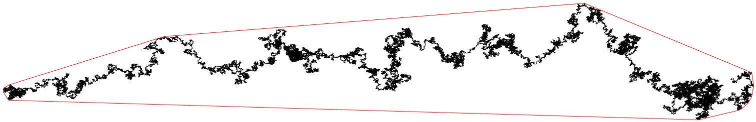
# Iterated-logarithm laws for convex hulls of random walks with drift



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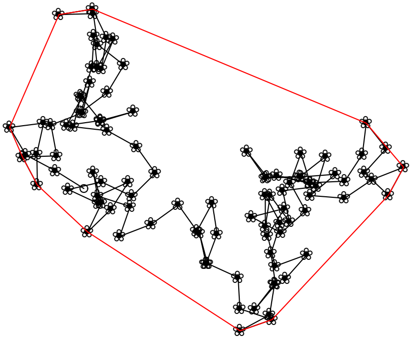
# Introduction

*On each of n unsteady steps, a drunken gardener drops a seed. Once the flowers have bloomed, what is the area of the garden enclosed by the minimal-length fence?*




## Acknowledgements.

*Thanks to James McRedmond and Vlad Vysotsky for sharing several ideas related to this work.*

# Introduction

Let $Z, Z_1, Z_2, \ldots \in \mathbb{R}^d$ ($d \geq 2$) be independent and identically distributed.

The $Z_k$ will be the increments of the random walk $S_n$, $n \geq 0$, started at the origin $\mathbf{0}$ in $\mathbb{R}^d$, defined by

$$S_0 = \mathbf{0}, \quad \text{and} \quad S_n = \sum_{k=1}^{n} Z_k \quad \text{for } n \geq 1.$$

We are interested in the convex hull

$$\mathcal{H}_n := \text{hull}\{S_0, \ldots, S_n\},$$

i.e., the smallest convex set that contains $\{S_0, \ldots, S_n\}$.

In particular, the $n \to \infty$ limit behaviour of the random variables

- $V_d(\mathcal{H}_n) =$ the volume of $\mathcal{H}_n$;
- $D(\mathcal{H}_n) =$ the diameter of $\mathcal{H}_n$;
- other intrinsic volumes.

# Outline

# Drift: zero vs. non-zero

Standing assumption: $\mathbb{E}\,\|Z\| \in (0, \infty)$.

For the mean drift vector of the walk we write $\mu = \mathbb{E}\,Z$.

There is going to be a clear distinction between the zero drift case ($\mu = \mathbf{0}$) and the non-zero drift case ($\mu \neq \mathbf{0}$).

For a qualitative result, observe that $\mathcal{H}_\infty := \cup_{n \geq 0} \mathcal{H}_n$ exists (by monotonicity) and $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) \in \{0, 1\}$ (by Hewitt–Savage zero–one law).

**Theorem** (López Hernández, W., 2021).
*We have* $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 1$ *if* $\mu = \mathbf{0}$ *and* $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 0$ *if* $\mu \neq \mathbf{0}$.

# Law of large numbers

View $\mathcal{H}_n$ as a sequence in the metric space of convex, compact subsets of $\mathbb{R}^d$ containing $\mathbf{0}$, with Hausdorff metric. Let $\ell_\mu := \text{hull}\{\mathbf{0}, \mu\}$, the line segment from $\mathbf{0}$ to $\mu$.

A consequence of the strong law of large numbers plus continuity:

**Proposition** (cf. Lo, McRedmond, Wallace, 2018).
*As $n \to \infty$, $n^{-1}\mathcal{H}_n \to \ell_\mu$, a.s.*

In non-zero drift case, this tells us the first-order asymptotic shape of convex hull, and (by continuity) implies that, e.g.,

$$\lim_{n\to\infty} n^{-1} D(\mathcal{H}_n) = \|\mu\|, \text{ and } \lim_{n\to\infty} n^{-d} V_d(\mathcal{H}_n) = 0, \text{ a.s.}$$

# Zero-drift case

When $\mu = 0$, the strong laws says only $n^{-1}\mathcal{H}_n \to \{\mathbf{0}\}$, a.s.

