Extra problems for "Group Theory"

1. If $[A, B] = B$ then calculate $e^{i\alpha A} e^{-i\alpha B}$.

2. Consider A_4 and its subalgebra $su(5)$. Among its representations is one with highest-weight given by the Dynkin labels $[0, 1, 0, 0]$.
   (a) Construct all the weights corresponding to this representation.
   (b) Write down the Young tableau for this representation. What does this tableau tell you about the way in which you could construct the $[0, 1, 0, 0]$ representation using fundamental representations?
   (c) Compute the dimension of this representation using the Young tableau dimension formula.
   (d) What can you conclude about the multiplicities of the weights obtained above?

3. Consider an algebra defined with N generators t_a which obey a Lie product relation
   $$[t_a, t_b] = i f_{abc} t_c \,. \tag{1}$$
   Derive the conditions on f_{abc} so that the Lie product is well-defined.

4. The algebra B_2 has Cartan matrix
   $$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} . \tag{2}$$
   (a) Which invariant information about roots is contained in this matrix? Give this information explicitly.
   (b) Write down all non-zero commutation relations of this algebra, using a suitable basis of generators.

5. Argue the following in general, and illustrate using the root system for A_2:

   (a) The difference $\alpha_1 - \alpha_2$ of two simple roots is not a root.

   (b) If we extend the set of simple roots with $-\theta$, the negative of the highest root, then removing any root again produces a simple root system.

6. Consider the algebra G_2.

   (a) Construct the root space of this algebra. Draw the root space diagram.

   (b) Give an expression for the highest root θ in terms of the simple roots.

   (c) Consider the extended root system given by the simple roots together with $-\theta$. What would be the corresponding Dynkin diagram? (this diagram is not one of the allowed Dynkin diagrams for finite-dimensional Lie algebras).

   (d) Argue that $A_1 + A_1$ (the trivial direct sum of two $sl(2)$ algebras) is a (regular) subalgebra of G_2. Argue the same for A_2.