^2 + m^2)G_F(x_1 - z_1) = -i\hbar\delta^{(4)}(x_1 - z_1)\,. \qquad (4.51)$$

All the propagators in the square brackets of (4.50) get removed this way (note: the only thing that remains of each propagator is the $\hbar$, by virtue of the right-hand side

of (4.51)). After doing the integral over the $x_i$, we are then left with [3]

$$\begin{aligned}
&\langle 0|\, T\Big(\hat{a}_{\rm out}(\vec{k_3})\hat{a}_{\rm out}(\vec{k_4})\,\hat{a}^\dagger_{\rm in}(\vec{k_1})\hat{a}^\dagger_{\rm in}(\vec{k_2})\Big)\,|0\rangle \\
&\qquad = \frac{(ig/\hbar)^2\hbar^4}{\sqrt{2\omega_{k_1}}\cdots\sqrt{2\omega_{k_4}}}\int \mathrm{d}z_1\mathrm{d}z_2 G_F(z_1-z_2)\Big[\; e^{iz_1(k_1+k_2)}e^{iz_2(-k_3-k_4)} \\
&\qquad\qquad\qquad\qquad + e^{iz_1(k_1-k_3)}e^{iz_2(k_2-k_4)} \\
&\qquad\qquad\qquad\qquad + e^{iz_1(k_1-k_4)}e^{iz_2(k_2-k_3)}\Big]\,.
\end{aligned} \tag{4.52}$$

The final integrals over the vertex operator positions can be done by using the momentum space representation (3.44) of the Feynman propagator. All $z_1, z_2$ dependence will then sit in exponentials, and we can integrate these variables out to produce

$$\begin{aligned}
&= \frac{(ig/\hbar)^2\hbar^4}{\sqrt{2\omega_{k_1}}\cdots\sqrt{2\omega_{k_4}}}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(2\pi)^8\hbar}{k^2+m^2-i\epsilon}\Big[\; \delta^{(4)}(k-k_1-k_2)\,\delta^{(4)}(k+k_3+k_4) \\
&\qquad\qquad + \delta^{(4)}(k-k_1+k_3)\,\delta^{(4)}(k-k_2+k_4) \\
&\qquad\qquad + \delta^{(4)}(k-k_1+k_4)\,\delta^{(4)}(k-k_2+k_3)\Big] \\
&= \frac{(ig)^2\hbar^3\,(2\pi)^4\delta^{(4)}(k_3+k_4-k_1-k_2)}{\sqrt{2\omega_{k_1}}\cdots\sqrt{2\omega_{k_4}}} \\
&\qquad \times\left[\frac{1}{(k_1+k_2)^2+m^2}+\frac{1}{(k_1-k_3)^2+m^2}+\frac{1}{(k_1-k_4)^2+m^2}\right].
\end{aligned} \tag{4.53}$$

This is our final expression for the transition amplitude. It contains an overall momentum conserving delta function, multiplied by three terms which express the different ways in which the particles can interact. It is usually expressed using the so-called *Mandelstam variables*,

$$s=-(k_1+k_2)^2\,,\quad t=-(k_1-k_3)^2\,,\quad u=-(k_1-k_4)^2\,. \tag{4.54}$$

Mandelstam variables are useful for the description of 4-particle processes, as they are Lorentz *scalars* adapted to the three possible momentum exchange channels.

and for obvious reasons the three terms in (4.53) are then called the $s$, $t$, and $u$-channel respectively. The manipulations described above can be summarised by a new set of Feynman rules, formulated directly in momentum space. These rules are given in table 4.1.

Finally, you may wonder about the way in which all of these transition amplitudes depend on Planck's constant $\hbar$. We have so far just carried them along, but in fact there is an important relation between the *order in* $\hbar$ and the *number of loops*. To see this, note that every propagator carries a factor of $\hbar$, and every vertex carries a factor $\hbar^{-1}$. Thus, a diagram with $E$ external lines, $I$ internal lines and $v$ vertices has a power

$$\hbar^{E+I-v}\,. \tag{4.55}$$

On the other hand, if there are $v$ vertices in the diagram, then there must be at least $v-1$ internal lines to glue them together, otherwise the diagram would be

[3] If you instead consider a $1\to 3$ or $3\to 1$ process, the only difference in the expression is an overall sign and a different distribution of signs for the momenta. In general, ingoing particles lead to $e^{ikx}$ factors with positive exponent, while outgoing ones have a negative exponent (cf. the difference between (4.44) and (4.45)).

1. Associate with each external particle of momentum $\vec{k}$ a factor $\hbar/\sqrt{2\omega_k}$.
2. Associate with each propagator a factor $\dfrac{-i\hbar}{p_\mu p^\mu + m^2 - i\epsilon}$.
3. Associate with each three-point vertex a factor $(-i/\hbar)g$.
4. Impose momentum conservation at each vertex.
5. Amputate the external legs (remove the propagators to the external points).
6. Integrate over every undetermined loop momentum $\displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$.
7. Divide by the symmetry factor.

**Table 4.1:** Feynman rules in momentum space for a theory with a three-point interaction term $-(g/3!)\phi^3$ in the action. The result is a transition amplitude for a $m \to n$ process, or equivalently $\langle 0|\hat{a}_{\text{out}}(k_1^{\text{out}}) \cdots \hat{a}_{\text{out}}(k_n^{\text{out}})\hat{a}^\dagger_{\text{in}}(k_1^{\text{in}}) \cdots \hat{a}^\dagger_{\text{in}}(k_m^{\text{in}})|0\rangle$.

disconnected. Any additional internal line will create a loop. So we have $L = I - (v-1)$ and thus the diagram has a power

$$\hbar^{E-1+L}\,. \tag{4.56}$$

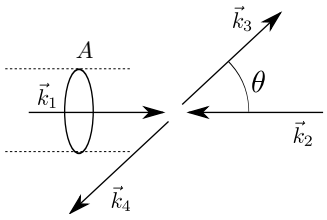
Thus, if we keep the structure of the diagram fixed (the number of external lines), then the power of $\hbar$ corresponds to the number of loops.

(end of lecture 9)

▸ ***Summary:*** The LSZ formula expresses amplitudes between in and out states of particles with given momenta in terms of 'amputated' correlators of fields in position space.

▸ ***See also:*** The discussion of the four-point function has been adapted from M. Srednicki, "Quantum field theory", Cambridge, 2007 chapters 5 and 10.

## 4.5. Cross sections and decay rates




Setup of the $2 \to 2$ scattering process and the definition of the angle $\theta$. The angle $\phi$ around the beam axis is suppressed here.

Having obtained an expression for the scattering amplitude of two particles, what remains to do in order to make contact with an actual experiment is to express this amplitude in measurable quantities. So what does an experimentalist measure in a particle collision experiment? The basic quantities accessible to experiment are *cross sections* and *decay rates*. The former typically have to do with processes in which more than one particle sits in the initial state, so that we can scatter them. The latter deal with what happens when a single particle decays into two or more other particles.

Let us first discuss cross sections, relevant for e.g. the $2 \to 2$ process which we discussed in the previous section. In an experiment, we will have one beam of particles of momentum $k_1$ colliding with a beam of particles of momentum $k_2$. We will be interested in computing the transition rate to an outgoing state, with one bunch of particles of momentum $k_3$ and one of momentum $k_4$. The setup is depicted in figure 4.5.

We start from the amplitude for the transition of an in-state with two particles (with momentum $k_1$ and $k_2$) to an out-state of two particles (with momenta $k_3$

and $k_4$). We computed it in the previous section, and will from now on denote it by $\langle 34|12\rangle$,

$$\begin{aligned}\langle 34|12\rangle &= \langle 0|\, T\Big(\hat{a}_{\rm out}(\vec{k_3})\hat{a}_{\rm out}(\vec{k_4})\,\hat{a}^\dagger_{\rm in}(\vec{k_1})\hat{a}^\dagger_{\rm in}(\vec{k_2})\Big)\,|0\rangle \\ &= \widetilde{\langle 34|12\rangle} \times (2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\prod_{i=1}^4 \frac{1}{\sqrt{2\omega_i}}\,. \end{aligned} \tag{4.57}$$

On the second line we have introduced $\widetilde{\langle 34|12\rangle}$ by splitting off the overall momentum conserving Dirac delta function, as well as some normalisation factors, which are always present in any amplitude. The transition *probability* is given, as usual in quantum mechanics, by the square of the norm, divided by the appropriate state normalisations. We are interested in the rate, so we also divide by the total time interval, and get[4]

$$\text{transition rate} = \frac{\text{transition probability}}{\text{total time}} = \frac{\big|\langle 34|12\rangle\big|^2}{\langle 12|12\rangle\langle 34|34\rangle}\frac{1}{T}\,. \tag{4.58}$$

Both the numerator and denominator are singular. In the numerator we will get the square of the delta function,

$$\Big|(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\Big|^2 = (2\pi)^4\delta^4(0)\times(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\,. \tag{4.59}$$

This can be rewritten, however, by noticing that the first factor is simply the four-volume of space-time (i.e. $V\times T$ where $V$ is the volume of space and $T$ the total length of the time interval, both of which of course go to infinity).[5] The norms of the states in the denominator are also singular. Single-particle states have norm $V$ and for the two-particle states we have

$$\langle 12|12\rangle = \langle 34|34\rangle = V^2\,, \tag{4.61}$$

by a similar argument. The transition rate is thus

$$\text{transition rate} = (2\pi)^4\delta^{(4)}(k_1+k_2-k_3-k_4)\times\Big|\widetilde{\langle 34|12\rangle}\Big|^2\times V^{-3}\prod_{i=1}^4\frac{1}{2\omega_i}\,. \tag{4.62}$$

First ingredient: the transition rate.

Now there are two more things to do. Firstly, in a real experiment we can never measure particles with exactly fixed momenta. What we should do is collect all particles in some small range of three-momenta. In a square box of size $V = L^3$, three-momenta are quantised as $\vec{k} = (2\pi n)/L$. In the limit of large volume a sum over momenta becomes

$$\sum_n \to \frac{V}{(2\pi)^3}\int \mathrm{d}^3k\,. \tag{4.63}$$

We should thus multiply our rate by a factor of this type for each outgoing particle. Secondly, we need to normalise our probability by dividing by the incoming flux.

Second ingredient: integrating over a region of outgoing momenta.

[4] All manipulations here are rather singular if we consider an infinitely extended space-time, so we will temporarily put things in a space-time box of finite three-volume $V$ and time extent $T$, and take the limit $V\to\infty$, $T\to\infty$ at the end.

[5] This is easiest to see by noting that the Dirac delta is given by the integral expression

$$(2\pi)^4\delta^{(4)}(p) = \int \mathrm{d}^4x\, e^{-ip_\mu x^\mu}\,. \tag{4.60}$$

Evaluating this at $p_\mu = 0$ then relates $\delta(0)$ to the volume of space-time.

In the rest frame of particle two, the incoming flux is simply the number of particles of momentum $\vec{k}_1$, per unit area of the beam, per unit time.

$$\text{incoming flux} = \frac{\#\text{ of incoming particles}}{A\,T} = \frac{\#\text{ of incoming particles} \times |\vec{v}_1|}{V}\,, \tag{4.64}$$

where we multiplied and divided by $|\vec{v}_1|$ to arrive at the last expression. So the flux of one particle is $|\vec{v}_1|/V$. More generally, when the target is not at rest, we would have to use the relative velocity of the two incoming particles, and the flux is $|\vec{v}_1 - \vec{v}_2|/V$. In terms of momenta, this is

$$\text{incoming flux} = \left|\frac{\vec{p}_1}{\omega_1} - \frac{\vec{p}_2}{\omega_2}\right| \frac{1}{V}\,. \tag{4.65}$$

Third ingredient: normalise with respect to the ingoing flux.

Putting all these things together we define the *differential cross section* as

$$\begin{aligned} \mathrm{d}\sigma_{12\to 34} &= \frac{\text{transition rate}}{\text{unit incoming flux}} \times \text{momentum intervals} \\ &= \left|\widetilde{\langle 34|12\rangle}\right|^2 \frac{1}{4\omega_1\omega_2|\vec{v}_1 - \vec{v}_2|} \underbrace{\frac{\mathrm{d}^3k_3}{(2\pi)^3 2\omega_3} \frac{\mathrm{d}^3k_4}{(2\pi)^3 2\omega_4} \times (2\pi)^4\delta^{(4)}(k_1 + k_2 - k_3 - k_4)}_{\mathrm{d}\Pi_2}\,, \end{aligned} \tag{4.66}$$

(where we indicated the phase space factor for the two outgoing particles with $\mathrm{d}\Pi_2$). This formula is easily generalised to more than two particles in the outgoing state: simply add additional momentum integration measure factors, and add the final state momenta to the overall delta function. The total cross section is obtained by integrating the expression above over the outgoing momenta.

For the $2 \to 2$ process we are looking at here, we can still simplify the expression a bit. First, let us go to the centre of mass of the ingoing system, that is, choose a frame in which $\vec{k}_1 = -\vec{k}_2$. The centre-of-mass energy is $\omega_{\text{cm}} = \omega_1 + \omega_2$. In this system the phase space measure for the outgoing particles can be written as

$$\begin{aligned} \mathrm{d}\Pi_2 &= \frac{1}{4(2\pi)^2\omega_3\omega_4}\delta(\omega_{\text{cm}} - \omega_3 - \omega_4)\delta^{(3)}(\vec{k}_3 + \vec{k}_4)\mathrm{d}^3k_3\mathrm{d}^3k_4 \\ &= \frac{1}{4(2\pi)^2\omega_3\omega_4}\delta(\omega_{\text{cm}} - \omega_3 - \omega_4)\,|\vec{k}_3|^2\,\mathrm{d}|\vec{k}_3|\,\mathrm{d}\Omega_{\text{cm}}\,. \end{aligned} \tag{4.67}$$

In the first step we separated the energy conserving and momentum conserving delta functions. In the next step, we integrated out $\vec{k}_4$ and introduced spherical coordinates for $\vec{k}_3$. The differential solid angle is $\mathrm{d}\Omega_{\text{cm}} = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$, where the $\theta$ angle is indicated in figure 4.5 and $\phi$ is the angle around the beam axis. We can now still integrate out the norm $|\vec{k}_3|$ and get rid of the last remaining delta function.[6] This produces for the two-particle phase space the expression

$$\mathrm{d}\Pi_2 = \frac{|\vec{k}_3|}{4(2\pi)^2\sqrt{s}}\,\mathrm{d}\Omega_{\text{cm}}\,, \tag{4.69}$$

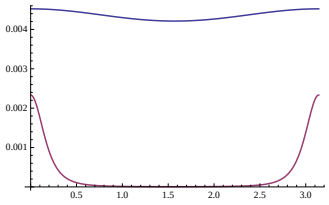
[6] This makes use of

$$\delta(f(x)) = \frac{\delta(x)}{|f'(x)|} \quad \text{and} \quad \frac{\partial}{\partial|\vec{k}_3|}(\omega_3 + \omega_4 - \omega_{\text{cm}}) = \frac{|\vec{k}_3|}{\omega_3} + \frac{|\vec{k}_3|}{\omega_4} = \frac{|\vec{k}_3|\,\omega_{\text{cm}}}{\omega_3\omega_4}\,. \tag{4.68}$$

where we have used $\sqrt{s} = \omega_{\text{cm}}$. If we now insert this back into (4.66) and integrate over $\phi$ to get a factor of $2\pi$, we end up with the following compact expression for the differential cross section,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \frac{2\pi}{4\omega_1\omega_2|\vec{v}_1 - \vec{v}_2|} \frac{|\vec{k}_3|}{(2\pi)^2 4\omega_{\text{cm}}} \left|\widetilde{\langle 34|12\rangle}\right|^2 = \frac{1}{32\pi\omega_{\text{cm}}^2} \left|\widetilde{\langle 34|12\rangle}\right|^2, \tag{4.70}$$

where the last equality holds because the four masses of the particles are equal. We will analyse the behaviour of this cross section in terms of the angle $\theta$ further in the exercises; a plot is given on the right.




Differential cross section $\mathrm{d}\sigma/\mathrm{d}\theta$ for the $2 \to 2$ process for $g = m = 1$ and two values of $\omega$: the top curve has $\omega = 2.1$ (which is near the lower bound $\omega^2 = 4m^2$) and the bottom curve has $\omega = 8$ (the latter scaled by a factor 15 to make it more visible). For small energies (non-relativistic) the distribution is isotropic, while for large energies (relativistic) there are peaks in the forward and backward direction.

(end of lecture 10)

▸ ***Summary:*** The differential scattering cross section of a $2 \to 2$ process is given by (4.66).

▸ ***See also:*** There is an infinite number of ways to explain cross sections and decay rates; the text here is based on chapter 11 of [4], page 150-154 of [1] and page 104-106 of [6].