New standing assumption: $\mathbb{E}(\|Z\|^2) \in (0, \infty)$.

Let $\Sigma := \mathbb{E}(ZZ^\top)$ denote the increment covariance matrix.

A consequence of Donsker's theorem plus continuity:

**Proposition** (cf. W., Xu, 2015; Lo, McRedmond, Wallace, 2018).
*Suppose that* $\mu = \mathbf{0}$*. For* $b : [0,1] \to \mathbb{R}^d$ *the trajectory of a standard Brownian motion,* $n^{-1/2}\mathcal{H}_n \xrightarrow{\mathrm{d}} \Sigma^{1/2} \operatorname{hull} b[0,1]$.

A consequence is that (for $\Sigma =$ identity, say)

$$n^{-1/2} D(\mathcal{H}_n) \xrightarrow{\mathrm{d}} \operatorname{diam} b[0,1], \text{ and } n^{-d/2} V_d(\mathcal{H}_n) \xrightarrow{\mathrm{d}} V_d(\operatorname{hull} b[0,1]).$$

For $d = 2$, the expected area of the Brownian convex hull is $\mathbb{E}\, V_2(\operatorname{hull} b[0,1]) = \pi/2$ (El Bachir, 1983). We don't know the expected diameter (cf. McRedmond, Xu, 2017).

# Scaling limit in the case with drift

How to go beyond law of large numbers when $\mu \neq \mathbf{0}$? To get a non-degenerate scaling limit, we now must scale space by factor $1/n$ in the direction of the drift and by factor $1/\sqrt{n}$ in the orthogonal directions.

Take $d = 2$ so we can draw a picture.

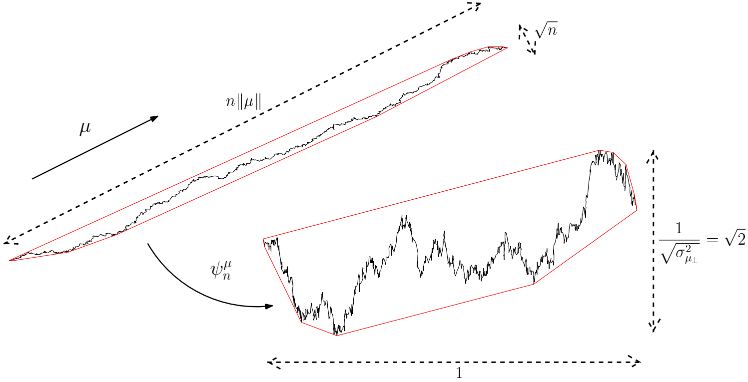
$$\text{Then, let } \varphi_n^\mu(x) = \left( \frac{x \cdot \hat{\mu}}{n\|\mu\|}, \frac{x \cdot \hat{\mu}_\perp}{\sqrt{n\sigma^2_{\mu_\perp}}} \right);$$

Here $\sigma^2_{\mu_\perp} = \mathbb{E}\left[ (Z \cdot \hat{\mu}_\perp)^2 \right]$.

Let $\tilde{b}$ denote the process on $\mathbb{R}^2$ given by $\tilde{b}(t) = (t, w(t))$, where $w$ is standard Brownian motion on $\mathbb{R}$.

The analogue of Donsker's theorem says that $\varphi_n^\mu(X_n)$ converges weakly to $\tilde{b}$ as $n \to \infty$; proof combines the functional LLN and CLT (cf. W. & Xu, 2015).

# Scaling limit in the case with drift




The affine map $\varphi_n^\mu$ preserves the convex hull, so:

**Theorem** (W. & Xu, 2015).
*If* $\mu \neq \mathbf{0}$ *and* $\sigma^2_{\mu_\perp} > 0$*, then as* $n \to \infty$*,* $\varphi_n^\mu(\mathcal{H}_n)$ *converges weakly to* hull $\tilde{b}[0,1]$.

