PROJECT IV 2025-26

HOMOTOPY THEORY and the SPHERES

John Hunton

In the module Geometric Topology III we saw the notion of *homotopy* and *homotopy equivalence*. These are the notions that roughly correspond to the intuitive idea of considering spaces to be equivalent if you can continuously deform one into the other without cutting or gluing. It is a powerful way of thinking about topological spaces, and for getting to grips with `deep' aspects of spaces that don't depend on metric notions of size, shape, etc. As such it lies at the heart of a range of mathematical problems, such as the existence or nature of solutions to equations, stable or unstable phenomena in dynamics, the possible vector fields you can have on surfaces, for which n the spaces $\mathbf{R}^n$ have a nice algebraic structure (answer: only n=1, 2, 4 and 8, being the reals, the complex numbers, the quaternions and the Cayley numbers), and so on. There are applications to a range of other disciplines, including for example biology, economics and computing. The main theory in the subject includes some of the most elegant ideas in mathematics, being a complex mix of both the geometric and the algebraic.

Also in Geometric Topology III we saw the *Fundamental Group* of a space X, which can be thought of as the (pointed) homotopy classes of maps from the circle S^1 to X. In the same way we can consider the set of homotopy classes from the n-sphere S^n to X. Remarkably, the simple question of computing the homotopy classes from one sphere to another is to this day unsolved. This basic problem has provided motivation for study for decades, and while much is known it still represents one of the fundamental unsolved problems in mathematics.

While a project on this topic needs to start with the basic theory, it can subsequently be explored in a number of ways, from its 'purest' aspects to its more applied.

PREREQUISITES
Geometric Topology III - MATH3491 is probably necessary. Depending on how you chose to develop the project, the module Algebraic Topology IV, MATH4161, may be helpful as well.

RESOURCES
A good place to start is the book
Allen Hatcher *Algebraic Topology*, CUP, especially Chapter 4. This book (which is also recommended for MATH4161 Algebraic Topology IV) is available free of charge from www.math.cornell.edu/~hatcher.
A concise book that mentions some of the main topics and applications is
Robert Ghrist *Elementary Applied Topology*, 2014, which can be found in the library at 514GHR. You could also browse the library shelves from 514.2 onwards for a variety of classic texts.

EMAIL: John Hunton (mailto:john.hunton@durham.ac.uk)