# 5
# Path integrals

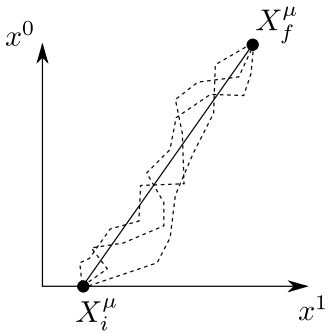
## 5.1. Quantum mechanics à la Feynman

The goal of the present chapter is to develop an alternative language for computations in quantum mechanics and quantum field theory, originally developed mainly by Feynman and Schwinger. Their way of computing amplitudes in quantum theories does not make use of operators or Hilbert spaces. Instead, it is entirely based on one of the most important principles in quantum mechanics, that of *superposition*.

The superposition principle tells us that, if there is more than one way in which a process can occur, the total amplitude that it will occur is simply the sum of the amplitudes for each of the contributions. This is a very general statement, true for any quantum system. For the time being, let us illustrate it at the level of a single free particle. If there is more than one way for it to propagate from an initial point $X_i$ to a final point $X_f$, then the total amplitude is a sum over all paths. Feynman's assumption (which can be motivated more rigorously) is that the contributions from each of these paths differ only by a phase,

$$\text{propagation amplitude} = \text{normalisation} \times \sum_{\text{all paths } X(\tau)} e^{i\cdot(\text{phase}[X(\tau)])} \,. \tag{5.1}$$

We have parameterised the path by some parameter $\tau$ which will have to be specified later. Of course, the number of paths is typically infinite, so the sum is generically an integral over a (very large) space of paths, and making sense of the 'sum' is often a challenging task. The so-called *path integral formalism* of quantum mechanics states that we can find an expression for the phase purely in terms of the classical paths, and then compute the amplitude as a weighted sum over classical objects, instead of dealing with operators and state vectors in Hilbert space (as we have done until now).




The propagation amplitude $G(X_f, X_i)$ is determined by a sum over all possible paths (dashed). In the classical limit $\hbar \to 0$, only a single path contributes (solid).

We can figure out what the phase should be by considering the classical limit $\hbar \to 0$. In the classical limit, only one path should contribute. Since the exponent is oscillatory, the dominant path will be the one for which the phase is stationary (the stationary phase appoximation). This condition reads

$$\text{classical path:} \quad \frac{\delta}{\delta X(\tau)} \Big( \text{phase}[X(\tau)] \Big) = 0 \,. \tag{5.2}$$

On the other hand, we already know a similar expression for the classical path, namely (2.17). That is, the classical path is described by the classical equation of motion, and the latter is given by

$$\text{classical path:} \quad \frac{\delta}{\delta X(\tau)} \Big( S[X(\tau)] \Big) = 0 \,, \tag{5.3}$$

where $S[X(\tau)]$ is the action, as a functional of the classical path. Moreover, the larger the action, the more dramatic the phase cancellations will be for non-stationary phases. It thus makes sense to identify the phase with the classical action *divided* by Planck's constant $\hbar$,

$$\text{phase}[X(\tau)] = \frac{1}{\hbar} S[X(\tau)] \,. \tag{5.4}$$

When $\hbar \to 0$ the only paths which will then contribute to the integral will be the classical ones (the stationary phase approximation becomes exact).[1]

We will now first show how make the expression (5.1) precise, and how to explicitly perform the sum over paths for the simple free particle system (section 5.2). We will then generalise the path integral formalism to more complicated quantum theories (section 5.3), in particular to the quantum field theories we have seen in the previous chapters.

## 5.2. Path integrals and propagators

The simplest process for which we can try to compute the amplitude using Feynman's "sum over paths" is that of the propagation of a single particle. We have already computed the amplitude for the propagation of a particle from a space-time point $X_i^\mu$ to a space-time point $X_f^\mu$ using canonical quantisation. The result is given by (3.44) and its Fourier transform is (3.45), which we repeat here for convenience,

$$G_F(k) = \frac{-i\hbar}{k_\mu k^\mu + m^2 - i\epsilon} \,. \tag{5.7}$$

We would like to derive this expression by computing the sum of amplitudes for each and every path that the particle can take between $X_i^\mu$ and $X_f^\nu$, and then Fourier transforming it.

We will label the paths by some parameter $\tau$, which takes the value $\tau_i$ at the start point $X_i^f$ of the path, and the value $\tau_f$ at the end point $X_f$ of the path. For a given value of $\tau$ inbetween, the position of the particle is given by $X^\mu(\tau)$. According to the path integral prescription, the amplitude for this process to take place is given by the formal expression

$$G_F(X_f(\tau_f), X_i(\tau_i)) = \int_{\text{paths}} \mathcal{D}X \, \exp\left[\frac{i}{2\hbar}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left(\dot{X}^\mu \dot{X}_\mu - m^2\right)\right] . \tag{5.8}$$

What we mean by the integral over paths will be made precise below. The expression in the exponent is simply $i$ times the classical action for a relativistic massive particle.[2] The length of the world-line in parameter space, $T := \tau_f - \tau_i$, is not a

[1] More generally, the stationary phase approximation is an approximation for integrals of the form

$$\int_{-\infty}^{\infty} f(x) e^{ig(x)} \mathrm{d}x \,, \tag{5.5}$$

where $f(x)$ is a slowly varying function of $x$. In the region where $g(x)$ is rapidly varying, the oscillatory contributions will cancel out because the prefactor $f(x)$ is almost constant. Therefore, the contributions to this integral will come from those regions where $g(x)$ is slowly varying as well. The dominant contribution will be in the region where

$$\mathrm{d}g(x)/\mathrm{d}x \approx 0 \,. \tag{5.6}$$

Expanding around such points gives the stationary phase approximation.

[2] Strictly speaking we should worry about the fact that this expression for the action is not invariant under reparameterisation of the world-line parameter $\tau$. Instead of dealing with those issues here, we will postpone them to chapter 6 where we can put them into a wider context.

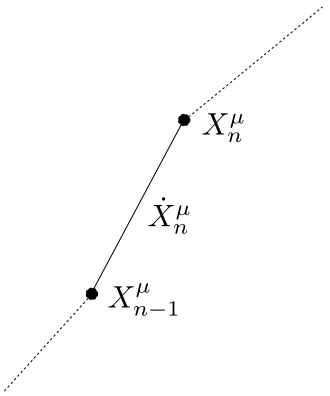
physical quantity, so we will also integrate over it. This gives

$$G_F(X_f, X_i) = \int_0^\infty \mathrm{d}T \int_{X(\tau_i)=X_i}^{X(\tau_i+T)=X_f} \mathcal{D}X \, \exp\left[\frac{i}{2\hbar}\int_0^T \mathrm{d}\tau \left(\dot{X}^\mu \dot{X}_\mu - m^2 + i\epsilon\right)\right] . \quad (5.9)$$

We have added a small imaginary piece to the mass-square, i.e. we have replaced $m^2 \to m^2 - i\epsilon$. This will be necessary later to make the $T$ integral converge, and we will see that it is related to the $i\epsilon$ which appears in the Feynman propagator (5.7). In addition, in order to compare with (5.7), we will in the end want to Fourier transform on the initial and final position variables.

In order to do the path integral, we will cut up the range of $\tau$ into $N$ small equal-sized segments $\delta$, such that $T = N\delta$. In the end we will take $N \to \infty$ to obtain the continuum limit.[3] The integration over the paths now becomes an $(N-1)$-fold integration over the positions $X_n^\mu := X^\mu(\tau = n\delta)$, for $n = 1 \ldots (N-1)$. The endpoints are $X_i^\mu = X_0^\mu$ and $X_f^\mu = X_N^\mu$. This is once more illustrated in the figure; note that the first segment is labelled with $n = 1$ and starts at $X_0^\mu$. The discretised approximation to the path integral (5.9) reads

$$G_F(X_f, X_i) = \int_0^\infty \mathrm{d}T \, \mathcal{C}(N,\delta) \int \mathrm{d}^4X_1 \cdots \mathrm{d}^4X_{N-1} \times \exp\left[\frac{i\delta}{2\hbar}\sum_{n=1}^{N}\left(\dot{X}_n^\mu \dot{X}_{n\,\mu} - m^2 + i\epsilon\right)\right] . \quad (5.10)$$




The $n$th segment of the path, which starts at $X_{n-1}^\mu$ and ends at $X_n^\mu$.

The normalisation constant $\mathcal{C}(N,\delta)$ is part of the definition of the path integral measure, and will be fixed at the very end. The quadratic term in $X_n^\mu$ in the expression above is a bit problematic: if we approximate the derivative by a finite difference according to

$$\dot{X}_n^\mu = \frac{1}{\delta}\left(X_n^\mu - X_{n-1}^\mu\right) . \quad (5.11)$$

we will get a bunch of cross terms involving the position variables for different values of $\tau$. So let us use a small trick to get rid of these quadratic terms. We do it by inserting $N$ copies of unity in the form of Gaussian integrals,

$$\left(\frac{-\delta}{2\pi\hbar}\right)^{2(N-1)} \prod_{m=1}^{N-1}\left[\int \mathrm{d}^4\alpha_m \, \exp\left(-\frac{i\delta}{2\hbar}\alpha_m^2\right)\right] = 1 . \quad (5.12)$$

The symbols $\alpha_m^\mu$ are four-vectors.[4] We can make the quadratic terms $X_n^\mu X_{\mu m}$ disappear by shifting these new variables according to $\alpha_n^\mu \to \alpha_n^\mu - \dot{X}_n^\mu$ (which leaves the integration measure invariant); this shift changes the exponent of (5.10) as

$$\frac{i\delta}{2\hbar}\sum_{n=1}^{N}\left(-\alpha_n^2 + \dot{X}_n^\mu \dot{X}_{n\,\mu} - m^2\right) \to \frac{i\delta}{2\hbar}\sum_{n=1}^{N}\left(2\alpha_n^\mu \dot{X}_{n\,\mu} - (\alpha_n^2 + m^2)\right) . \quad (5.13)$$

All the $X_n^\mu$-dependence now sits in the first term inside the sum above. If we insert the discretised derivative (5.11), these terms become

$$\frac{i}{\hbar}\sum_{n=1}^{N}\alpha_n^\mu(X_n^\mu - X_{n-1}^\mu) = \frac{i}{\hbar}\left(\alpha_N^\mu X_{N\,\mu} - \alpha_1^\mu X_{0\,\mu}\right) - \sum_{n=1}^{N-1}\frac{i}{\hbar}(\alpha_{n+1}^\mu - \alpha_n^\mu)X_{n\,\mu} . \quad (5.14)$$

[3] This limit is the hard bit in all path integral problems; one should worry about whether the limit is convergent, and one should worry about whether the result depends on how you have discretised the problem in the first place. For most path integrals we encounter later, these are unsolved problems.

[4] We are slightly cheating here because the argument of the exponent is imaginary rather than real; this can be repaired without changing the end result so we will ignore this issue here.

The last term contains all the integration variables. Making use of

$$\int \mathrm{d}^4 X_n \, \exp\left[-\frac{i}{\hbar}(\alpha_{n+1}-\alpha_n)\cdot X_n\right] = (2\pi\hbar)^4\,\delta^4(\alpha_{n+1}-\alpha_n) \tag{5.15}$$

allows us to integrate them all out and also get rid of all integrations over the $\alpha_n^\mu$ variables, because they all get set equal to each other by the Dirac delta function above. This simplifies the remaining terms in (5.14) as well, and what we are left with is (we will ignore overall constants as they can be adjusted by changing $\mathcal{C}(N,\delta)$)

$$G_F(X_f,X_i) \propto \int_0^\infty \mathrm{d}T \int \frac{\mathrm{d}^4\alpha}{(2\pi)^4}\exp\Big[-\frac{i}{2\hbar}T(\alpha^2+m^2-i\epsilon)+\frac{i}{\hbar}\alpha\cdot(X_f-X_i)\Big]. \tag{5.16}$$

We can now once more shift the integration variable $\alpha^\mu \to \alpha^\mu + (X_f-X_i)^\mu/T$ so as to turn this into a Gaussian integral. After doing that integral we are left with

$$G_F(X_f,X_i) \propto \int_0^\infty \mathrm{d}T \left(\frac{2\pi\hbar}{T}\right)^2 \exp\Big[-\frac{im^2+\epsilon}{2\hbar}T+\frac{i}{2\hbar T}(X_f-X_i)^2\Big]. \tag{5.17}$$

This is as simple as it is going to get in position space. In order to make contact with (5.7) we still need to do a Fourier transform to momentum space,

$$G_F(P_f,P_i) = \int_{-\infty}^\infty \mathrm{d}X_f \mathrm{d}X_i \, \exp\left(\frac{i}{\hbar}P_iX_i+\frac{i}{\hbar}P_fX_f\right) G_F(X_f,X_i)\,. \tag{5.18}$$

Of course we expect that the initial and final momenta are not independent, and indeed we will see a Dirac delta function appear in a minute. We thus finally get, for our path integral version of the momentum space propagator,

$$\begin{aligned} &G_F(P_f,P_i) \\ &\quad\propto \int_0^\infty \frac{\mathrm{d}T}{T^2}\int_{-\infty}^\infty \mathrm{d}X_f\mathrm{d}X_i\,\exp\left(\frac{i}{\hbar}P_iX_i+\frac{i}{\hbar}P_fX_f+\frac{i}{2\hbar\,T}(X_f-X_i)^2-\frac{im^2+\epsilon}{2\hbar}T\right) \\ &\quad= \int_0^\infty \frac{\mathrm{d}T}{T^2}\int_{-\infty}^\infty \mathrm{d}X_f\mathrm{d}X_i\,\exp\Big(\frac{i}{2\hbar\,T}(X_f-X_i+P_f\,T)^2-\frac{i}{2\hbar}(P_f)^2\,T \\ &\qquad\qquad +\frac{i}{\hbar}(P_i+P_f)X_i-\frac{im^2+\epsilon}{2\hbar}T\Big), \end{aligned} \tag{5.19}$$

where on the second line we have split off a Gaussian part. The integral over $X_f$ results in a $(2\hbar)^2T^2$ factor, so that the $T^{-2}$ which is already present gets cancelled. The $X_i$ integral then produces a momentum-conservation delta function, and the final result is

$$\begin{aligned} G(P_f,P_i) &\propto -\frac{1}{2}\delta(P_f+P_i)\int_0^\infty \mathrm{d}T\,\exp\Big(-\frac{i}{2\hbar}(P_f)^2T-\frac{im^2}{2\hbar}T-\frac{1}{2\hbar}\epsilon T\Big) \\ &= \delta(P_f+P_i)\frac{-i\hbar}{(P_f)^2+m^2-i\epsilon}\,. \end{aligned} \tag{5.20}$$

This is, indeed, the Feynman propagator for a scalar field in momentum space. With an appropriate choice of the path integral measure (i.e. $\mathcal{C}(N,\delta)$), the proportionality becomes an equality.

We thus see that the quantum mechanical properties of a single relativistic particle are encoded in an infinite-dimensional sum over classical paths, weighted appropriately by $i$ times the value that the classical takes on each path. No operators or Hilbert spaces are required.

A disadvantage of this method is of course that we have no control over interactions. In order to compute a $1 \to 2$ process, for instance, we would have to artificially glue together three particle paths, and the classical action of the relativistic particle does not say anything about the strength of that coupling (string theory solves this problem in an elegant way). For the remainder of the present chapter, we will instead focus on a path integral treatment of the field theories we have seen in previous chapters. It should be noted though that there is one range of applications for which the formalism presented here *is* very useful: the computation of quantum effects of particles in background fields. We will see an example of this in the exercises.

(end of lecture 11)

▸ ***Summary:*** The Feynman sum-over-paths for the relativistic particle leads to the Klein-Gordon propagator (this path integral form is also known as the Schwinger, or proper-time, or world-line representation of the propagator).

▸ ***See also:*** Polchinski volume I, chapter 5 and 7. R. P. Feynman, "Mathematical formulation of the quantum theory of electromagnetic interaction", *Phys. Rev.* **80** (1950) 440–457. J. S. Schwinger, "On gauge invariance and vacuum polarization", *Phys. Rev.* **82** (1951) 664–679.

## 5.3. Path integrals in field theory

Having seen how path integrals can be used for single particles, we will now generalise the path integral idea to field theories. The basic idea is the same as in the previous section: in order to determine the quantum amplitude for one field configuration to change into another one, we have to sum over all possible intermediate field configurations.