# Scaling limit in the case with drift

By continuity and scaling of volumes (one coordinate by the LLN scaling $n$, the other $d-1$ coordinates by the CLT scaling $\sqrt{n}$) this leads to distributional limit for volumes:

**Corollary** (W. & Xu, 2015; McRedmond, 2019).
*Suppose that $\mu \neq \mathbf{0}$ and $\sigma^2_{\mu_\perp} > 0$. Then, as $n \to \infty$,*

$$n^{-(d+1)/2} \|\mu\|^{-1} (\sigma^2_{\mu_\perp})^{-1/2} V_d(\mathcal{H}_n) \xrightarrow{\mathrm{d}} V_d(\text{hull}\, \tilde{b}[0,1]).$$

W. & Xu (2015) show that, when $d = 2$, $\mathbb{E}\, V_2(\text{hull}\, \tilde{b}[0,1]) = \frac{1}{3}\sqrt{2\pi}$.

This scaling limit strategy does not work so nicely for diameter or perimeter length when $\mu \neq \mathbf{0}$, because $\varphi_n^\mu$ does not act in a sensible way on lengths. This leads to another story (and a different class of limit phenomena): W. & Xu (2015) for perimeter, McRedmond & W. (2018) for diameter.

# Outline

# Iterated-logarithm laws: Overview

We want to study a.s. behaviour of upper envelope of e.g. $V_d(\mathcal{H}_n)$: we seek appropriate versions of the law of the iterated logarithm (LIL) from classical fluctuation theory.

In the zero-drift case, the answer is an elegant theorem due to KHOSHNEVISAN (1992), using STRASSEN'S (1964) functional LIL. For example, when $d = 2$, $\mu = \mathbf{0}$, and $\Sigma = I$ (identity), Khoshnevisan shows that area satisfies

$$\limsup_{n\to\infty} \frac{V_2(\mathcal{H}_n)}{n \log \log n} = \frac{1}{\pi}, \text{ a.s.}$$

The constant $1/\pi$ arises from solving a variational problem (this is typical for a Strassen-type argument).

The analogue of this result for Brownian motion had already been obtained in a remarkable paper of LÉVY (1955), who anticipated to some extent the functional LIL of STRASSEN (1964).

# Iterated-logarithm laws: Overview

In the non-zero drift case, Khoshnevisan's LIL does not apply. Our result is:

**Theorem** (Cygan, Sandrić, Šebek, W., 2023).
*If $d = 2$, $\mu \neq \mathbf{0}$, and $\Sigma = I$,*

$$\limsup_{n \to \infty} \frac{V_2(\mathcal{H}_n)}{n^{3/2}\sqrt{\log \log n}} = \frac{\|\mu\|}{\sqrt{6}}, \text{ a.s.}$$

Our general result covers all intrinsic volumes and (like Khoshnevisan's) is founded on Strassen's functional LIL, modified appropriately to apply to walks with non-zero drift; in our setting, as in Khoshnevisan's, limiting constants can often be characterized by variational problems, but in only a limited number of instances is the solution known.

The variational problem also arises in the context of large deviations, where it was solved using an alternative approach by Akopyan & Vysotsky (2021).

# Strassen's theorem

Let $\mathcal{C}_d$ denote the set of continuous $f : [0,1] \to \mathbb{R}^d$, and let $\mathcal{C}_d^0$ denote the subset of those $f \in \mathcal{C}_d^0$ for which $f(0) = \mathbf{0}$. Define the linearly-interpolated random walk trajectory

$$Y_n(t) := S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}, \text{ for } t \in [0,1].$$

Then $Y_n \in \mathcal{C}_d^0$ for every $n \in \mathbb{Z}_+$. The KHINCHIN scaling function for the classical LIL is

$$\ell(n) := \sqrt{2n \log \log n} \text{ for } n \geq 3.$$

The symmetric, non-negative definite matrix $\Sigma$ has a unique symmetric, non-negative definite square-root $\Sigma^{1/2}$, which acts as a linear transformation of $\mathbb{R}^d$.

