Energy-constrained random walk with boundary replenishment

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Stochastic Processes under Constraints

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Introduction

Topic is a random walk in a finite domain with a finite energy capacity, which loses energy in the interior and gains energy at the boundary.

Caricature of an animal nourished by localized resources, which must roam into resource-scarce regions for social or territorial imperatives.

Random walks have been used for over a century to model animal movement. But a basic aspect of population dynamics is the flow of energy. Individuals consume resource and subsequently expend energy in somatic growth, maintenance, reproduction, foraging, and so on.

The distribution of scarce resource imposes constraints on animal movement: flights of butterflies between flowers, elk movements between feeding craters, elephants moving between water sources...

Will return later to a brief discussion of motivation and related models of animal foraging and resource depletion, where there has been a lot of recent activity in both mathematical ecology, physics, and elsewhere. But first we describe our particular probability model.

Outline

Energy-constrained random walk

We will study a random walk in a finite domain of size N (spatial constraint) with a finite energy capacity M, which loses energy in the interior and gains energy at the boundary. We study the lifetime, i.e., time until energy is exhausted.

Notation:

- For $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$, let $I_N := \mathbb{Z} \cap [0, N]$ (discrete interval) and $I_\infty := \mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$ (discrete half line).
- We write $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and then $I_N, N \in \overline{\mathbb{N}}$ includes both finite and infinite cases.
- The boundary ∂I_N of I_N is defined as $\partial I_N := \{0, N\}$ for $N \in \mathbb{N}$, and $\partial I_\infty := \{0\}$ for $N = \infty$; the interior is $I_N^\circ := I_N \setminus \partial I_N$.

Our random walk will be a Markov chain $\zeta := (\zeta_0, \zeta_1, \ldots)$, where $\zeta_n := (X_n, \eta_n) \in I_N \times I_M$.

X_n = location of the random walker, and η_n = energy level. Parameters M = (finite) energy capacity, N = (possibly infinite) size of domain.

Energy-constrained random walk

The transition law is as follows.

- *Energy-consuming random walk:* If $i \in I_M \setminus \{0\}$ and $x \in I_N^\circ$,

$$\mathbb{P}(X_{n+1} = X_n \pm 1,\, \eta_{n+1} = \eta_n - 1 \mid X_n = x, \eta_n = i) = \frac{1}{2}.$$

- *Extinction through exhaustion:* If $x \in I_N^\circ$, then

$$\mathbb{P}(X_{n+1} = x,\, \eta_{n+1} = 0 \mid X_n = x, \eta_n = 0) = 1.$$

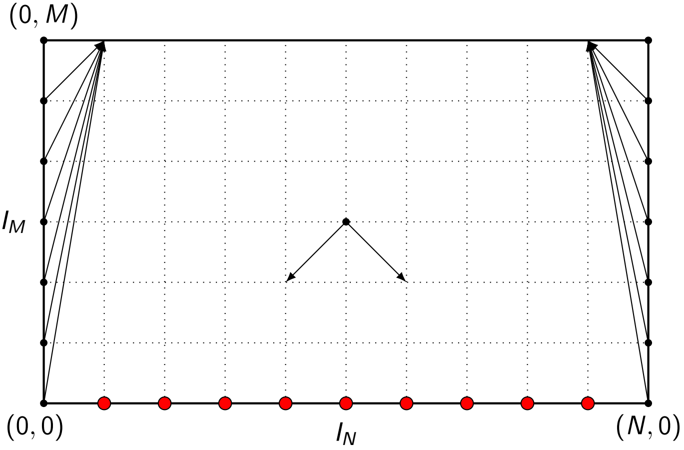
- *Boundary reflection and energy replenishment:* If $i \in I_M$, $x \in \partial I_N$, and $y \in I_N^\circ$ is the unique y such that $|y - x| = 1$, then

$$\mathbb{P}(X_{n+1} = y,\, \eta_{n+1} = M \mid X_n = x, \eta_n = i) = 1.$$

For convenience we take $X_0 = 1$, $\eta_0 = M$. We write $\mathbb{P}^{N,M}$ and $\mathbb{E}^{N,M}$ for probability and expectation under the law of the corresponding Markov chain ζ with spatial domain I_N ($N \in \overline{\mathbb{N}}$) and energy domain I_M ($M \in \mathbb{N}$).