There are some small changes in the details of this generalisation. First of all, we are dealing with a somewhat different type of boundary conditions. In the case of a particle we were interested in dealing with a path for which $X^\mu(\tau_i)$ and $X^\mu(\tau_f)$ were fixed. In the case of a field theory, we will be interested in "paths" for which the field $\phi(t, \vec{x})$ approaches zero at spacelike and timelike infinity. In the path integral, we implement this using a simple trick. Consider for simplicity the free field theory of a real scalar. The claim is that we have the following correspondence between the vacuum-to-vacuum amplitude and a path integral expression,

$$\langle 0|0\rangle \leftrightarrow \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar} S[\phi] - \frac{\epsilon}{\hbar}\int \mathrm{d}^4x \, \phi(x)^2\right) . \tag{5.21}$$

We will of course again have to make precise what we mean with the integral over field configurations on the right hand side, and how we can compute it. For the time being, just observe that the additional term in the exponent (with $\epsilon$ a small positive constant, as before) has the consequence that the integrand is exponentially suppressed with a factor $\exp(-\int \mathrm{d}^4x \, \phi^2)$. If the field does not approach zero at infinity, this factor will lead to a large suppression. This addition to the integrand thus implements the boundary conditions on the field $\phi(x)$, and the path integral can be taken over all field configurations.

A second new ingredient concerns the type of computations we want to do. In field theory, we are mainly interested in correlation functions, as we have seen in section 4.2. In the path integral formulation, correlation functions are computed by simply inserting fields into the path integral. For a two-point correlation function, we have the correspondence

$$\langle 0|\, T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle \leftrightarrow \int \mathcal{D}\phi \, \phi(x_1)\phi(x_2) \, \exp\left(\frac{i}{\hbar} S[\phi] - \frac{\epsilon}{\hbar}\int \mathrm{d}^4x \, \phi(x)^2\right) . \tag{5.22}$$

Note that the fields which are inserted in the integrand are just classical objects; the quantum effects are taken care of by the integration over all possible field configurations $\phi(x)$. Again, take this as a definition for now; we will make a closer connection between the two sides later.

We will deal with correlation functions such as (5.22) in the next section, and first focus on the vacuum-to-vacuum amplitude (5.21). A useful generalisation of that amplitude is the "vacuum-to-vacuum amplitude in the presence of a source". It is defined as

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar} S[\phi] + \int \mathrm{d}^4x \, J(x)\phi(x)\right) , \tag{5.23}$$

so that $Z[J=0] = \langle 0|0\rangle$. What we have done here is to simply add an interaction term to the action, which contributes whenever the external source field $J(x)$ is non-zero. The reason for doing this is that we can use $Z[J]$ to generate correlation functions, simply by taking functional derivatives with respect to the source. For instance,

$$\langle 0|\, T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle = \frac{1}{Z[J]}\left(\frac{\delta}{\delta J(x_1)}\right)\left(\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0} . \tag{5.24}$$

We will do that in section 5.4. For the time being, let us focus on $Z[J]$ itself.

In the free theory, $Z[J]$ is extremely simple, as we can do the path integral over $\phi$ explicitly. For a real scalar field this goes as follows. We start from

$$Z_0[J] = \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar}\int \mathrm{d}^4x \left[\frac{1}{2}\phi(\Box - m^2 + i\epsilon)\phi + \frac{\hbar}{i}J\phi\right]\right) . \tag{5.25}$$

By completing the square, we can absorb the $J\phi$ term into the quadratic term. This requires the shift

$$\phi(x) = \tilde{\phi}(x) + \int \mathrm{d}^4y \, G_F(x-y)J(y) , \tag{5.26}$$

where $G_F(x-y)$ is the Feynman propagator which satisfies (see (3.46))

$$(\Box - m^2 + i\epsilon)G_F(x-y) = \frac{\hbar}{i}\delta^{(4)}(x-y) . \tag{5.27}$$

What is left of $Z_0[J]$ is then

$$Z_0[J] = \int \mathcal{D}\tilde{\phi} \, \exp\left\{\frac{i}{\hbar}\int \mathrm{d}^4x \left[\frac{1}{2}\tilde{\phi}(\Box - m^2 + i\epsilon)\tilde{\phi}\right] + \frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \, J(x)G_F(x-y)J(y)\right\} . \tag{5.28}$$

The slightly awkward double integral looks somewhat nicer when written in momentum space. We can now do the Gaussian path integral over $\tilde{\phi}$, which is nothing else but $Z_0[J=0]$. We thus finally get

$$Z_0[J] = Z_0[J=0] \times \exp\left\{\frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \, J(x)G_F(x-y)J(y)\right\} . \tag{5.29}$$

You can check that this is compatible with (5.24) and (3.39).

For interacting theories the story is of course more complicated, and generically it is not possible to do the path integral explicitly. We hence resort, as in the previous chapter, to a perturbative analysis valid for small values of the coupling constant which sets the interaction strength. Let us therefore write the action as

$$S[\phi, J] = S_0[\phi] + \lambda S_I[\phi] + \frac{\hbar}{i}\int \mathrm{d}^4x \, J(x)\phi(x) , \tag{5.30}$$

where $S_0[\phi]$ is the part of the action quadratic in the fields, and $S_I[\phi]$ is the rest. We have assumed that a small coupling constant $\lambda$ can be extracted from this latter part. This enables us to expand the exponential in powers of $\lambda$, as

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar} S_0[\phi] + \int \mathrm{d}^4x \, J(x)\phi(x)\right) \exp\left(\frac{i\lambda}{\hbar} S_I[\phi]\right) \\ &= \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar} S_0[\phi] + \int \mathrm{d}^4x \, J(x)\phi(x)\right) \\ &\quad \times \left(1 + \frac{i\lambda}{\hbar} \int \mathrm{d}^4y_1 \, \mathcal{L}_I\left[\phi(y_1)\right] + \frac{1}{2}\left(\frac{i\lambda}{\hbar}\right)^2 \int \mathrm{d}^4y_1 \, \mathcal{L}_I\left[\phi(y_1)\right] \int \mathrm{d}^4y_2 \, \mathcal{L}_I\left[\phi(y_2)\right] + \ldots\right) . \end{aligned} \tag{5.31}$$

The insertions of the interacting Lagrangian are nothing more than insertions of (integrated) polynomials of the field. We can rewrite those insertions using the same trick as used in (5.24) to write correlation functions: simply take a functional derivative with respect to the source. So we can also write

$$\begin{aligned} Z[J] = \Bigg(1 &+ \frac{i\lambda}{\hbar} \int \mathrm{d}^4y_1 \, \mathcal{L}_I\left[\frac{\delta}{\delta J(y_1)}\right] \\ &+ \frac{1}{2}\left(\frac{i\lambda}{\hbar}\right)^2 \int \mathrm{d}^4y_1 \, \mathcal{L}_I\left[\frac{\delta}{\delta J(y_1)}\right] \int \mathrm{d}^4y_2 \, \mathcal{L}_I\left[\frac{\delta}{\delta J(y_2)}\right] + \ldots\Bigg) \\ &\times \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar} S_0[\phi] + \int \mathrm{d}^4x \, J(x)\phi(x)\right) . \end{aligned} \tag{5.32}$$

The only $\phi$-dependence now sits in the last factor, and we can thus do the path integral over $\phi$ just as in the computation of $Z_0[J]$. We obtain

$$\begin{aligned} Z[J] = Z_0[J=0] \times \Bigg(1 &+ \frac{i\lambda}{\hbar} \int \mathrm{d}^4y_1 \, \mathcal{L}_I\left[\frac{\delta}{\delta J(y_1)}\right] \\ &+ \frac{1}{2}\left(\frac{i\lambda}{\hbar}\right)^2 \int \mathrm{d}^4y_1 \, \mathcal{L}_I\left[\frac{\delta}{\delta J(y_1)}\right] \int \mathrm{d}^4y_2 \, \mathcal{L}_I\left[\frac{\delta}{\delta J(y_2)}\right] + \ldots\Bigg) \\ &\times \exp\left(\frac{1}{2} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \, J(x_1) G_F(x_1 - x_2) J(x_2)\right) . \end{aligned} \tag{5.33}$$

The expression above is quite complicated, but fortunately we can find a convenient graphical notation for the various manipulations which are contained in it. Let us see how this works by considering the order $\lambda^n$ term in this expansion, and assuming that the interaction looks like $\mathcal{L}_I(x) = -\phi^p(x)/p!$, i.e. there is only one type of interaction vertex. This term in the expansion of the partition function contains $n$ integrals over $y_1, \ldots, y_n$,

$$\begin{aligned} &\frac{1}{n!}\left(\frac{i\lambda}{\hbar}\right)^n \left(\underbrace{\int \mathrm{d}^4y_1 \int \mathrm{d}^4y_2 \cdots \int \mathrm{d}^4y_n}_{n \text{ integrals}} \frac{\delta^p}{\delta J(y_1)^p} \frac{\delta^p}{\delta J(y_2)^p} \cdots \frac{\delta^p}{\delta J(y_n)^p}\right) \\ &\qquad \times \exp\left(\frac{1}{2} \int \mathrm{d}^4x_1 \, \mathrm{d}^4x_2 \, J(x_1) G_F(x_1 - x_2) J(x_2)\right) . \end{aligned} \tag{5.34}$$

The variational derivatives bring down the exponent, and remove a $\int \mathrm{d}^4x_i \, J(x_i)$ from

it, so that we get an expression of the form

$$\left(\int \mathrm{d}^4y_1 \int \mathrm{d}^4y_2 \cdots \int \mathrm{d}^4y_n \, \frac{\delta^p}{\delta J(y_1)^p} \frac{\delta^p}{\delta J(y_2)^p} \cdots \frac{\delta^p}{\delta J(y_n)^p}\right) \exp\left(\ldots\right)$$

$$\downarrow$$

$$\left(\int \mathrm{d}^4y_1 \int \mathrm{d}^4y_2 \cdots \int \mathrm{d}^4y_n \, \frac{\delta^{p-1}}{\delta J(y_1)^{p-1}} \frac{\delta^p}{\delta J(y_2)^p} \cdots \frac{\delta^p}{\delta J(y_n)^p}\right) \int \mathrm{d}^4x_1 \, J(x_1) G_F(x_1 - y_1) \exp\left(\ldots\right) . \tag{5.35}$$

If we take a further derivative, we can either act again on the exponential, or act on the $J$ that was brought down earlier. The latter action corresponds to

$$\left(\int \mathrm{d}^4y_1 \int \mathrm{d}^4y_2 \cdots \int \mathrm{d}^4y_3 \, \frac{\delta^{p-1}}{\delta J(y_1)^{p-1}} \frac{\delta^p}{\delta J(y_2)^p} \cdots \frac{\delta^p}{\delta J(y_n)^p}\right) \int \mathrm{d}^4 \, x_1 J(x_1) G_F(x_1 - y_1) \exp\left(\ldots\right)$$

$$\downarrow$$

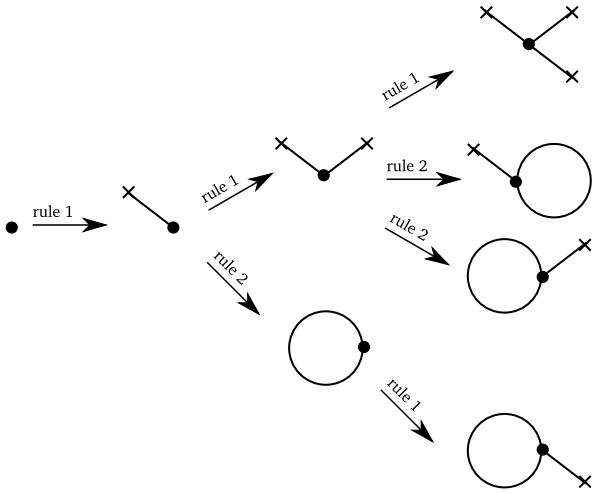
$$\left(\int \mathrm{d}^4y_1 \int \mathrm{d}^4y_2 \cdots \int \mathrm{d}^4y_3 \, \frac{\delta^{p-1}}{\delta J(y_1)^{p-1}} \frac{\delta^{p-1}}{\delta J(y_2)^{p-1}} \cdots \frac{\delta^p}{\delta J(y_n)^p}\right) G_F(y_2 - y_1) \exp\left(\ldots\right) . \tag{5.36}$$

Playing a bit with these procedures quickly leads to the following set of graphical rules. We represent every integration over an $y_i$ variable by a dot (these integration variables are dummies, so we do not have to label the dots). An integration which also contains a source (like the $\int \mathrm{d}^4x_1 \, J(x_1)$ above) is represented by a cross, and Feynman propagators are as usual connecting dots and crosses. At each differentiation step we pick a dot and

1. either add a cross to the diagram, and connect it with the dot,
2. or pick a cross, and replace it by the dot.

For each dot, this has to be done $p$ times ($p$, remember, being the order of the interaction term).

As an example, let us consider the order $\lambda$ term for a model with a $\phi^4$ interaction. The first three steps of the procedure above are depicted below:




After one more step, we arrive at the final set of diagrams representing $Z[J]$,

$$Z[J] = Z_0[J=0] \times \left[1 - \frac{i\lambda}{4!\,\hbar}\Big( 3 \; + \; 6 \; + \; \Big) + \mathcal{O}(\lambda^2)\right] \exp(\ldots) . \tag{5.37}$$

The overall numerical factors simply indicate in how many ways the associated graph can be built using the rules given above. Note that there is always still an exponential factor which depends on the source $J$. This will be important when we compute correlation functions. Also note once more that all dots and crosses correspond to integrals, which is why the 'external points' are not labelled.

Instead of doing all diagrams in one shot, as we did above, you can also obtain the symmetry factors for each diagram separately, by looking carefully at how it can be constructed. For example, the figure eight diagram above can be made by starting with a vertex with 4 lines sticking out, numbered 1 to 4. Then you can have the following pairings to make the figure eight:

$$(\{1,2\},\{3,4\}) \text{ or } (\{1,3\},\{2,4\}) \text{ or } (\{1,4\},\{2,3\}).$$

This explains the factor '3'. Similarly, to get the second graph, you can make the following pairings,

$$(\{1,J\},\{2,J\},\{3,4\}) \text{ or } (\{1,J\},\{3,J\},\{2,4\}) \text{ or } (\{1,J\},\{4,J\},\{2,3\}) \text{ or}$$
$$(\{2,J\},\{3,J\},\{1,4\}) \text{ or } (\{2,J\},\{4,J\},\{1,3\}) \text{ or } (\{3,J\},\{4,J\},\{1,2\}),$$

where the pairing with a $J$ means connecting the line to a cross (and these crosses, as explained above, represent identical integrated $J$'s factors). Hence a factor of 6. (end of lecture 12)

▸ ***Summary:*** In a free theory the generating functional $Z_0[J]$ can be computed explicitly and is given by (5.29). Interacting theories can be treated in perturbation theory. The key trick is to replace insertions of the field by variational derivatives with respect to the source, as in (5.32). The $\phi$ path integral then becomes a Gaussian integral.

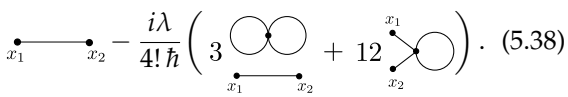
## 5.4. Feynman diagrams again

Having computed the generating functional $Z[J]$ in perturbation theory, we now want to extend our toolbox and compute correlation functions. This is now fairly simple, because we just have to work out expressions of the form (5.24), and use the expressions for $Z[J]$ worked out using the logic in the previous section.

Let us consider the two-point correlator (5.24) as our first example. When we take a variational derivative with respect to $J(x_1)$, it can act on two different places. Either it acts on the 'crosses' in (5.37), removing the integral and the $J$ factor from them. Or it acts on the exponential which multiplies the whole expression, bringing down a factor $\int \mathrm{d}^4 z\, G_F(x_1 - z) J(z)$. So we have a procedure which is quite similar to the one in the previous section. After we have taken both variational derivatives, we should set all remaining sources to zero; this means discarding any diagrams which still have crosses, and setting the exponential factor equal to one.

In a diagrammatic form, what we obtain to order $\lambda$ is

$$\left(\frac{\delta}{\delta J(x_1)}\right)\left(\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0} =$$




(5.38)

The labelled dots are un-integrated endpoints of propagators (in contrast to the unlabelled ones, which are integrated endpoints). Observe that this is precisely what we would have obtained from a construction of Feynman diagrams according to the rules in the previous chapter.[5] In particular, the symmetry factors work out precisely

[5] With the small difference that we have here spelled out the coupling constant $\lambda/\hbar$ explicitly, instead of absorbing it in the normalisation of the vertices. Similarly, we have kept all combinatorial factors, including the $1/4!$ of the vertex, explicit.

as required. For e.g. the first diagram, the total factor we found above is $3/4! = 1/8$. If we would try to compute that number using the logic of section 4.3, we would observe that the graph has a symmetry factor $2 \times 2 \times 2 = 8$, in agreement with the path integral computation.