Strassen's theorem is a statement about the a.s. limit points of the sequence $Y_n / \ell(n)$ in the metric space $\mathcal{C}_d^0$ (endowed with the supremum metric).

# Strassen's theorem

**Theorem** (Strassen's theorem for random walk).

*Let $d \in \mathbb{N}$ and $\mu = \mathbf{0}$. With probability 1, the sequence $Y_n/\ell(n)$ in $\mathcal{C}_d^0$ is relatively compact, and its set of limit points is $\Sigma^{1/2} U_d$.*

Here

$$U_d := \left\{ \text{a.c. } f : f(0) = \mathbf{0}, \int_0^1 \|f'(s)\|^2 \mathrm{d}s \leq 1 \right\}$$

is unit ball in Cameron–Martin space for the Wiener measure, and $f'$ is componentwise derivative.

In words, the theorem states that, a.s., (a) every subsequence of $Y_n/\ell(n)$ contains a further subsequence that converges, its limit being some $f \in \Sigma^{1/2} U_d$, and (b) for every $f \in \Sigma^{1/2} U_d$, there is a subsequence of $Y_n/\ell(n)$ that converges to $f$.

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Example: among $f \in U_d$, maximum $f(1) = 1$ achieved by $f(s) \equiv s$; so corollary to Strassen's theorem is the classical LIL: for $\Sigma = I$,

$$\limsup_{n \to \infty} \frac{S_n}{\ell(n)} = 1, \text{ a.s.}$$

# Strassen's theorem

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Example: also yields the extension that for $\Sigma = I$,

$$\liminf_{n \to \infty} \left| \frac{S_n}{\ell(n)} - \theta \right| = 0, \text{ a.s., if and only if } \theta \in [-1, 1].$$

# A Strassen theorem for non-zero drift

Idea: Use different scalings, like in the W. & Xu weak convergence result; this time LLN scaling in drift direction, LIL scaling in the rest.

WLOG, choose coordinates so that the standard orthonormal basis $(e_1, \dots e_d)$ of $\mathbb{R}^d$, $d \geq 2$, has $e_1$ in the direction of $\mu$.

Let $\Sigma_{\mu^\perp}$ denote the matrix obtained from $\Sigma$ by omitting the first row and column (reduced covariance matrix).

For $n \in \mathbb{N}$, define $\psi_n^\mu : \mathbb{R}^d \to \mathbb{R}^d$, acting on $x = (x_1, \dots, x_d)$, by

$$\psi_n^\mu(x_1, \dots, x_d) = \left( \frac{x_1}{n}, \frac{x_2}{\ell(n)}, \dots, \frac{x_d}{\ell(n)} \right).$$

Let $I_\mu : [0,1] \to \mathbb{R}_+$ denote the function $I_\mu(t) = \|\mu\| t$, and set

$$W_{d,\mu,\Sigma} := \{ g = (I_\mu, \Sigma_{\mu^\perp}^{1/2} f) : f \in U_{d-1} \}, \text{ for } d \geq 2.$$

# A Strassen theorem for non-zero drift

**Theorem** (Cygan, Sandrić, Šebek, W., 2023).
*Suppose that $d \geq 2$ and $\mu \neq \mathbf{0}$. With probability 1, the sequence $\psi_n^\mu(Y_n)$ in $\mathcal{C}_d^0$ is relatively compact, and its set of limit points is $W_{d,\mu,\Sigma}$.*

**Proof.**
Combine the strong LLN (in functional form) for the first component, with Strassen's LIL for the remaining $d-1$ components. □

**Corollary.**
*Suppose that $d \geq 2$ and $\mu \neq \mathbf{0}$. Let $G$ be a real-valued, continuous function on compact, convex sets. Then*

$$\limsup_{n\to\infty} G(\psi_n^\mu(\mathcal{H}_n)) = \sup_{g \in W_{d,\mu,\Sigma}} G(\operatorname{hull} g[0,1]), \text{ a.s.}$$

Note: Not necessarily immediate to use, because of the involved nature of the $\psi_n^\mu$ map.