Energy-constrained random walk

The process $\zeta = (X, \eta) \in I_N \times I_M$ can be viewed as a two-dimensional random walk with a mixed reflecting/absorbing boundary:



Energy-constrained random walk

We study the total lifetime:

$$\lambda := \min\{n \in \mathbb{Z}_+ : X_n \in I_N^\circ, \eta_n = 0\}, \qquad (1)$$

where $\min \emptyset := +\infty$. Relation to classical fluctuation theory for random walk, and ruin/insurance models, see e.g. ASMUSSEN's book (and later).

The Markov chain is finite with absorbing states $(x, 0)$, $x \in I_N^\circ$, and all other states communicating (provided $N \geq 3$). Thus extinction is certain, i.e., $\mathbb{P}^{N,M}(\lambda < \infty) = 1$. Indeed:

Lemma.

Suppose that $N \in \overline{\mathbb{N}}$ with $N \geq 3$, that $M \in \mathbb{N}$. There exists $\delta > 0$ (depending only on M) such that $\mathbb{E}^{N,M}[\mathrm{e}^{\delta\lambda}] < \infty$.

We want to study the behaviour of the finite random variable λ as $N, M \to \infty$.

Outline

Limit regimes

Recall parameters $M =$ (finite) energy capacity, $N =$ (possibly infinite) size of domain. Consider $\lambda =$ (finite, random) lifetime.

Diffusivity means walk covers distance $\approx M^2$ in time M.

Suggests key comparison is M vs. N^2.

Exhaustion mechanism can be expressed as follows: Take ordinary reflected simple random walk on I_N. Consider durations of successive excursions away from the boundary; the first excursion duration to exceed M will be the excursion on which the walk becomes extinct.

In finite-N domain, the expected duration of an excursion started from near the boundary grows like N, but this is made up of high probability of a short $O(1)$ excursion, plus small probability ($\approx 1/N$) of a long excursion ($\approx N^2$) (by gambler's ruin). In $N = \infty$ domain, excursion duration $\approx 1/2$-stable, so per-excursion extinction probability $\approx M^{-1/2}$, but given duration is $< M$, (conditional) expected size is $\approx M^{1/2}$.

Limit regimes

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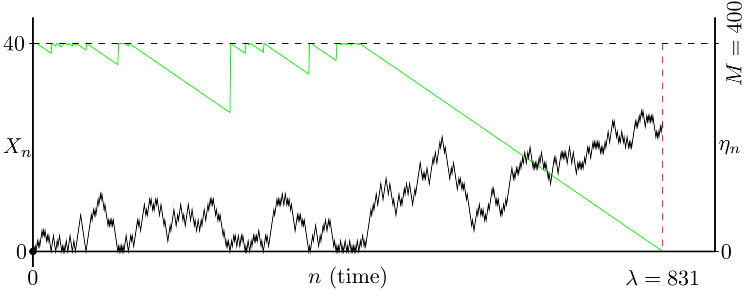
It turns out distinct phenomena are observed in three regimes:

- Meagre capacity. When $M \ll N^2$, macroscopic excursions will likely lead to exhaustion, so lifetime is made up of short excursions plus a terminal excursion.
- Confined space. When $M \gg N^2$, typical (even macroscopic) excursions are unlikely to exhaust the energy, so lifetime is made up of many typical excursions, until a rare very long excursion.
- Critical case. If $M \sim \rho N^2$ for a parameter ρ. Is the most delicate case, and the value of ρ is important.

Limit regimes: Meagre capacity



Simulation with $N = 1000$, $M = 400 \ll N^2$. Pictured is the trajectory of X_n (black, scale on left axis) and the trajectory of η_n (green, scale on right axis). Note that $N = 1000$ (and indeed any $N > M$ or so) may as well be $N = \infty$. We show $N \gg \sqrt{M}$ is in the same class too.

The meagre-capacity limit

Consider a sequence of models, indexed by M, in which $\lim_{M\to\infty} M/N_M^2 = 0$. In this regime, energy is unlikely to be sufficient to complete an excursion of macroscopic size.