(end of lecture 13)

## 5.5. Behaviour of the series expansion

We have so far assumed that the perturbation series in quantum field theory is well behaved, with a finite radius of convergence. Unfortunately, it is not.

The path integral provides us with a nice framework to discuss the behaviour of the perturbation series. This is because many aspects of the path integral are visible already in zero space-time dimensions, in which case it reduces to an ordinary integral. We can then study the behaviour of the integral as a function of the perturbation parameter $\lambda$.

So let us consider a 0-dimensional quantum field theory, in which there just a single point and $\phi$ is simply a number, not a field. The generating functional for the $\phi^4$ model is given by the integral

Path integrals in zero-dimensional quantum field theory are ordinary integrals, over one single degree of freedom.

$$Z_\lambda[J] = \int_{-\infty}^{\infty} e^{-\phi^2 - \lambda\phi^4 - J\phi} \, \mathrm{d}\phi \; . \tag{5.39}$$

It is tricky to find a solution for arbitrary $J$, so we will just look at an even simpler system, where $J = 0$. In that case, the integral above can be expressed in terms of a Bessel function,

$$Z_\lambda[J=0] = \frac{1}{2\sqrt{\lambda}} e^{\frac{1}{8\lambda}} K_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right) . \tag{5.40}$$

Here $K_n(y)$ is the modified Bessel function of the second kind. This step is impossible to do in full-fledged quantum field theory, but in this simple case we thus have access to a complete analytic answer for arbitrary value of the coupling constant $\lambda$. For $\lambda = 0$ the answer is of course $Z_{\lambda=0}[J=0] = \sqrt{\pi}$.

Now let us see what we would obtain if we would expand $Z_\lambda[J=0]$ in a perturbation series as we did in the previous sections. We would write

$$Z_\lambda[J=0] = \int_{-\infty}^{\infty} e^{-\phi^2} \sum_{k=0}^{\infty} \frac{(-\lambda\phi^4)^k}{k!} \mathrm{d}\phi \tag{5.41}$$

and then exchange the order of summation and integration. We can do all these integrals using a standard trick,

$$\int_{-\infty}^{\infty} \phi^{4k} e^{-\phi^2} \mathrm{d}\phi = \int_{-\infty}^{\infty} \frac{\partial^{2k}}{\partial a^{2k}} e^{-ax^2} \bigg|_{a=1} = \frac{\partial^{2k}}{\partial a^{2k}} \sqrt{\frac{\pi}{a}} \bigg|_{a=1} . \tag{5.42}$$

We can figure out what this order-$2k$ derivative yields by writing out the first few factors explicitly,

$$\begin{aligned} \frac{\partial^{2k}}{\partial a^{2k}} \sqrt{\frac{\pi}{a}} \bigg|_{a=1} &= \underbrace{\left(\frac{1}{2}\frac{3}{2}\right)\left(\frac{5}{2}\frac{7}{2}\right)\cdots}_{2k \text{ fractions}} \\ &= \underbrace{\left(\frac{1}{2}\frac{2}{2}\frac{3}{2}\right)\frac{4}{4}\left(\frac{5}{2}\frac{6}{6}\frac{7}{2}\right)\frac{8}{8}\cdots}_{4k \text{ fractions}} \\ &= \frac{(4k)!}{2^{2k}} \underbrace{\frac{1}{2}\frac{1}{4}\frac{1}{6}\cdots}_{2k \text{ fractions}} = \frac{(4k)!}{2^{2k}} \frac{1}{2^{2k}(2k)!} . \end{aligned} \tag{5.43}$$

Sticking this back into the expression (5.41) we get our perturbative expansion of the generating functional at $J = 0$, to all orders in perturbation theory,

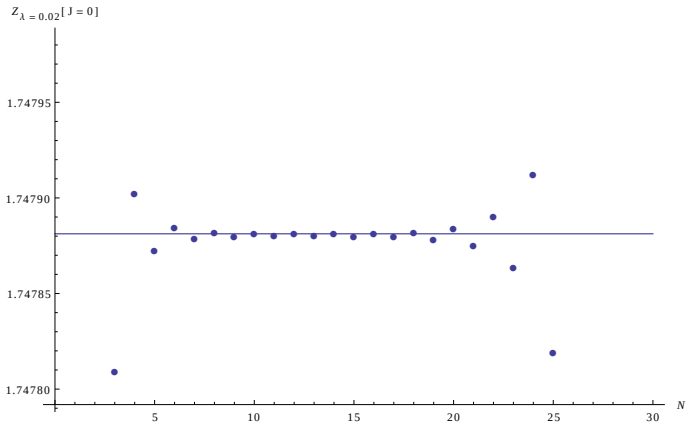
$$Z_\lambda[J=0] = \sum_{k=0}^{\infty} \sqrt{\pi}(-\lambda)^k \frac{(4k)!}{2^{4k}(2k)!k!}\,. \tag{5.44}$$

While this is a nice explicit result, it is also easy to show that this series is in fact *divergent*. Simply use Stirling's formula $k! \approx \sqrt{2\pi k}(k/e)^k$ to find that the terms in the sum above behave, for large $k$, as

$$\frac{(4k)!}{2^{4k}(2k)!k!} \approx \frac{4^k}{\sqrt{\pi k}}\left(\frac{k}{e}\right)^k \approx \frac{1}{\sqrt{2\pi}} 4^k k!\,. \tag{5.45}$$

The perturbation series in quantum field theory is an *asymptotic* series; it diverges when taken to all orders, but gives good approximations if the number of terms is not taken too large (as it turns out $N \sim 1/\lambda$).

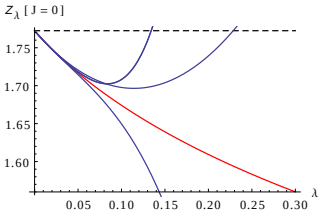
That clearly leads to a divergent series. However, let us do what a quantum field theory physicist would do, and look at the first few terms in the expansion.




Dots represent the value $Z_\lambda[J=0]$ at the arbitrary value $\lambda = 0.02$ when expanded to order $N$ in perturbation theory. The horizontal line is the exact value obtained from (5.40). Clearly visible is the fact that for small $N$ the sum first converges to the exact value, but then diverges again for larger values of $N$.

In figure 5.5 we have plotted the value $Z_{\lambda=0.02}[J=0]$, evaluated directly from the exact result (5.40) and evaluated by summing the $k = 0 \ldots N$ terms in the sum (5.44). You see the typical behaviour of a Feynman diagram expansion: the first few terms tend to make the answer converge, but then at some point the terms grow in size again and make the answer diverge.

▸ ***See also:*** The material in this section is based on [].

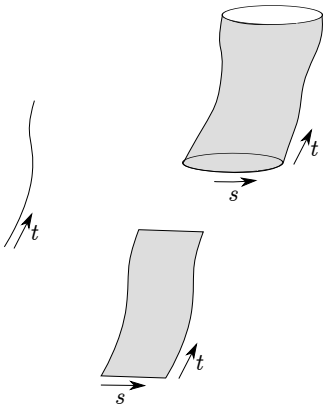



Comparison of the exact $Z_\lambda[J=0]$ of (5.40) (red curve in the middle) with successive orders in perturbation theory. The latter all grow unbounded for sufficiently large values of $\lambda$.

# 6
# Strings

In the previous chapter we have seen that the propagator of quantum scalar field can also be obtained by considering the path integral of a single relativistic particle. String theory is an attempt to generalise this idea, by replacing point particles by small one-dimensional objects, strings. The vibration modes of these strings correspond, as we will see, to various types of elementary particles. The strings can be open or closed. In both cases, the string 'sweeps out' a two-dimensional *world-sheet* as it evolves in time. See figure 6.

Although string theory has its share of problems in connecting to the real world, there are a few undisputed 'good' aspects to it. One is that string theory inevitably leads to a unification of the gravitational force with the electromagnetic, weak and strong forces. The other one is that strings can evolve consistently in geometrical backgrounds that are singular, and hence seem to be much better behaved in e.g. black hole space-times. We will discuss these two aspects, or at least the origins of them, in some detail in this chapter (several important recent results, such as the computation of black hole entropy and the correspondence between strings and gauge fields, will be skipped as they go far beyond the scope of the current text).




The world-line of a particle, parameterised by $\tau$, becomes the world-sheet of a closed string (top) or an open string (bottom), parameterised by $\tau$ and $\sigma$.

## 6.1. Relativistic particles again

Before we can discuss strings, we need to take one step back and have a closer look again at the relativistic particle, and in particular the *reparameterisation invariant* ways of describing them.

There are various ways in which one can write down an appropriate action for this particle (we have seen one such form in the previous chapter; we will see here how it fits into the bigger picture). The most natural action is actually simply the proper length of the world-line between the start and end points. If we parameterise the world-line by a parameter $\tau$, and denote the position of the particle in space-time at a given value of this parameter with $X^\mu(\tau)$, the action is given by

$$S_{\text{Nambu-Goto}} = -m \int_{\tau=\tau_i}^{\tau=\tau_f} \mathrm{d}\tau \sqrt{-\frac{\partial X^\mu(\tau)}{\partial \tau}\frac{\partial X_\mu(\tau)}{\partial \tau}}\,. \tag{6.1}$$

Importantly, this action is invariant under reparameterisation of the world-line, i.e. under transformations $\tau \to f(\tau)$.

The disadvantage of (6.1) is that the massless limit $m \to 0$ is ill-defined. In addition, the presence of the square root makes calculations rather messy. In order to circumvent this problem, a standard trick is to introduce an additional degree of

freedom, the so-called *einbein*. The action of this extended system is given by

$$S_{\text{Polyakov}} = \frac{1}{2}\int \mathrm{d}\tau \left[\frac{1}{e(\tau)}\frac{\partial X^\mu(\tau)}{\partial\tau}\frac{\partial X_\mu(\tau)}{\partial\tau} - e(\tau)\, m^2\right]. \tag{6.2}$$

When $e(\tau) = 1$, this reduces to the action which we have been using in the previous chapter. The equation of motion for $e(\tau)$ is algebraic (there are no derivatives on $e(\tau)$ anywhere); one finds

$$-\frac{1}{e(\tau)^2}\frac{\partial X^\mu(\tau)}{\partial\tau}\frac{\partial X_\mu(\tau)}{\partial\tau} - m^2 = 0 \qquad \text{(mass-shell constraint)}. \tag{6.3}$$

This is a *constraint*. Its interpretation becomes manifest when expressed in terms of the momentum conjugate to $X^\mu(\tau)$,

$$P_\mu(\tau) = \frac{\delta S}{\delta \dot{X}^\mu(\tau)} = \frac{1}{e(\tau)}\dot{X}_\mu(\tau)\,, \quad \to \quad P_\mu(\tau)P^\mu(\tau) = -m^2\,. \tag{6.4}$$

This is simply the mass-shell condition for the particle. By taking (6.3), solving for $e(\tau)$ and substituting it back into the action, we recover (6.1). However, it is more useful to use reparameterisation invariance to fix $e(\tau)$ to a specific function and thereby remove it from the action altogether (which is, again, what we have implicitly done in the previous chapter).

The action (6.2) is still invariant under world-line reparameterisations. The charge associated to constant translations in $\tau$ is the world-sheet Hamiltonian. It is obtained in the usual way as the Noether charge associated to the symmetry

$$\tau \to \tau + c \qquad \text{(world-line translations)}. \tag{6.5}$$

The Noether charge which we obtain in this way turns out to be exactly equal to the left-hand side of (6.3). This is a general result: systems which are reparameterisation invariant have a vanishing Hamiltonian.

Apart from the symmetry associated to translations on the world-line (global reparameterisations), there are also symmetries associated to translations in space-time. These give rise to more useful charges. The symmetry is

$$X^\mu(\tau) \to X^\mu(\tau) + \xi^\mu \qquad \text{(target-space translations)}. \tag{6.6}$$

The charge associated to this symmetry is simply $P_\mu$ as written in (6.4). This is the conserved target-space energy-momentum which squares to $-m^2$, and the zero-th component is the (non-vanishing) energy.

▸ ***Summary:*** Because the way we parameterise the world-line of a particle does not contain any physics, the action of a relativistic particle should be reparameterisation invariant. There are two ways to write down such an action, (6.1) and (6.2). The Noether charge associated to world-line translations is the world-line Hamiltonian, which vanishes by virtue of the constraint. The Noether charge associated to space-time translations is the space-time four-momentum, which does not vanish.

## 6.2. Reparameterisation invariant strings

Let us now turn to strings. For point particles, we used the proper length of the world-line as the action. Therefore, it is only natural to expect that the dynamics of a string is determined by the proper *area* of its world-surface,

$$S = T\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_{\sigma_L}^{\sigma_R}\mathrm{d}\sigma\,\sqrt{-\det\partial_\alpha X^\mu\partial_\beta X_\mu}\,. \tag{6.7}$$

This requires some explanation. The $X^\mu(\tau,\sigma)$ objects are vector-valued fields, which map a given point $(\tau,\sigma)$ on the two-dimensional world-sheet to a point $X^\mu(\tau,\sigma)$ in space-time. The determinant is, more explicitly, given by

$$\det \partial_\alpha X^\mu \partial_\beta X_\mu = \det \begin{pmatrix} \partial_\tau X^\mu \partial_\tau X_\mu & \partial_\tau X^\mu \partial_\sigma X_\mu \\ \partial_\sigma X^\mu \partial_\tau X_\mu & \partial_\sigma X^\mu \partial_\sigma X_\mu \end{pmatrix} . \tag{6.8}$$

The parameter $T$ denotes the *string tension*.[1] It is also often expressed using $\alpha'$, the *string slope*, using the relation

$$\alpha' := \frac{1}{2\pi T} . \tag{6.9}$$

As we already announced, strings can be open, in which case the two boundary points $\sigma = \sigma_L$ and $\sigma = \sigma_R$ are distinct, or closed, when $\sigma_L = \sigma_R$.

Although one can get quite far with the quantisation of the string by using the Nambu-Goto form (6.7) (see e.g. the book [21]), it at some point becomes impractical. Therefore, we again introduce an auxiliary field on the world-volume to turn the action into one of polynomial form,

$$S = -\frac{T}{2} \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_{\sigma_L}^{\sigma_R} \mathrm{d}\sigma \, \sqrt{-h} h^{\alpha\beta} \, \partial_\alpha X^\mu \partial_\beta X_\mu . \tag{6.10}$$

The field $h_{\alpha\beta}$ is the analogue of the einbein $e(\tau)$; it is called the *world-sheet metric*. It is a two-dimensional metric which lives on the world-sheet, and its only role is to make sure that the action is invariant under reparameterisations. When we do such a reparameterisation, both the derivatives and the world-sheet metric transform, in such a way that the action remains unchanged.

The equations of motion for the world-volume scalar fields $X^\mu$ and the world-volume metric $h_{\alpha\beta}$ are

$$\begin{aligned} \frac{\delta S}{\delta X^\mu} &= \sqrt{-h} h^{\alpha\beta} n_\alpha \partial_\beta X^\mu \Big|_{\sigma=\sigma_L}^{\sigma=\sigma_R} + \partial_\alpha \left( \sqrt{-h} h^{\alpha\beta} \partial_\beta X^\mu \right) = 0 , && \text{eq. of motion} , \\ T_{\alpha\beta} &= -\frac{1}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} = \frac{1}{2} \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu = 0 , && \text{constraints} , \end{aligned} \tag{6.11}$$

where $n_\alpha$ is the normal to the boundary. Recall that in general relativity, the variation of the matter action with respect to the metric yields the energy momentum tensor. The same thing happens here: vanishing of the equation of motion for $h_{\alpha\beta}$ is the same as the statement that the energy-momentum tensor of the world-volume theory should vanish. Note that the equation of motion contains both a bulk and a boundary term. The latter of course vanishes for closed strings, but it is non-trivial for open strings.

Apart from diffeomorphisms, which transform the fields $X^\mu$ as if they are scalars on the world-volume, there is one further symmetry of (6.10) which is extremely important: the action is invariant under local rescalings

$$h_{\alpha\beta}(\tau,\sigma) \to f(\tau,\sigma) \, h_{\alpha\beta}(\tau,\sigma) . \tag{6.12}$$

Such transformations are also known as *Weyl transformations*. Together the transformations act on the metric as

$$h_{\alpha\beta} \to h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + \Lambda \, h_{\alpha\beta} , \tag{6.13}$$

[1] Because the string tension is an overall factor in front of the action, it will appear inside the path integral multiplying $i/\hbar$ in the Feynman weight $\exp[(i/\hbar)S]$. We can thus always absorb $\hbar$ in $T$, and will set $\hbar = 1$ in the remainder of this chapter.

where $\nabla$ is the covariant derivative associated to $h_{\alpha\beta}$.