# Application to volumes

**Theorem** (Cygan, Sandrić, Šebek, W., 2023).
*Suppose that $d \geq 2$ and $\mu \neq \mathbf{0}$. Then, a.s.,*

$$\limsup_{n\to\infty} \frac{V_d(\mathcal{H}_n)}{\sqrt{2^{d-1} n^{d+1} (\log\log n)^{d-1}}} = \|\mu\| \cdot \sqrt{\det \Sigma_{\mu^\perp}} \cdot \lambda_d,$$

*where*

$$\lambda_d := \sup_{f \in U_{d-1}} V_d(\text{hull}\{(t, f(t)); t \in [0,1]\}).$$

**Theorem** (Akopyan & Vysotsky, 2021; Cygan, Sandrić, Šebek, W., 2023).
*When $d = 2$, the constant takes value $\lambda_2 = \sqrt{3}/6$.*

Together, these results give the LIL for area of the planar convex hull stated earlier.

# Link to large deviations

One proof of Strassen's theorem for Brownian motion is via Schilder's theorem from large deviations.

As kindly pointed out to us by Vlad Vysotsky, the Strassen-type functional in the variational problem for $\lambda_2$ is exactly the large-deviations rate function for a degenerate, non-centred Gaussian distribution. The solution to this isoperimetric problem is given in Proposition 2.15 of Akopyan & Vysotsky (2021). The proof (by approximating the degenerate distribution by non-degenerate ones) is omitted, but is it is fully covered in Theorem 1 of Vysotsky (2023).

# General intrinsic volumes

For $k \in \{1, \ldots, d\}$, let $V_k(\mathcal{H}_n)$ denote the $k$th intrinsic volume of $\mathcal{H}_n$.
($V_d =$ volume, $V_{d-1} \approx$ surface area, etc.)

**Theorem** (Cygan, Sandrić, Šebek, W., 2023).
*Suppose that $d \geq 2$ and $\mu \neq \mathbf{0}$. Let $k \in \{1, 2, \ldots, d\}$. Then there exists a constant $\Lambda \in (0, \infty)$, depending on $d$, $k$, and the law of $Z$, such that, a.s.,*

$$\limsup_{n\to\infty} \frac{V_k(\mathcal{H}_n)}{\sqrt{2^{k-1} n^{k+1} (\log\log n)^{k-1}}} = \Lambda.$$

- Case $k = d$ is the LIL for volumes. For other $k$, $V_k$ does not scale so nicely through $\psi_n^\mu$, so the proof is less direct, and the constant less explicit.
- Proof uses some further ingredients, including a zero–one law.

# Outline

# The planar constant: isoperimetric problem

We turn back to:

**Theorem** (Cygan, Sandrić, Šebek, W., 2023).
*When $d = 2$, the constant in the LIL for area is $\lambda_2 = \sqrt{3}/6$.*

Recall that $\lambda_2$ was characterized via

$$\lambda_2 = \sup_{f \in U_1} V_2(\mathrm{hull}\{(t, f(t)); t \in [0,1]\}),$$

where $U_1$ was the Strassen ball, i.e., a.c. $f : [0,1] \to \mathbb{R}$ with $f(0) = \mathbf{0}$ and

$$\Gamma(f) := \int_0^1 f'(s)^2 \mathrm{d}s \leq 1.$$

Denoting by $\bar{f}$, $\underline{f}$ the least concave majorant and greatest convex minorant, respectively, of $f$, we can write

$$V_2(\mathrm{hull}\{(t, f(t)); t \in [0,1]\}) = A(f) := \int_0^1 \left(\bar{f}(s) - \underline{f}(s)\right) \mathrm{d}s.$$

