

TOPOLOGY AND APERIODICITY

Supervisor: John Hunton

Project research area: Geometry and Topology

Description

Mathematics is very good at describing both phenomena that are very symmetric, and phenomena that are pretty random, but there are a lot of things in between: more explicitly, while group theory is good at categorising symmetric patterns, and probability and statistics for more random or indeterminate systems, the idea of a pattern that is close to being symmetric, but not quite, is better thought about using *topology*.

Examples of the sort of object in mind are things like the *Penrose tiling*, which has a high degree of structure, and a lot of local similarities, but no overall global translational symmetry.

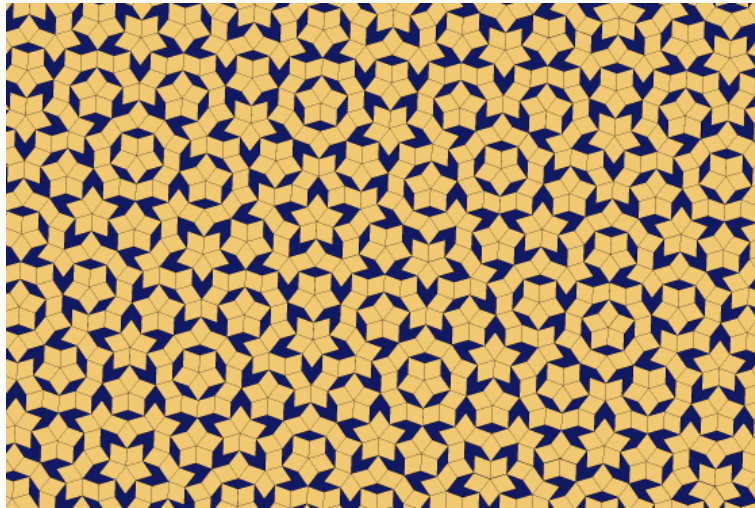


FIGURE 1. *Patch of a Penrose tiling. The tiling has no translational symmetries, but arbitrarily large finite patches can be found repeated throughout the plane.*

The Penrose tiling is among the best known patterns with this structured but non-symmetric nature, but in fact there are an uncountable number of similarly rich yet aperiodic structures.

Another set of examples come from the world of chaos theory. Here a classic example is the *Lorenz attractor*: this describes the long term behaviour of a certain weather model Lorenz was exploring, and gave rise to what we understand as chaos, sometimes referred to as the *butterfly effect* – systems that are highly dependent on initial conditions in that if you make the slightest change to the starting point, wildly different behaviour in the long run can ensue.

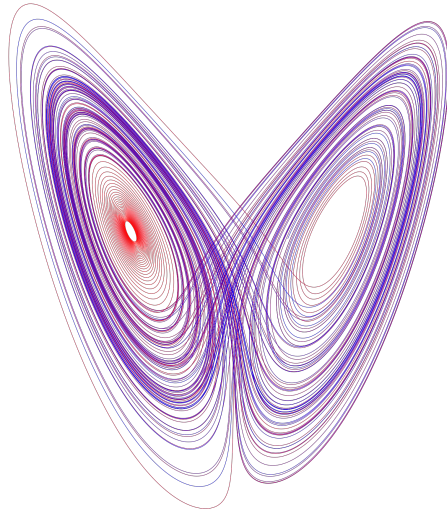


FIGURE 2. *Sketch of the Lorenz attractor. Any two line components in this space, no matter how close they are at some point, will eventually diverge – one will go round the left loop while the other the right.*

Another example could be the position of all the planets in the sky today: a pattern that will never exactly come again (so, does not repeat exactly over time), but something close to today's configuration may occur on many future (and past) occasions.

Yet another family of examples (though actually secretly connected to the above) arise from *substitutions*. To take the case of the Fibonacci substitution, consider the sequence of A's and B's formed as follows by the rule

$$A \mapsto AB \quad B \mapsto A.$$

To enact this rule, we begin with an A; replace it with AB, then in this new word, replace the A by AB, and the B by A. Continue indefinitely:

A
AB
ABA
ABAAB
ABAABABA
ABAABABAABAAB
ABAABABAABAABAABAABA
 ⋮

Letting the word grow to infinity (this is well defined as the increasing initial segment remains the same in all subsequent lines), it produces a never repeating sequence, and yet one

where finite sequences appear again and again.¹ Many beautiful examples of similar constructions in higher dimensions also exist – try, for example, looking up *Chair* or *Ammann Beenker* tilings on the web.

As a final, historical, note the catch-all term *aperiodic order* (literally objects such as the above that are both aperiodic – lacking symmetries – and yet strongly structured) originally arose from the remarkable X-ray diffraction experiments of the early 1980's that formed the discovery of so-called *Quasicrystals*, which in due course came to be modelled with some of the patterns introduced above.

This project will look at examples of such complex patterns, how their properties can be translated into topological terms, and how tools from topology help to examine them.

Project

The initial stage of the project shall concentrate on looking at the range of patterns that come under the heading of Aperiodic Order. It will then think about how topology can be introduced to say something about these objects.

After that there will be a number of ways the project can be taken, from studying particular examples, to general theorems in the area.

Mode of operation and evidence of learning for the individual project

After the initial stages, the project will typically focus on reading material from a variety of texts relevant to the particular topic chosen and interests of the student. The emphasis will be on mathematical precision and the development of deep conceptual understanding. Students will demonstrate their understanding through exploring examples and communicating clearly the theory and examples in both written and oral formats.

Pre-requisites and co-requisites Geometric Topology III is desirable as a pre-requisite, though not completely necessary. The language of metric and topological spaces from Topology II is very relevant, so that module would be highly desirable. Algebraic Topology IV may be a useful co-requisite.

Additional information

If you would like more information about this project, discuss its scope and/or its pre- and co-requisites, please contact me at john.hunton@durham.ac.uk.

Resources

Some sample or introductory resources are as follows.

- N. Frank, *A primer of substitution tilings of the Euclidean plane*, Expo. Math. 26 (2008), no. 4, 295-326. Also available via her webpage [Natalie Frank's webpage](#).
- M. Senechal, *Quasicrystals and Geometry*, Cambridge University Press (1996).

¹To explain the link of this to the classical Fibonacci numbers and Fibonacci's original work, consider A standing for Adult rabbit, B for Baby rabbit. After each time interval each baby becomes an adult and each adult gives rise to a baby. The numbers of A's at each stage (or B's for that matter) describe the well-known Fibonacci numbers. Also, the ratios of A's to B's stage by stage give a sequence of numbers converging to the Golden Ratio ϕ , which in turn implies that this sequence additionally encodes the trajectory of a billiard ball on a square table running at an angle of ϕ to the sides: there is a lot of maths behind this simple substitution rule!

- L. Sadun, *Topology of Tiling Spaces*, American Mathematical Society, University Lecture Series 46 (2008).
- The Wikipedia article on Aperiodic Tilings [Aperiodic Tilings](#) also gives an excellent overview of the some of the sorts of objects we can use topology to analyse.
- There are the slides of a nice general talk introducing the subject at [Ian Putnam Talk](#).
- For some background on complex patterns, try the classic Branko Grünbaum and G. C. Shephard, *Tilings and Patterns: An Introduction*, W.H.Freeman & Co (1989).