Reparameterisation invariance allows us to choose a coordinate system such that the metric is, at any point, proportional to the two-dimensional Minkowski metric,

$$h_{\alpha\beta} \overset{\text{reparameterisation}}{\longrightarrow} e^{\varphi(\tau,\sigma)}\,\eta_{\alpha\beta} \quad \text{(conformal gauge)}\,. \tag{6.14}$$

Weyl transformations allow us to go one step further, and change the pre-factor to be one at all points on the world-sheet,

$$h_{\alpha\beta} \overset{\substack{\text{reparameterisation}\\ \text{\& rescaling}}}{\longrightarrow} \eta_{\alpha\beta} \quad \text{(flat gauge)}\,. \tag{6.15}$$

In two dimensions, the metric is therefore completely non-dynamical.[2]

In the conformal gauge the equation of motion and constraint become rather simple,

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0\,, \qquad (\dot{X}^\mu)^2 + (X'^\mu)^2 = 0\,, \qquad \dot{X}^\mu X'_\mu = 0\,. \tag{6.16}$$

(we will discuss the boundary terms in a moment). The equation of motion is a simple wave equation, with a general solution given by arbitrary functions of $\tau+\sigma$ and $\tau-\sigma$,

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)\,. \tag{6.17}$$

The functions $X_L$ and $X_R$ have to be chosen as to satisfy the constraints (the 2nd and 3rd equation in (6.16)). We will see examples of various classical solutions in the next section. For future reference, let us introduce world-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, and write (6.16) as

$$\partial_+\partial_- X^\mu = 0\,, \qquad (\partial_+ X^\mu)^2 = 0\,, \qquad (\partial_- X^\mu)^2 = 0\,. \tag{6.18}$$

In this coordinate system the world-sheet metric takes an off-diagonal form,

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{6.19}$$

Even if we fix the metric to be (6.15) or (6.19), there are still combined reparameterisations and Weyl rescalings which leave the metric invariant. Any coordinate transformation which is such that it makes the metric scale by a factor only can be undone by a Weyl transformation. Because an infinitesimal coordinate transformation transforms the metric $\eta_{\alpha\beta}$ according to

$$\eta_{\alpha\beta} \to \eta_{\alpha\beta} + \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha\,, \tag{6.20}$$

the equation which determines the left-over coordinate freedom reads

$$\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = \Lambda\eta_{\alpha\beta}\,. \tag{6.21}$$

Writing this in world-sheet light-cone coordinates, it implies that we can have a coordinate transformation of the form

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+)\,, \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-)\,. \tag{6.22}$$

[2] Contrast this to the situation in general relativity, where the Einstein-Hilbert term $\int\sqrt{-g}R$ is what gives dynamics to the metric; this term does not appear in the string action.

In terms of the $\tau$ and $\sigma$ coordinates, this reads

$$\begin{aligned}\tilde{\tau} &= \frac{1}{2}\big(\tilde{\sigma}^+(\tau+\sigma) + \tilde{\sigma}^-(\tau-\sigma)\big)\,,\\ \tilde{\sigma} &= \frac{1}{2}\big(\tilde{\sigma}^+(\tau+\sigma) - \tilde{\sigma}^-(\tau-\sigma)\big)\,.\end{aligned} \tag{6.23}$$

The first line simply states that $\tilde{\tau}$ is a solution of the free wave equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)\tilde{\tau} = 0\,. \tag{6.24}$$

Since $\tilde{\tau}$ can be chosen to be anything as long as it satisfies the wave equation, and since this is also exactly the equation of motion for any of the string coordinates, we can set $\tau$ equal to e.g. $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^1)$. This choice

$$\tau \sim X^+ \tag{6.25}$$

is called the *light-cone gauge*. Another choice is given by

$$\tau \sim X^0\,. \tag{6.26}$$

This is the *static gauge*. The proportionality constant will, in both cases, be chosen later.

(end of lecture 14)

▸ ***Summary:*** The equations of motion of the string reduce, in the conformal gauge, to a wave equation for the embedding coordinates $X^\mu$. In addition there are two constraints. Using residual gauge invariance, we can still fix $\tau$ to be equal to any of the target space coordinates. Light-cone gauge ($\tau = X^+/p_+$) and static gauge ($\tau = X^0/\kappa$) are the two most common choices.

## 6.3. The classical theory, its solutions and charges

We will soon start quantising the fluctuations of the relativistic string around the point-like configuration (where all points of the string sit at the same location in space). However, there are many interesting aspects of the classical theory as well. So let us first take a look at some of the solutions to the equations of motion (6.11) of the bosonic string.

We will always gauge fix the world-sheet diffeomorphism freedom i.e. choose a coordinate system, so that $h_{\alpha\beta}$ is fixed. Our convention will be that the range of the space-like coordinate $\sigma$ is restricted to

$$\begin{aligned}\text{closed string}: &\quad 0 \le \sigma \le 2\pi\,,\\ \text{open string}: &\quad 0 \le \sigma \le \pi\,.\end{aligned} \tag{6.27}$$

(there are various historical reasons for choosing these ranges to be unequal for closed and open strings). In addition, we have to impose $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$ for the closed string to make the endpoints meet. You could choose any other range but it would not matter for the physical results.

One typical solution of the open string is the "folded string" solution, given by

$$\begin{aligned}X^0 &= L\,\tau\,,\\ X^1 &= L\,\cos\sigma\,\cos\tau\,,\\ X^2 &= L\,\cos\sigma\,\sin\tau\,,\end{aligned} \tag{6.28}$$

with all other coordinates vanishing. One can verify that this solution makes the boundary term in (6.11) vanish, and one can also verify that this solution satisfies not only the equation of motion but also the constraints.

A simple solution of the closed string equations of motion is the pulsating string,

$$\begin{aligned} X^0 &= R\,\tau\,, \\ X^1 &= R\,\cos\sigma\,\cos\tau\,, \\ X^2 &= R\,\sin\sigma\,\cos\tau\,. \end{aligned} \tag{6.29}$$

Again, this solution satisfies both the equations of motion and the constraints.

For open strings, we can make the boundary term in (6.11) vanish by setting $\partial_\sigma X^\mu = 0$ at the endpoints. Such a boundary condition is called a *Neumann* boundary condition.[3] However, we can also simply impose that $\delta X^\mu$ vanishes at the endpoint of the open string. This in particular requires that there is no change of $X^\mu$ under time evolution, $\partial_\tau X^\mu = 0$. This type of boundary condition is called a *Dirichlet* boundary condition. Both of these are important for open strings.

Instead of exploring the space of solutions on the basis of isolated examples, let us now turn to a more systematic expansion of the $X^\mu$ in modes. For closed strings, we will write $X^\mu = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$. For open strings, the coordinates which satisfy Neumann boundary conditions will be labelled $X^\mu_N$, while the ones satisfying Dirichlet boundary conditions will be denoted $X^\mu_D$. We then find the following generic expansions:

$$\text{closed string}: \quad \begin{cases} X^\mu_R = \dfrac{1}{2}x^\mu + \dfrac{1}{4\pi T}(\tau-\sigma)\,p^\mu + \dfrac{i}{\sqrt{4\pi T}}\displaystyle\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}\,, \\ X^\mu_L = \dfrac{1}{2}x^\mu + \dfrac{1}{4\pi T}(\tau+\sigma)\,p^\mu + \dfrac{i}{\sqrt{4\pi T}}\displaystyle\sum_{n\neq 0}\frac{1}{n}\tilde\alpha^\mu_n e^{-in(\tau+\sigma)}\,. \end{cases} \tag{6.30a}$$

$$\text{open string}: \quad \begin{cases} X^\mu_N = x^\mu + \dfrac{1}{\pi T}p^\mu\,\tau + \dfrac{i}{\sqrt{\pi T}}\displaystyle\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\tau}\cos(n\sigma)\,, \\ X^\mu_D = x^\mu + \dfrac{i}{\sqrt{\pi T}}\displaystyle\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\tau}\sin(n\sigma)\,. \end{cases} \tag{6.30b}$$

Here $x^\mu$, $p^\mu$ and all the $\alpha^\mu_n$ and $\tilde\alpha^\mu_n$ are constant vectors. The normalisation of these constants is of course arbitrary. The sums run over both positive and negative integers $n$. Because the $X^\mu$ have to be real, we have the following conditions on the oscillator coefficients,

$$(\alpha_n)^\dagger = \alpha_{-n}\,, \qquad (\tilde\alpha_n)^\dagger = \tilde\alpha_{-n}\,. \tag{6.31}$$

The variables $x^\mu$ and $p_\mu$ are the centre-of-mass position and momentum respectively: averaging $X^\mu$ and $P_\mu$ over the string yields

$$\begin{aligned} X^\mu(\tau=0) &= \frac{1}{2\pi}\int \mathrm{d}\sigma\, X^\mu(\tau=0,\sigma) = x^\mu\,, \\ P_\mu(\tau) &= T\int \mathrm{d}\sigma \dot X_\mu = p_\mu\,. \end{aligned} \tag{6.32}$$

[3] The solution for the folded closed string (6.28) can be interpreted as a solution for an open string if you let $\sigma \in [0\pi\rangle$. The solution then has Neumann boundary conditions, and is called the "rigid rod" solution for the open string.

where the integration regions should be chosen appropriately for the open and closed string.

It is also useful to express the constraints in terms of the oscillators. To this extent, one customarily defines generators $L_m$ and $\tilde{L}_m$, which are simply Fourier modes of the stress tensor components. To be precise,

$$\text{closed string}: \quad \begin{cases} L_m := 2T \int_0^{2\pi} \mathrm{d}\sigma\, e^{-im\sigma} T_{--} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n} \alpha^\nu_n \eta_{\mu\nu}\,, \\ \tilde{L}_m := 2T \int_0^{2\pi} \mathrm{d}\sigma\, e^{+im\sigma} T_{++} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}^\mu_{m-n} \tilde{\alpha}^\nu_n \eta_{\mu\nu}\,, \end{cases} \tag{6.33a}$$

$$\text{open string}: \quad L_m := 2T \int_0^{\pi} \mathrm{d}\sigma \left( e^{im\sigma} T_{++} + e^{-im\sigma} T_{--} \right) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n} \alpha^\nu_n \eta_{\mu\nu}\,. \tag{6.33b}$$

Here we have introduced the notation $\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = 1/\sqrt{4\pi T}\, p^\mu$ for the closed string and $\alpha^\mu_0 = 1/\sqrt{\pi T}\, p^\mu$ for the open string in order to make the notation more uniform. For the directions in which the open string has Dirichlet boundary conditions, the $\alpha^\mu_0$ are zero, as there is no $p^\mu$ for these directions in the expansion (6.30).

The constraints, in particular $L_0$, can be used to determine the mass-squared of a string. With mass-square we mean the square of the centre-of-mass momentum, $p_\mu p^\mu$. From (6.33) we see that the linear combination of constraints $L_0 + \tilde{L}_0 = 0$ relates the mass-square to the oscillators,

The mass of a string state is determined in terms of the oscillatory modes, by virtue of the reparameterisation constraint.

$$M^2 = -p_\mu p^\mu = \frac{2}{\alpha'} \left( \sum_{n\geq 1} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu} + \sum_{n\geq 1} \tilde{\alpha}^\mu_{-n} \tilde{\alpha}^\nu_n \eta_{\mu\nu} \right) =: \frac{2}{\alpha'} \left( N + \tilde{N} \right)\,, \tag{6.34}$$

(you can express this in various other ways by combining it with $L_0 - \tilde{L}_0 = 0$). This is new compared to the point particle: the mass is determined by the oscillations of the string. If the string does not oscillate, i.e. if $N = \bar{N} = 0$, then it behaves as a *massless* particle.

We can also find the angular momentum in terms of the oscillators,

$$\begin{aligned} J^{\mu\nu} &= T \int_0^{2\pi} \mathrm{d}\sigma \left( X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu \right) \\ &= x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left[ (\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n) + (\tilde{\alpha}^\mu_{-n} \tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n} \tilde{\alpha}^\mu_n) \right]\,. \end{aligned} \tag{6.35}$$

Let us now look at the Poisson brackets of the string. The starting point is as always the canonical bracket

$$\{ X^\mu(\sigma),\, P_\nu(\sigma') \}_{\mathrm{PB}} = \delta^\mu_\nu \delta(\sigma - \sigma')\,. \tag{6.36}$$

The momentum density $P_\mu(\sigma)$ is obtained as

$$P_\mu(\sigma) = \frac{\delta S}{\delta \dot{X}^\mu(\sigma)} = \frac{1}{2\pi\alpha'} \dot{X}^\mu(\sigma)\,. \tag{6.37}$$

If we insert the oscillator expansion of the field (6.30) into (6.37), and then write out (6.36), we find the following Poisson brackets between the various oscillators of

the closed string:

$$\begin{aligned}
\{\alpha_m^\mu,\, \alpha_n^\nu\}_{\mathrm{PB}} &= -i\, m\, \delta_{m+n} \eta^{\mu\nu}\,, \\
\{\tilde\alpha_m^\mu,\, \tilde\alpha_n^\nu\}_{\mathrm{PB}} &= -i\, m\, \delta_{m+n} \eta^{\mu\nu}\,, \\
\{\alpha_m^\mu,\, \tilde\alpha_n^\nu\}_{\mathrm{PB}} &= 0\,, \\
\{x^\mu,\, p_\nu\}_{\mathrm{PB}} &= \delta^\mu_\nu\,.
\end{aligned} \tag{6.38}$$

This shows once more that the left-moving sector and the right-moving sector are independent, and it also shows that the centre-of-mass position $x^\mu$ and momentum $p_\mu$ are canonically conjugate. These Poisson brackets will be the starting point of the quantisation of the string in the next section.

(end of lecture 15)

Using the Poisson brackets just derived, we can find the Poisson algebra generated by the modes of the constraint, $L_m$ and $\tilde L_m$. One finds

$$\begin{aligned}
\{L_m,\, L_n\}_{\mathrm{PB}} &= -i\,(m-n)\, L_{m+n}\,, \\
\{\tilde L_m,\, \tilde L_n\}_{\mathrm{PB}} &= -i\,(m-n)\, \tilde L_{m+n}\,, \\
\{L_m,\, \tilde L_n\}_{\mathrm{PB}} &= 0\,.
\end{aligned} \tag{6.39}$$

This algebra is called the classical *Virasoro algebra* (or more appropriately, the *Witt* algebra). We will see that the quantum constraints satisfy a similar algebra, though with important quantum corrections.

In the light-cone gauge $X^+ \sim \tau = \frac{1}{2}(\sigma^+ + \sigma^-)$, it is possible to solve for one of the oscillators in terms of all the others. The proportionality constant between $X^+$ and $\tau$ can be related to the centre-of-mass momentum $p_\mu$ using the second line of (6.32), giving

$$\begin{aligned}
\text{closed string}: &\quad X^+ = \alpha'\, p^+\, \tau\,, \\
\text{open string}: &\quad X^+ = 2\alpha'\, p^+\, \tau\,.
\end{aligned} \tag{6.40}$$

In particular, in the light-cone gauge we have set $\alpha_n^+ = \tilde\alpha_n^+ = 0$ for $n > 0$ and also $x^+ = 0$. Using these relations, we find that in the light-cone gauge the constraints (6.18) become

$$\begin{aligned}
\text{closed string}: &\quad \partial_\pm X^- = \frac{1}{\alpha' p^+} (\partial_\pm X^i)^2\,, \\
\text{open string}: &\quad \partial_\pm X^- = \frac{1}{2\alpha' p^+} (\partial_\pm X^i)^2\,.
\end{aligned} \tag{6.41}$$

This allows us to solve for the $\alpha_n^-$ and $\tilde\alpha_n^-$ oscillators in terms of the other ones,

$$\begin{aligned}
\text{closed string}: &\quad \begin{cases} \alpha_n^- = \dfrac{1}{\sqrt{2\alpha'}\, p^+} \left( \displaystyle\sum_{m=-\infty}^{\infty} \alpha^i_{n-m} \alpha^i_m \right) \\ \tilde\alpha_n^- = \dfrac{1}{\sqrt{2\alpha'}\, p^+} \left( \displaystyle\sum_{m=-\infty}^{\infty} \tilde\alpha^i_{n-m} \tilde\alpha^i_m \right) \end{cases} \\
\text{open string}: &\quad \alpha_n^- = \frac{1}{2\sqrt{2\alpha'}\, p^+} \left( \sum_{m=-\infty}^{\infty} \alpha^i_{n-m} \alpha^i_m \right)\,.
\end{aligned} \tag{6.42}$$

In general gauges we can of course also solve the constraint, but eliminating one oscillator in terms of the others will then involve square roots and make life much more complicated.