# The planar constant: isoperimetric problem

We can express the variational problem to identify $\lambda_2$ as

$$\text{maximize } A(f) \text{ subject to } \Gamma(f) \leq 1,$$

where $f(0) = 0$ and

$$\Gamma(f) = \int_0^1 f'(s)^2 \mathrm{d}s; \quad A(f) = \int_0^1 \left( \bar{f}(s) - \underline{f}(s) \right) \mathrm{d}s.$$

**Theorem** (Cygan, Sandrić, Šebek, W., 2023).
*The optimal $f$ is $f = f^\star$ given by*

$$f^\star(u) = \sqrt{3}u(1-u), \ \text{for } 0 \leq u \leq 1,$$

*which has $\Gamma(f^\star) = 1$ and $A(f^\star) = \sqrt{3}/6$.*

We sketch the proof.

# The planar constant: isoperimetric problem

Three important reductions:

- Suffices to work with bridges, $f(0) = f(1) = 0$.
  Easy: a calculation shows the bridge $\hat{f}$ given by $\hat{f}(s) := f(s) - sf(1)$ has $A(\hat{f}) = A(f)$ and $\Gamma(\hat{f}) \leq \Gamma(f)$.
- Suffices to work with positive bridges, $f(s) > 0$ for $s \in (0, 1)$.
  Harder?: our proof uses symmetrization.
- Suffices to work with concave positive bridges.
  Easy: replace positive bridge by its concave majorant to decrease $\Gamma$.

Problem then reduces to

$$\text{maximize } \int_0^1 f(s)\mathrm{d}s \text{ subject to } \Gamma(f) \leq 1,$$

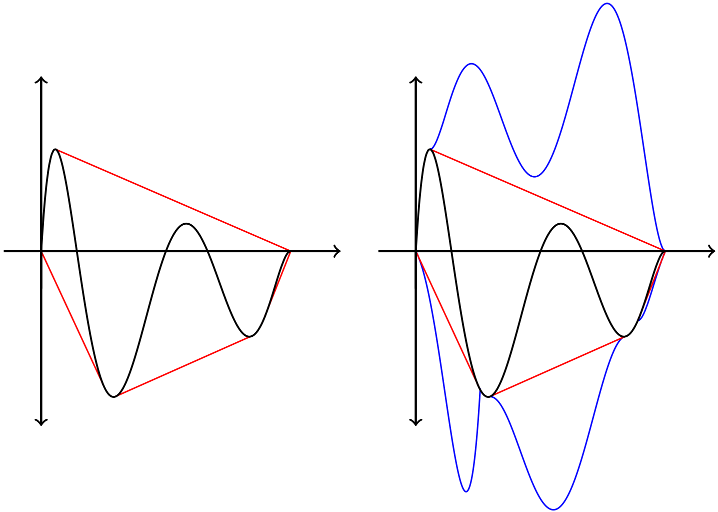
and to show that optimal $f$ is $f = f^\star$ given above.

This is a "Cameron–Martin" or "Strassen" version of the Dido problem of antiquity to find maximal enclosed area for a curve of given arc length; here arc length is replaced by Strassen cost $\Gamma$. Adjacent results by Schmidt (1940).

# The planar constant: isoperimetric problem

## Proposition.

*For every bridge $f$, there is a positive bridge $f^{\mathrm{s}}$ (produced by symmetrization) for which $\Gamma(f^{\mathrm{s}}) = \Gamma(f)$ and $A(f^{\mathrm{s}}) \geq A(f)$.*



# Outline

# Concluding remarks

- The infinite-variance, multidimensional case (when the random walk is in the domain of attraction of a *d*-dimensional stable law), distributional limit theory recently studied by Cygan, Sandrić, Šebek (2022). LIL-type behaviour still open.
- As hinted earlier, some functionals fall into a different class of limit theorems, e.g. perimeter in case $\mu \neq \mathbf{0}$ satisfies a CLT (W., Xu, 2015) and we would expect a LIL there, too, but existing approaches do not seem to apply.

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Thank you!

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