▸ ***Summary:*** The world-sheet theory of the string is a theory of two-dimensional free scalars, which can solved completely by expanding the $X^\mu$ in modes. The Fourier modes of the stress tensor generate the *Virasoro algebra*. In the light-cone gauge, we can solve for the $\alpha_n^-$ modes in terms of a quadratic expression in the $\alpha_n^i$.

## 6.4. Quantum strings

In order to quantise the fluctuations of the string around its point-like configuration, we should go to a world-sheet gauge for which all residual gauge freedom has been exhausted. But in order to actually write down the spectrum, it is best to do this in a way in which the constraints can be solved in a simple way. We have seen that this is the case in the light-cone gauge (see eq. (6.42)). The disadvantage of the light-cone gauge is that target-space Poincaré symmetry is no longer manifest.

However, let us first discuss some general issues concerning quantisation. As usual, we quantise by promoting all dynamical variables to operators and by promoting Poisson brackets to commutators (as we discussed in section 3.2). From (6.38) we thus obtain

$$\begin{aligned}
[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] &= m\,\delta_{m+n}\eta^{\mu\nu}\,, \\
[\hat{\tilde{\alpha}}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] &= m\,\delta_{m+n}\eta^{\mu\nu}\,, \\
[\hat{\alpha}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] &= 0\,, \\
[\hat{x}^\mu, \hat{p}_\nu] &= i\delta_\nu^\mu\,.
\end{aligned} \tag{6.43}$$

From these expressions one can see that $\alpha_m$ and $\alpha_m^\dagger$ satisfy, when rescaled by a factor of $1/\sqrt{m}$ the commutator relation for harmonic oscillator "creation" and "annihilation" operators. We will soon use them to create the spectrum by laddering from the ground state.

First, however, let us make a few statements about objects which are composite operators of the oscillators, such as the Virasoro generators $L_m$. These need more care, as the products of operators needs to be regulated. As usual, we will use normal ordering. However, we will allow for a normal-ordering constant. The starting point is the expression for the Virasoro generators,

$$\hat{L}_m = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} :\hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu:\,\eta_{\mu\nu}\right) - a\,\delta_{m\,0}\,, \tag{6.44}$$

and a similar one for $\hat{\tilde{L}}_m$. The operators $\hat{L}_m$ and $\hat{\tilde{L}}_m$ are called *Virasoro generators* and the quantum algebra which they generate is the *Virasoro algebra*. Already at this state we see an important consequence of the normal ordering constant: the constraint $(\hat{L}_0 + \hat{\tilde{L}}_0)|\text{phys}\rangle = 0$ now implies that a physical state will satisfy

The physical mass $M$ of a string quantum state sees the effect of the ordering ambiguity $a$ in the Virasoro generators.

$$M^2 = -p_\mu p^\mu = \frac{2}{\alpha'}\left(\hat{N} + \hat{\tilde{N}} - 2\,a\right)\,, \tag{6.45}$$

where we have used the number operator $\hat{N}$ which we have already seen in the classical theory in (6.34),

$$\hat{N} := \sum_{n\geq 1} \hat{\alpha}_{-n}^\mu\,\hat{\alpha}_n^\nu\,\eta_{\mu\nu}\,, \tag{6.46}$$

(ditto for the $\hat{\tilde{N}}$ operator). If we are forced to a positive normal-ordering constant $a$, this may thus introduce negative mass-squared states into the spectrum. And indeed, we will see soon that the quantum theory is only consistent (what this means

will become clear shortly) for the choice $a = 1$. Thus, the quantised bosonic string will exhibit a somewhat non-physical spectrum.

One immediate further question that one can raise now is whether the composite operators $\hat{L}_m$ still satisfy the commutation relations which we would expect from the Poisson brackets of the classical theory. For the Virasoro generators we find, after a somewhat tedious computation that

$$[\hat{L}_m, \hat{L}_n] = (m-n)L_{m+n} + \frac{d}{12} m(m^2-1)\delta_{m+n}\,. \tag{6.47}$$

The term proportional to $\delta_{m+n}$ was not present in the classical Poisson algebra (6.39). It is called an *anomalous* term. We will return to the quantum Virasoro algebra in one of the exercises.

## 6.5. Physical states and Lorentz covariance

After these general remarks, which hold true regardless of the way in which we fix the residual gauge freedom (6.22), let us now focus on the light-cone gauge, in which $\tau$ is fixed as in (6.40). We have already seen that at the classical level, the $\alpha_n^-$ oscillators can be expressed in terms of the transverse oscillators, and this is no different in the quantum theory, except for the appearance of the normal ordering constant:

$$\text{closed string}: \begin{cases} \hat{\alpha}_n^- = \dfrac{1}{\sqrt{2\alpha' p^+}} \left( \displaystyle\sum_{m=-\infty}^{\infty} :\hat{\alpha}_{n-m}^i \hat{\alpha}_m^i : -2\,a\,\delta_{n0} \right) \\ \hat{\tilde{\alpha}}_n^- = \dfrac{1}{\sqrt{2\alpha' p^+}} \left( \displaystyle\sum_{m=-\infty}^{\infty} :\hat{\tilde{\alpha}}_{n-m}^i \hat{\tilde{\alpha}}_m^i : -2\,a\,\delta_{n0} \right) \end{cases} \tag{6.48}$$

$$\text{open string}: \quad \hat{\alpha}_n^- = \frac{1}{2\sqrt{2\alpha' p^+}} \left( \sum_{m=-\infty}^{\infty} :\hat{\alpha}_{n-m}^i \hat{\alpha}_m^i : -2\,a\,\delta_{n0} \right) .$$

We do not have to worry about these expressions too much, since the mass operator follows immediately from the covariant expression (6.45) by using the fact that $\alpha_n^+ = 0$ for $n \neq 0$,

$$\begin{aligned} \text{closed string}: M^2 &= \frac{2}{\alpha'} \left( \hat{N}_{\text{lc}} + \hat{\tilde{N}}_{\text{lc}} - 2\,a \right) , \\ \text{open string}: M^2 &= \frac{1}{\alpha'} \left( \hat{N}_{\text{lc}} - a \right) . \end{aligned} \tag{6.49}$$

with the number operators $\hat{N}_{\text{lc}}$ and $\hat{\tilde{N}}_{\text{lc}}$ now only counting transverse oscillations. For open strings, this relation between mass and oscillator number takes care of $\hat{L}_0|\text{phys}\rangle = 0$ while for closed strings it takes care of $\hat{L}_0 + \hat{\tilde{L}}_0|\text{phys}\rangle = 0$. For closed strings, we still have to take care of the other linear combination, $\hat{L}_0 - \hat{\tilde{L}}_0|\text{phys}\rangle = 0$. This constraint means that the excitation level has to be the same in the left-moving sector as in the right-moving sector; it is called the *level-matching constraint*.

The level-matching constraint imposes that the excitation level has to be the same in the left- and right-moving sectors of the closed string.

Let us now look more closely at the space of physical states. Since we have solved for the constraints, any state constructed by acting with the creation operators $\hat{\alpha}_{-n}^i$ (for $n \geq 1$) on the vacuum will be physical. Let us first discuss the open string spectrum. Because we quantise around the no-oscillation state of the string, the ground state is simply labelled by the transverse momentum, i.e. we denote it $|p^i\rangle$, and its mass is given by

$$m^2|p^i\rangle = -\frac{a}{\alpha'}|p^i\rangle\,. \tag{6.50}$$

The first excited state is obtained by acting with $\hat{\alpha}^j_{-1}$ on $|p^i\rangle$. Its mass is

$$m^2\hat{\alpha}^j_{-1}|p^i\rangle = (1-a)\alpha^j_{-1}|p^i\rangle\,. \tag{6.51}$$

Let us try to understand what this state means in terms of a particle. It carries one transverse index, so it transforms as a vector under the transverse rotation group $\mathrm{SO}(d-2)$. This is how a massless vector particle would transform if the theory is Lorentz invariant. Therefore, Lorentz invariance requires $a = 1$. While this gives us a nice massless vector particle (i.e. a gauge field), it also implies that the state $|p^i\rangle$ actually has negative mass-squared: it is a tachyon. All other states which we can make, by using $\hat{\alpha}^i_{-n}$ oscillators with $n > 1$ or by acting multiple times, have a mass-square which is larger than zero. There is an infinite tower of them.

The story for the closed string is very similar. We now have two types of oscillators, corresponding to the left-moving and the right-moving modes. The lowest-mass state now has

$$m^2|p^i\rangle = -\frac{4a}{\alpha'}|p^i\rangle\,. \tag{6.52}$$

For the first excited state, we have to make sure that the level-matching constraint is satisfied. This implies that the first excited state is $\hat{\alpha}^j_{-1}\hat{\tilde{\alpha}}^k_{-1}|p^i\rangle$, with

$$m^2\hat{\alpha}^j_{-1}\hat{\tilde{\alpha}}^k_{-1}|p^i\rangle = 4\frac{(1-a)}{\alpha'}\hat{\alpha}^j_{-1}\hat{\tilde{\alpha}}^k_{-1}|p^i\rangle\,. \tag{6.53}$$

It is useful to decompose this state with respect to irreducible representations of $\mathrm{SO}(d-2)$,

$$\hat{\alpha}^j_{-1}\hat{\tilde{\alpha}}^k_{-1} = \underbrace{\hat{\alpha}^{[j}_{-1}\hat{\tilde{\alpha}}^{k]}_{-1}}_{\text{anti-symmetric}} + \underbrace{\left(\hat{\alpha}^{(j}_{-1}\hat{\tilde{\alpha}}^{k)}_{-1} - \frac{1}{d-2}\delta^{jk}\hat{\alpha}^l_{-1}\hat{\tilde{\alpha}}^l_{-1}\right)}_{\text{traceless symmetric}} + \underbrace{\frac{1}{d-2}\delta^{jk}\hat{\alpha}^l_{-1}\hat{\tilde{\alpha}}^l_{-1}}_{\text{trace}}\,. \tag{6.54}$$

The anti-symmetric, symmetric traceless and trace pieces transform as irreducible representations of $\mathrm{SO}(d-2)$. If we set $a = 1$ these pieces correspond to a massless anti-symmetric tensor gauge field, a graviton and a scalar field respectively. (end of lecture 16)

Having constructed the spectrum, and having interpreted the lowest-mass states in terms of fields which transform covariantly under the full Lorentz symmetry $\mathrm{SO}(d-1,1)$, we have found that we need to set the normal ordering constant $a = 1$. We would really have to show that the theory is covariant under the full Lorentz group. This requires a lot more work. However, there is a short argument that gives us the normal ordering constant $a$ by direct computation. It arises from

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{\infty}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{\infty}:\alpha^i_{-n}\alpha^i_n: + \frac{D-2}{2}\sum_{n=1}^{\infty}n\,. \tag{6.55}$$

This second sum is divergent and needs to be regularised. We do this using so-called zeta-function regularisation, which uses the definition of the zeta function

$$\zeta(s) := \sum_{n=1}^{\infty}n^{-s}\,. \tag{6.56}$$

This sum converges for $\mathrm{Re}(s) > 1$, and has a unique analytic continuation to $s = -1$, where it takes the value $\zeta(-1) = \frac{-1}{12}$. We thus find that

$$a = \frac{D-2}{24}\,. \tag{6.57}$$

This gives us the first sign that Lorentz covariance requires the bosonic string (both open and closed) to live in 26 dimensions.

▸ ***Summary:*** The spectrum of the open bosonic string contains a $m^2 < 0$ scalar tachyon, a massless vector, and a tower of massive particles. The spectrum of the closed bosonic string has a scalar tachyon, and at the massless level a spin-two particle, a spin-one particle in the form of an anti-symmetric tensor and a scalar. Only in $d = 26$ and with normal ordering constant $a = 1$ do we find Poincaré invariance at the quantum level.

▸ ***See also:*** D. Lüst and S. Theisen, "Lectures on string theory", Springer, 1989, chapter 3, in particular also the appendix.

## 6.6. No-ghost theorem

A cleaner way to derive the condition on the dimension of space-time can be obtained by going back to the covariant formulation. In contrast to the formulation in the light-cone gauge, where we solve for the constraints before quantising, we now have to impose the Virasoro constraints as quantum operator equations, i.e.

$$\hat{L}_m|\text{phys}\rangle = 0\,. \tag{6.58}$$

Interestingly, it turns out that for generic values of the dimension $d$, it is possible to find physical states which have negative norm. By requiring that these go away, we can then find a condition on the dimension.[4].

To see how this works, consider the following open string state,

$$|\chi\rangle = \left[\hat{\alpha}_{-1}\cdot\hat{\alpha}_{-1} + A\,\hat{\alpha}_0\cdot\hat{\alpha}_{-2} + B\,(\hat{\alpha}_0\cdot\hat{\alpha}_{-1})^2\right]|k\rangle\,,$$

where $|k\rangle$ is a physical eigenstate of the momentum operator,

$$\hat{p}^\mu|k\rangle = k^\mu|k\rangle\,.$$

With $A$ and $B$ still arbitrary, we can already obtain the mass-square eigenvalue of this state from the $\hat{L}_0$ constraint, and find that $m^2 = (2-a)/\alpha' = 1/\alpha'$.

The other constraints can only be satisfied if we tune $A$ and $B$ appropriately. By imposing

$$\hat{L}_1|\chi\rangle = 0 \quad \text{and} \quad \hat{L}_2|\chi\rangle = 0\,,$$

we can find expressions for the coefficients $A$ and $B$ in terms of the dimension of spacetime $d$. Using the intermediate results

$$\begin{aligned} L_1\alpha^\mu_{-1}\alpha^\mu_{-1}|k\rangle &= 2\,\alpha^\mu_{-1}\alpha^\mu_0|k\rangle\,. \\ L_1\alpha^\mu_0\alpha^\mu_{-2}|k\rangle &= 2\,\alpha^\mu_0\alpha^\mu_{-1}|k\rangle \\ L_1\alpha^\mu_0\alpha^\nu_0\alpha^\mu_{-1}\alpha^\nu_{-1}|k\rangle &= 2\,\alpha^\mu_0\alpha^\mu_0\alpha^\nu_0\alpha^\nu_{-1}|k\rangle\,. \end{aligned} \tag{6.59}$$

one finds that the constraints hold provided

$$A = \frac{d-1}{5}\,, \quad B = \frac{d+4}{10}\,. \tag{6.60}$$

The constraints therefore fix the form of the state if we know the value of $d$.

[4]More formally, one can show that for these values of $d$ physical states are the sum of a positive norm state (a so-called *DDF state*) and a null physical state.

If we now compute the norm of $|\chi\rangle$ we find

$$\langle\chi|\chi\rangle = 2d + 2A^2p^2 + 2B^2p^4 + 4Bp^2\,. \tag{6.61}$$

With $p^2 = -2$ and the values for $A$ and $B$ above, this becomes

$$\langle\chi|\chi\rangle = \frac{-2}{25}(26 - 27d + d^2)\,. \tag{6.62}$$

For $d = 26$ this vanishes, while it is negative for $d > 26$. This shows that, at least for the state $|\chi\rangle$ analysed here, the dimension of space-time has to be at most $d = 26$. A more careful analysis along these lines fixes this number completely.

## 6.7. T-duality

We have seen that the action of the string is based on a two-dimensional field theory for fields $X^\mu(\tau,\sigma)$. These map a point $(\tau,\sigma)$ on the string world-sheet to a point in space-time. Interestingly, such field theories have unexpected properties whenever the $X^\mu$ take values on a compact manifold (instead of on $\mathbb{R}^n$, as we have assumed so far). Physically, the simplest case of such a situation is when a closed string moves on a manifold of which one direction is a circle.

Let us focus on the action for the coordinate $X^1$ which we will take to be compact. Its contribution to the action is, as usual,

$$S[X^1] = T\int \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma\, \left[\frac{1}{2}(\partial_\tau X^1)^2 - \frac{1}{2}(\partial_\sigma X^1)^2\right]\,. \tag{6.63}$$

The expansion of $X^1(\tau,\sigma)$ in modes will be similar to that in (6.30a). We will ignore the oscillation modes in the circle direction. However, we need to take into account that it is possible for the string to 'wind' an arbitrary number of times around the circle. Winding is expressed as a term in the expansion of $X^1$ which is linear in $\sigma$. The non-oscillatory part of the mode expansion thus reads

$$X^1(\tau,\sigma) = \frac{1}{2\pi T}\Big(p^1\,\tau + w^1\,\sigma\Big)\,. \tag{6.64}$$

In terms of left and right movers, $X^1 = X^1_R + X^1_L$, this reads

$$\begin{aligned} X^1_R(\tau,\sigma) &= \frac{1}{4\pi T}(\tau-\sigma)(p^1 + w^1)\,,\\ X^1_L(\tau,\sigma) &= \frac{1}{4\pi T}(\tau+\sigma)(p^1 - w^1)\,. \end{aligned} \tag{6.65}$$

Now of course we cannot just take arbitrary values for $p^1$ and $w^1$. For $p^1$, which is the centre-of-mass momentum of the string along the circle, we can use our knowledge of the spectrum of a particle in a box. In the quantum theory the momentum eigenvalues have to be quantised in order for the wave function to be single-valued. Specifically, with $\hat{p} = (1/i)\partial_x$ and a wave function $\psi \propto \exp(inx/R)$, the spectrum of the momentum operator is

$$p^1 = \frac{n}{R}\,,\quad n \in \mathbb{Z}\,. \tag{6.66}$$

A similar condition holds true for the spectrum of $w^1$. The fact that the string can wind around the circle is expressed as the equivalence condition

$$X^1(\tau,\sigma+2\pi) = X^1(\tau,\sigma) + 2\pi m R\,, \tag{6.67}$$

where $m$ indicates the number of windings. Comparing this with (6.64) shows that

$$w^1 = 2\pi T\, mR\,, \quad m \in \mathbb{Z}\,. \tag{6.68}$$

Note that the winding energy quantum grows with the radius, while the momentum quantum decreases with the radius.

Let us now consider the mass-squared operator $M^2$. For an observer in the non-compact part of the space-time, the effective mass-squared is given by $M^2 = -p_0 p^0 - \sum_{i=2}^{25} p_i p^i$, that is, without the momentum along the circle. Remember that in section 6.3 we used the constraint $L_0 + \tilde{L}_0 = 0$ to express $M^2$ in terms of the oscillators (see equation (6.34)). In order to generalise that discussion to the present situation, we just need to know the contributions to $L_0$ and $\tilde{L}_0$ which come from $X^1$. Using (6.11) for the stress tensor, and inserting this in (6.33a), we find that the constraint $L_0 + \tilde{L}_0 = 0$ now reads

$$L_0 + \bar{L}_0 = \frac{1}{4\pi T}\left(p_0 p^0 + \sum_{i=2}^{25} p_1 p^1\right) + \frac{1}{8\pi T}\left(p^1 - w^1\right)^2 + \frac{1}{8\pi T}\left(p^1 + w^1\right)^2 + N + \tilde{N}\,. \tag{6.69}$$

From here we thus conclude that the mass-squared for the 25-dimensional observer is given by

$$M^2 = (p^1)^2 + (w^1)^2 + \frac{2}{\alpha'}(N + \tilde{N}) = \left(\frac{n}{R}\right)^2 + (2\pi T\, mR)^2 + \frac{2}{\alpha'}(N + \tilde{N})\,. \tag{6.70}$$

This expression has an important property: it is invariant under the transformation

$$R \to \frac{1}{2\pi T\, R}\,, \quad n \leftrightarrow m\,. \tag{6.71}$$

This transformation, also called *T-duality*, inverts the radius of the circle. Momentum modes along the circle get mapped to winding modes around the circle. Physically, this invariance means that if we look at the energy spectrum of string states, there is no way to say whether the target space is a circle of radius $R$ or a circle of radius $1/2\pi TR$.

# 7
# Loops and scales

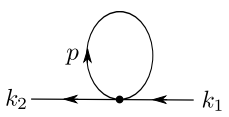
Before we can discuss Feynman diagrams with loops – the main topic of the present chapter – we need to take a step back and think once more about the meaning of the scattering amplitudes which we have been computing in chapter 4. Consider for instance the four-particle amplitude (4.53). By itself, it is not very helpful for an experimentalist. The reason is that it contains a free parameter $g$, and the theory does not predict its value. So if we want to use this expression to say something about the four-particle amplitude for arbitrary momenta, we will first have to determine $g$, for instance by doing a single experiment at some fixed values of the momenta. The same is actually true for $m$. The mass is physically extracted from the behaviour of the two-point correlator (more on that below). This physical mass is not a priori the same thing as the parameter $m$.

When we consider diagrams with loops, this distinction between *physically measurable quantities* on the one hand and *parameters in the Lagrangian* on the other hand becomes even more important and subtle. The process of 'tuning' of the parameters such that the physically measurable quantities take the right value is called *renormalisation*.

## 7.1. Regularisation

We have seen how to compute tree-level diagrams in chapter 4. In these diagrams, all the momenta on the internal lines are fixed by the momentum-conserving Dirac delta functions associated to the vertices. For diagrams with loops, there are not enough such constraints, and we are left with additional momentum integrals. The integration momenta are associated to the particles which "go around in the loops". Unfortunately, these integrals are generically divergent, so before we can compute them, we need to make sense of these divergences.

Consider as an example the loop diagram which occurs in the computation of the $1 \to 1$ process in a scalar theory (we have encountered this diagram in position space in (4.37)). In momentum space, this graph gives a correction to the propagator, given by




$$= \frac{1}{2}\delta^4(k_1+k_2)\left(-\frac{i}{\hbar}\lambda\right)\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{-i\hbar}{p^2+m^2-i\epsilon}\,. \tag{7.1}$$

Since the numerator of the integrand only depends on the norm $p^2$, we can use

spherical coordinates, integrate out the angular piece, and write the integral as

$$\mathrm{vol}(S^3)\int_0^\infty \frac{\mathrm{d}p}{(2\pi)^4}p^3\frac{-i\hbar}{p^2+m^2-i\epsilon}\,. \tag{7.2}$$

For large values of the norm $p$, this integral behaves as $\int \mathrm{d}p\, p$, which is quadratically divergent. But "infinity" is obviously not the right answer.

However, nobody expects theories of particle physics to be valid up to arbitrary energy scales. What is much more reasonable is that at some very high energy or mass scale (some very large value of $p^2$), new interactions and new particles start playing a role. The only requirement on a physically relevant theory is that, *as long as* we stay away from this high energy scale (i.e. as long as the momenta of the scattering particles are much smaller than this scale), the physics is not influenced by these new particles and interactions.

So what we should really do is to cut off the loop integral at some large but finite value of $p$ (let us call it $\Lambda$), and then impose that the results which we get do not depend on $\Lambda$.[1] The process of cutting off the integral is called *regularisation* (it obviously makes the integral finite, regular). The process of imposing that the physical results are independent of $\Lambda$ is called *renormalisation* (for reasons which will become clear soon).

With momentum cut-off regularisation, the integral becomes

$$\mathrm{vol}(S^3)\int_0^\Lambda \frac{\mathrm{d}p}{(2\pi)^4}p^3\frac{-i\hbar}{p^2+m^2-i\epsilon} = \frac{-i\hbar}{2}\,\mathrm{vol}(S^3)\left(\Lambda^2+m^2\log\left[\frac{m^2}{\Lambda^2+m^2}\right]\right). \tag{7.3}$$

This expression is finite for any finite value of $\Lambda$, and it clearly shows the quadratic divergence if we would take $\Lambda\to\infty$. Apart from the quadratic term there is also a piece containing a logarithm of the cut-off scale.

Regularisation can of course be done in many other ways. The procedure outlined above – cutting off the momentum integrals at the upper end – is one of them. Another often used procedure is to change the dimension in which we evaluate the integral, from $d = 4$ to $d = 4+\epsilon$ for some small value of $\epsilon$. The latter is called dimensional regularisation, and we will encounter it in one of the exercises (it is not as useful to build an intuitive understanding of regularisation and renormalisation, hence we will focus on cut-off regularisation here). A third and often-used method is lattice regularisation, in which space-time is descretised to a lattice of points. This method is useful for doing computer simulations, but less practical for analytic approaches.

As it stands, the expression for the two-point function is now $\Lambda$ dependent. But that is only superficially so, since we have not yet determined how the *parameter* $\lambda$ should be chosen. This is where *renormalisation* enters.

▸ ***See also:*** For a lucid exposition of the physics behind "regularisation" and "renormalisation", see chapter III of A. Zee, "Quantum field theory in a nutshell", 2003.

## 7.2. Renormalisation

Now that we know how to make our integrals finite, let us look at the renormalisation procedure. Instead of discussing the formal theory, let us get straight down to

[1] There is a subtletly related to the fact that for light-like momenta $p_\mu$, we always have a vanishing norm $p_\mu p^\mu = 0$, even when the energy of the massless particle can be arbitrarily large. We will avoid this problem by evaluating all integrals using a rotation of the $p_0$ integration into the complex plane, so that we are effectively dealing with a Euclidean theory.

business and focus on a simple example: the renormalisation of the real scalar field theory with $\phi^4$ interaction to one-loop order.

Let us first compute the two- and four-point amplitude which we will use to determine our physical mass and coupling constant. For the two-point function to one-loop order we have

$$\Gamma^{(2)}(k_1,k_2) = \quad + \qquad \begin{aligned} &= k_1^2 + m^2 + \frac{1}{2}\left(-\frac{i}{\hbar}\lambda\right)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-i\hbar}{p^2+m^2-i\epsilon}\,. \end{aligned} \tag{7.4}$$

(If you compare this with the Feynman rules in table 4.1 you will notice that we have not written the normalisation factors $1/\sqrt{2\omega}$ nor the overall momentum conserving delta function; these are the same for both graphs and we will ignore them here for simplicity as only the relative factors will be important). We have evaluated this integral in the previous section, so let us go straight on to the next set of graphs.

For the four-particle scattering amplitude $\Gamma^{(4)}$ we have, again to one loop order,

$$\Gamma^{(4)}(k_i) = \quad + \quad + \quad + \qquad \begin{aligned} &= \left(-\frac{i}{\hbar}\lambda\right) + V(k_1+k_3) + V(k_1+k_2) + V(k_1+k_4)\,, \end{aligned} \tag{7.5}$$

where the integral $V(q)$ is given by

$$V(q) = \frac{1}{2}\left(-\frac{i}{\hbar}\lambda\right)^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-i\hbar}{p^2+m^2-i\epsilon}\frac{-i\hbar}{(q+p)^2+m^2-i\epsilon}\,. \tag{7.6}$$

Evaluating this integral is a bit messy. The high-energy behaviour, however, does not depend on the mass $m$ as long as it is small enough. So let us use the approximation $m \ll \Lambda$ and also $m \ll k_i$ so that we can effectively set $m \approx 0$. The next step is then to rewrite the four-dimensional momentum integral over $p$ in a form in which we can reduce it to a scalar integral. This is done using the 'Feynman parameter trick', which is a method to combine the product of denominators into one. The general expression is given in (7.20). For our integral it leads to the conversion

$$\begin{aligned} &\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2-i\epsilon}\frac{1}{(q+p)^2-i\epsilon} \\ &\qquad = \int_0^1 \mathrm{d}x_1\mathrm{d}x_2 \int \frac{\mathrm{d}^4p}{(2\pi)^4}\delta(1-x_1-x_2)\frac{1}{\left[x_1(p^2-i\epsilon)+x_2((p+q)^2-i\epsilon)\right]^2} \\ &\qquad = \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{\left[p^2+2x\,p\cdot q + x\,q^2 - i\epsilon\right]^2} \\ &\qquad = \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{\left[l^2+x(1-x)q^2-i\epsilon\right]^2}\,. \end{aligned} \tag{7.7}$$

The step from the second to the third line can be made because the Dirac delta sets $x_1 = 1 - x_2$. In the last step we have introduced a new integration variable $l = p + xq$. This results in the loop momentum only occurring quadratically, so that we can reduce the integral to a trivial spherical volume integral times an integral over the norm of $l$, which can be done (see section 7.5). Only the $x$ integral is then left, which is also doable.

The upshot of the analysis is that

$$\begin{aligned}\Gamma^{(2)}(k_i) &= k_1^2 + m^2 - i\hbar\pi^2\lambda\Lambda^2 + \mathcal{O}(\hbar^2)\,,\\ \Gamma^{(4)}(k_i) &= \left(\frac{-i}{\hbar}\lambda\right) + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)\right] + \mathcal{O}(\hbar)\,,\end{aligned} \tag{7.8}$$

where we have re-introduced the Mandelstam variables. The coefficient $C$ is a positive number (the value of which is not important here). The suppressed terms are not only higher order in $\hbar$, but also higher-order in $\lambda$.

Now what do these expressions in (7.8) mean? On the left-hand sides, we have a physically measurable quantity. On the right-hand side, we have functions of the a priori undetermined numbers $m$ and $\lambda$, as well as the cut-off scale $\Lambda$. If an experimentalist would ask you to compute the probability for a certain complicated scattering process to take place, the only thing which makes sense is to express that answer in terms of the two quantities which sit on the *left-hand side* of the expressions. Motivated by the lowest order terms, we are led to define a physical mass and physical coupling by

$$m^2_{\text{phys}} = \Gamma^{(2)}(k=0)\,, \qquad \lambda_{\text{phys}} = i\hbar\Gamma^{(4)}(k_i = k_i^{(0)})\,. \tag{7.9}$$

These reduce to $m^2_{\text{phys}} = m^2$ and $\lambda_{\text{phys}} = \lambda$ at lowest order in $\hbar$, but then receive corrections. In the case of the coupling constant, these corrections depend on the momenta, so we need to evaluate the four-point amplitude at a particular value $k_i^{(0)}$ for these momenta to define what we mean with "physical coupling". Any value is as good as any other; one usually chooses $s = t = u = \mu^2$.

Of course, we want $m_{\text{phys}}$ and $\lambda_{\text{phys}}$ to be independent of the cut-off $\Lambda$: this is our requirement that extremely high-energy effects in loops should not influence low-energy scattering amplitudes. Doing this requires careful tuning of $m$ and $\lambda$ as $\Lambda$ is changed. Instead of doing that, let us simply try to express all amplitudes directly in terms of $m_{\text{phys}}$ and $\lambda_{\text{phys}}$. For the physical coupling we have

$$\lambda_{\text{phys}}(\mu) = \lambda - 3\hbar C\lambda^2\log\left(\frac{\Lambda^2}{\mu^2}\right) + \mathcal{O}(\lambda^3)\,. \tag{7.10}$$

This coupling measures the strength of the interaction at one particular momentum scale $\mu$. To this order in $\lambda$ we can invert this to get

$$\lambda = \lambda_{\text{phys}}(\mu) + 3\hbar C\lambda_{\text{phys}}(\mu)^2\log\left(\frac{\Lambda^2}{\mu^2}\right) + \mathcal{O}\big(\lambda_{\text{phys}}(\mu)^3\big)\,. \tag{7.11}$$

If we now plug this back into the amplitude $\Gamma^{(4)}$ (for arbitrary momenta, not just $s = t = u = \mu$), we get the result

$$\Gamma^{(4)} = -\frac{i}{\hbar}\lambda_{\text{phys}}(\mu) + iC\lambda^2_{\text{phys}}(\mu)\left[\log\left(\frac{\mu^2}{s}\right) + \log\left(\frac{\mu^2}{t}\right) + \log\left(\frac{\mu^2}{u}\right)\right] + \mathcal{O}\big(\lambda_{\text{phys}}(\mu)^3\big)\,. \tag{7.12}$$

This result is quite remarkable: all dependence on the cut-off has disappeared! Once we have measured the strength of the coupling at one particular momentum scale $\mu$, the expression above will tell us the strength at any other value of the momenta.

Note that since $\lambda_{\text{phys}}$ is finite, we see that we are forced to make $\lambda$ go to zero in the limit that we take the cut-off $\Lambda$ to infinity. In more standard language, the *bare* coupling $\lambda$ vanishes in the limit in which the cut-off is removed. This is good to keep in the back of your mind, however, once you express amplitudes in terms of $\lambda_{\text{phys}}$ you do not have to worry about it anymore. This process of tuning the bare parameters such as to obtain the correct physical amplitudes is called *renormalisation* (somewhat of a misnomer since we never normalised the bare parameters before).

The feature that the cut-off does not show up in any amplitude when it is expressed in terms of physical coupling constants is a property of so-called *renormalisable* theories. Theories which are *non-renormalisable* do not have this property, and physical quantities will come out to depend on $\Lambda$. This simply means that knowledge of physics at or beyond the scale $\Lambda$ will be necessary in order to understand the low-energy physics which we are interested in.

▸ ***Summary:*** The physical mass $m_{\text{phys}}$ and coupling constant $g_{\text{phys}}$ are defined by measurable quantities, in our case $\Gamma^{(2)}(k^2 = 0) = m^2_{\text{phys}}$ and $\Gamma^{(4)}(k_i = 0) = g_{\text{phys}}$. Imposing that these are independent of the cut-off $\Lambda$ requires that the bare parameters $m^2$ and $g$ depend on $\Lambda$, in a divergent way.

▸ ***See also:*** The renormalisation of $\phi^4$ theory, as well as many more computational details, are given in e.g. M. Srednicki, "Quantum field theory", Cambridge, 2007 and M. Peskin and D. Schroeder, "An introduction to quantum field theory", Perseus, 1995.

## 7.3. Renormalisation group flow

Let us now consider once more the physical coupling $\lambda_{\text{phys}}(\mu)$. It gives us the strength of the interaction when the particles which are involved have a momentum $s = t = u = \mu$.[2] An interesting thing to work out is the dependence of this coupling on the energy scale $\mu$. We can use (7.10) to work out the way in which the value of the couplings at two different energy scales are related. We find

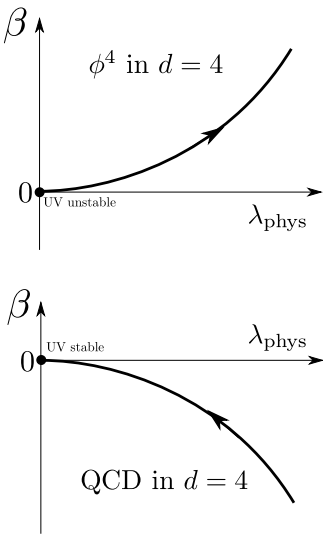
$$\lambda_{\text{phys}}(\mu') = \lambda_{\text{phys}}(\mu) + 3C\lambda_{\text{phys}}(\mu)^2 \log\left(\frac{\mu'^2}{\mu^2}\right) + \mathcal{O}(\lambda_{\text{phys}}(\mu)^3)\,. \tag{7.13}$$

The coupling thus has a logarithmic dependence on the overall energy scale. Such an energy-dependent physical coupling is called a *running coupling constant*.[3]

It is common to express the scale dependence of physical couplings in the form of a differential equation. By differentiating equation (7.13) with respect to $\mu'$ we see that is equivalent to

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda_{\text{phys}}(\mu) = 6C\lambda_{\text{phys}}(\mu)^2 + \mathcal{O}\big(\lambda_{\text{phys}}(\mu)^3\big)\,. \tag{7.14}$$

This kind of equation is called a *renormalisation group flow equation*. It describes the flow of the coupling constant as $\mu$ is changed. The word 'group' is a bit out of place here, as the only group which plays a role here is that of one-dimensional translations $\mu \to \mu + \delta\mu$.




The beta functions for the $\phi^4$ and QCD theories in four dimensions (to lowest order in perturbation theory). In the top plot the fixed point at $\lambda_{\text{phys}} = 0$ is UV unstable, while in the bottom plot it is UV stable.

[2] This is not even actually physically reachable, but for computational purposes it will do.

[3] What "energy" means in this definition can be quite confusing. In the example we discussed here, we *defined* the coupling to be the value of the scattering amplitude for the particular process in which $s^2 = t^2 = u^2 = \mu^2$. But the amplitude is a function of more than one energy scale, and we could have taken something else as our definition.

The object on the left-hand side of (7.14) is an important one, and called the *beta function* for the coupling $\lambda$,

$$\beta_\lambda := \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \lambda_{\text{phys}} \,. \tag{7.15}$$

If the coupling depends on the energy scale, the beta function is non-zero. The beta function says what happens to the coupling when the energy scale is *increased*.

If we want to understand how couplings behave as we go towards high energies (the UV) or low energies (the IR), it is convenient to make a plot of $\beta$ as a function of the coupling. The bit of information (7.14) which we have just extracted at lowest order in perturbation theory is represented in the top plot of figure 7.3. For any value of $g$, the change of $g$ is positive as the energy scale is increased (at least for small enough values of $g$).[4]

This is to be contrasted with the coupling constant for the strong nuclear force (QCD). We have not discussed that theory in these lectures, but a perturbative analysis produces a plot as in the bottom half of figure 7.3. For any value of the coupling, an increase of the energy scale makes the coupling smaller, until it vanishes completely in the high-energy limit. This behaviour is called "asymptotic freedom", and tells us that quarks become free in the high-energy limit.

▸ ***See also:*** A. Zee, "Quantum field theory in a nutshell", 2003 chapter VI.8, M. Le Bellac, "Quantum and statistical field theory", Oxford University Press, 1991 chapter 7.

## 7.4. Power counting

We have seen examples divergent integrals in the previous sections, and explored their behaviour by working them out explicitly. However, there is a quick way to determine whether integrals are likely to be divergent, which does not involve doing them explicitly. This method goes under the name of *power counting*.

For power counting we simply use the fact that every loop is associated to a four-dimensional integral which goes like $p^4$ and every propagator goes like $1/p^2$. The *superficial degree of divergence* is defined as

$$D = 4L - 2I \,, \tag{7.16}$$

where $L$ is the number of loops and $I$ the number of propagators or internal lines in the diagram (i.e. excluding external propagators which get removed by the LSZ formula). It is superficial because often integrals behave in more subtle ways than indicated by the power counting argument. Nevertheless, it can be helpful as a rough guideline. If $D = 0$ we have an integral of the form $\int^\Lambda \mathrm{d}x x^{-1}$ which is likely to diverge as $\log \Lambda$. If $D > 0$ we will get a power-like divergence $\Lambda^D$. For negative $D$ the integral is likely to be finite (at least in the UV region).

If we use the fact that the number of loops equals

$$L = I - (v - 1) \tag{7.17}$$

where $v$ is the number of vertices, and if we also use the fact that the number of vertices in a theory with $\phi^n$ interactions is given by

$$v = \frac{E + 2I}{n} \,, \tag{7.18}$$

[4] There is a similar running of the mass $m_{\text{phys}}$, but since we have at various points in this section used the massless approximation, we will not look at it here.

we can express $D$ in a sometimes more useful form. This reads

$$D = 4 - E + (n-4)v\,. \tag{7.19}$$

For the example $\phi^4$ theory which we discussed in the previous sections we have $n = 4$, and we immediately find that $D = 2$ for the one-loop correction to the two-point function and $D = 0$ for the one-loop correction to the four-point function. These numbers agree with the type of divergences which we have computed explicitly.

## 7.5. Integrals

(The material in this section is for information only, do not learn these formulas by heart). Feynman parameters can be used to combine products of denominators into one factor. In general it states that

$$\begin{aligned}\frac{1}{A_1^{\alpha_1}\,A_2^{\alpha_2}\,\cdots\,A_2^{\alpha_2}} &= \frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} \\ &\times \int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_n\,\frac{x_1^{\alpha_1-1}\,x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\delta(1-x_1-x_2\cdots x_n)}{(x_1A_1+x_2A_2+\cdots+x_nA_n)^{\alpha_1+\alpha_2+\cdots+\alpha_n}}\,.\end{aligned} \tag{7.20}$$

Here $\Gamma$ is the Euler gamma function, which for positive integers satisfies $\Gamma(n) = (n-1)!$.

The momentum integral can then be done by making use of

$$\int_0^\infty \frac{u^\alpha\,\mathrm{d}u}{(u+r)^\beta} = r^{\alpha+1-\beta}\frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{\Gamma(\beta)}\,. \tag{7.21}$$

This will leave the integral over the Feynman parameter, which can typically only be re-written in terms of hypergeometric functions, which is rather unilluminating. In the massless limit one can reduce to gamma functions which is sometimes useful,

$$\int_0^1 \mathrm{d}u\,u^{\alpha-1}(1-u)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,. \tag{7.22}$$

The above will simplify further in the large $\Lambda$ limit which we have been looking at in the main text. The relevant integrals are then

$$\begin{aligned}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+c^2-i\epsilon)^3} &= \frac{-i}{32\pi^2c^2}\,, \\ \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+c^2-i\epsilon)^2} &= \frac{i}{16\pi^2}\left(\log(\frac{\Lambda^2}{c^2})-1+\ldots\right)\,.\end{aligned} \tag{7.23}$$

The second one is related to the first one by differentiating with respect to $m^2$.

# 8
# Assorted material

## 8.1. What you should be able to do on the exam

**chapter 2** Compute equations of motion, canonical momenta, transform fields and show that an action is invariant, understand the Noether procedure and know how to compute Noether charges; all for classical mechanics systems as well as for real and complex scalar field theories.

**chapter 3** Understand the logic behind free field expansions such as (3.26), know what Poisson brackets are, know what Dirac quantisation is, know how to expand fields in modes and derive the commutation relations of the mode operators from the canonical commutators, do oscillator algebra with quantum fields and operators built from them, know what normal ordering means, understand the meaning of the quantum operators $\hat{a}$ and $\hat{b}$, know what a propagator is and be able to compute it from the oscillator expansion.

**chapter 4** Know how to compute correlators of an interacting theory by only using correlators of a free theory, compute correlators of time-ordered free fields using oscillator algebra, know Wick's theorem and how to prove it for small numbers of fields, know what Feynman diagrams are and how to associate them to correlators, compute symmetry factors of diagrams, know the Feynman rules for real and complex scalar fields, how to determine them from a Lagrangian, and how to use them to translate graphs to mathematical expressions, know how to determine the order of $\hbar$ of a graph, know the building blocks of a scattering amplitude and the Lehmann-Symanzik-Zimmermann reduction formula.

**chapter 5** Understand the concept of path integration, do simple path integrals of Gaussian type, know how to write down correlators of fields in path integral language, know what it means to couple the system to an external source and why that is useful in computing correlators, know how to compute $Z[J]$ by completing a square and integrating out $\phi$, be able to express $Z[J]$ for an interacting theory in terms of diagrams, know how to compute correlation functions from $Z[J]$ by differentiating w.r.t. $J$, and be able to do this also at the graphical level using the graphs of $Z[J]$.

**chapter 6** Understand reparameterisation invariance, know the Nambu-Goto and Polyakov forms of the actions for particles and strings, compute the corresponding equations of motion, understand the symmetries of the string worldsheet action, know the string equations of motion in flat gauge, understand

what light-cone and static gauge mean, be able to verify and interpret classical solutions to the equations of motion, know about left/right-movers and the expansion of fields in oscillators, be able to use the constraints to compute the mass of a string state, compute with quantum Virasoro charges and their algebra, know about the normal ordering issues.

**chapter 7** Know how to write down expressions for loop diagrams, know how to determine whether graphs diverge and what is their superficial degree of divergence, understand the concept of cut-off integrals, understand the difference between bare and renormalised couplings, understand the logic behind renormalisation, know what renormalisability means physically, know what is a renormalisation group equation and a beta function.

## 8.2. Special relativity reminder

This is a brief reminder of some of the concepts and notation used in special relativity. The key postulate is that light always travels with the same speed $c$, independent of the velocity of the observer. The distance travelled by light satisfies

$$\Delta t = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\,, \tag{8.1}$$

for any coordinate system $t, x, y, z$ corresponding to an observer moving with constant velocity. The interval between two space-time events (not necessarily on the same light beam's world line) is defined as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,. \tag{8.2}$$

This is the *Minkowski* interval. Time and space are unified into space-time. The coordinates are unified in the notation $x^\mu$ where $\mu = 0, 1, 2, 3$. We have $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$.

Let us now consider vector and gradient objects in this four-dimensional space. Vectors are simply arrows, which associate a direction to the point at which they are located. We denote them with

$$v^\mu = (v^0, v^1, v^2, v^3)\,. \tag{8.3}$$

Such vectors transform, under a coordinate transformation which takes us from original coordinates $x^\mu$ to new coordinatecompactly s $x'^\mu$, as

$$v^\mu \to v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} v^\nu\,. \tag{8.4}$$

This transformation rule is called *contra-variant* (the reason is essentially that the components of a vector transform in the opposite way from basis vectors).

Gradients of functions also carry an index,

$$\frac{\partial f}{\partial x^\mu} = \Big(\frac{\partial f}{\partial x^0}, \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\Big)\,. \tag{8.5}$$

By the chain rule, such gradients transform under a coordinate transformation as

$$\frac{\partial f}{\partial x^\mu} \to \frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial f}{\partial x^\nu}\,. \tag{8.6}$$

Note that the transformation matrix here is different from the one of vectors. The transformation behaviour of gradients is called *co-variant*. One often uses the shorthand notation

$$\frac{\partial}{\partial x^\mu} = \partial_\mu f\,, \tag{8.7}$$

which is only ambiguous if $f$ depends on more than one coordinate.

From the expressions above, it is also clear that if we multiply a contra-variant object with a co-variant one and sum over the index values, we get an expression which does not change under a coordinate transformation,

$$\sum_{\mu=0}^{3} v^\mu \frac{\partial f}{\partial x^\mu}\,. \tag{8.8}$$

One typically uses the Einstein summation convention in which the summation symbol is suppressed, and summation is understood whenever there is both a 'lower'

and an 'upper' index of the same name. Such a summation is called an *index contraction* or simply *contraction*.

The space-time interval can be summarised by making use of a *two-tensor*, the *Minkowski metric*,

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,. \tag{8.9}$$

Using this metric we can convert upper indices to lower ones,

$$v_\mu = \eta_{\mu\nu} v^\nu\,. \tag{8.10}$$

In particular, the coordinate-independent four-dimensional squared norm of a vector is given by making a double contraction $|v|^2 = v_\mu v^\mu = \eta_{\mu\nu} v^\mu v^\nu$.

▸ ***See also:*** See also "Introduction to tensor calculus" by Kees Dullemond and Kasper Peeters [27].

## 8.3. Variational derivatives reminder

Definition of the *variational derivative*.

Variational derivatives (also called functional derivatives) are derivatives of functionals $F[\phi]$ with respect to the function(s) on which they depend (here $\phi$). Formally, one writes

$$\int \frac{\delta F}{\delta \phi(x)} \delta\phi(x)\mathrm{d}x = \lim_{\epsilon\to 0} \frac{F[\phi(x)+\epsilon\delta\phi(x)] - F[\phi(x)]}{\epsilon}\,. \tag{8.11}$$

which defines the derivative $\delta F/\delta\phi(x)$. But where does this definition[1] come from?

It is probably best intuitively understood to be a generalisation of ordinary partial derivatives of functions of multiple variables. Consider a function $F(\phi_1,\ldots,\phi_n)$ of $n$ variables $\phi_i$. One can write down the partial derivative with respect to any single variable $\phi_k$,

$$\frac{\partial F}{\partial \phi_k} = \lim_{\epsilon\to 0} \frac{F(\phi_1,\ldots,\phi_k+\epsilon,\ldots,\phi_n) - F(\phi_1,\ldots\phi_n)}{\epsilon}\,, \tag{8.12}$$

We can rewrite this slightly by changing $\phi_k$ not by $\epsilon$, but by some constant $\delta\phi_k\cdot\epsilon$. The expression then reads

$$\frac{\partial F}{\partial \phi_k}\delta\phi_k = \lim_{\epsilon\to 0} \frac{F(\phi_1,\ldots,\phi_k+\epsilon\delta\phi_k,\ldots,\phi_n) - F(\phi_1,\ldots\phi_n)}{\epsilon}\,, \tag{8.13}$$

(with *no* sum over $k$ implied). Because we have the freedom to take $\delta\phi_k$ different for any value of $k$, we can now write down a kind-of "derivative in all directions simultaneously": change all $\phi_i$ on the right hand side above, and sum over $k$,

$$\sum_k \frac{\partial F}{\partial \phi_k}\delta\phi_k = \lim_{\epsilon\to 0} \frac{F(\phi_1+\epsilon\delta\phi_1,\ldots,\phi_n+\epsilon\delta\phi_n) - F(\phi_1,\ldots\phi_n)}{\epsilon}\,, \tag{8.14}$$

The connection to (8.11) is now made by taking the limit of an infinite number of variables $\phi_i$, i.e. take $n\to\infty$. Instead of labelling the variables by a discrete index $i$, we then label them by a continuous (real) variable $x$.

[1] Note that neither the left-hand side nor the right-hand side depend on $x$. On the left-hand side you can see this because there is an explicit integral over $x$. On the right-hand side, there is no $x$-dependence because $F[\phi(x)]$ does not actually depend on $x$, only on $\phi(x)$. If you are confused: an example of an $F[\phi(x)]$ would be the action of a field $\phi(x)$.

In physics, people are often sloppy with the choice of $\delta\phi(x)$. This is supposed to be a small change, to be added to $\phi(x)$. But what we do instead is to just take $\delta\phi(x) = \delta(x-y)$. The key formula (8.11) then reads

$$\int \frac{\delta F}{\delta \phi(x)} \delta(x-y) \mathrm{d}x = \frac{\delta F}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{F[\phi(x) + \epsilon\delta(x-y)] - F[\phi(x)]}{\epsilon} \,. \tag{8.15}$$

This works fine most of the time, but you may run into trouble in cases where $F[\phi(x) + \epsilon\delta(x-y)]$ is not well-defined, for instance because this formal change of the argument of $F$ leads to products of Dirac delta functions.

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