Theorem (Partly Bacher & Sportiello).
Suppose $\lim_{M\to\infty} M/N_M^2 = 0$. *Then, for a random variable* $\xi \sim \mathrm{DM}(1/2)$, *it holds that*

$$\frac{\lambda}{M} \xrightarrow{\mathrm{d}} 1 + \xi, \text{ as } M \to \infty. \quad (2)$$

- $\mathrm{DM}(1/2)$ is a Darling–Mandelbrot distribution; see next slide.
- For $N = \infty$ the result is contained in Bacher & Sportiello (2016) (we indicate the connection later). In our paper we extend also to more general initial states (X_0, η_0), where the influence of the first excursion changes the limit (omit this here for simplicity).

The Darling–Mandelbrot distribution

To describe the limit distribution in the theorem, define

$$\mathcal{I}(t) := t \int_0^1 u^{-1/2} \mathrm{e}^{ut} \mathrm{d}u, \text{ and } t_0 := \inf\{t \in \mathbb{R} : \mathrm{e}^t - \mathcal{I}(t) \leq 0\}. \quad (3)$$

Then $t_0 \approx 0.8540326566$, and mgf of $\xi \sim \mathrm{DM}(1/2)$ is

$$\mathbb{E}[\mathrm{e}^{t\xi}] = \varphi_{\mathrm{DM}}(t) := \frac{1}{\mathrm{e}^t - \mathcal{I}(t)}, \text{ for } t < t_0. \quad (4)$$

- Appearance of $\mathrm{DM}(1/2)$ here is due to its role in fluctuation theory of 1/2-stable variables (Darling, 1952), that we explain later.
- Constant t_0 also related to asymptotics of maximum excursion duration of random walk (Csáki, Erdős, Révész, 1985).
- Recently $\mathrm{DM}(1/2)$ appeared in analysis of anticipated rejection algorithms, which is where Bacher & Sportiello (2016) saw it.
- $\mathrm{DM}(1/2)$ has a density continuous on $(0, \infty)$, is non-analytic at integer points, and has no elementary closed form, but has an infinite series representation: see Lew (1994) and Louchard (1999).

The confined-space limit

The second regime is when energy is plentiful, so the walk typically makes many visits to the boundary and many macroscopic excursions.

Let $\mathcal{E}_1$ denote a unit-mean exponential random variable.

Theorem.

Suppose that $\lim_{M\to\infty} N_M = \infty$ *and* $\lim_{M\to\infty} M/N_M^2 = \infty$. *Then, as* $M \to \infty$,

$$\frac{4\lambda}{N_M^2} \cos^M(\pi/N_M) \xrightarrow{\mathrm{d}} \mathcal{E}_1. \quad (5)$$

- Note $\cos^M(\pi/N_M) = \exp(-\frac{\pi^2 M}{2N_M^2}(1+o(1))) \to 0$.
- $\mathcal{E}_1$ limit is a manifestation of rare-event driver of the asymptotics: metastability (cf Aldous, 1989; Kalashnikov, 1997).
- This result can also be extended to more general initial conditions (X_0, η_0), which here do not influence the limit statement as no individual excursion is significant.

The critical case

Finally, we treat the critical case. Here both short and long excursions contribute, and there are a (random) finite number of macroscopic excursions, each of which contributes.

Theorem.
Suppose that there is $\rho \in (0, \infty)$ such that $\lim_{M\to\infty} N_M = \infty$ and $\lim_{M\to\infty} M/N_M^2 = \rho$. Then, as $M \to \infty$,

$$\frac{\lambda}{M} \xrightarrow{\mathrm{d}} 1 + \xi_\rho,$$

where ξ_ρ is a random variable with an infinitely divisible distribution on $\mathbb{R}_+$ defined via the moment generating function $\mathbb{E}[\mathrm{e}^{s\xi_\rho}] = \phi_\rho(s)$, $s < s_\rho$, that we describe on the next slide.

The critical case

To define ϕ_ρ, introduce the decreasing function $H : (0, \infty) \to \mathbb{R}_+$ by

$$H(y) := \sum_{k=1}^{\infty} h_k(y), \text{ where } h_k(y) := \exp\left\{ -\frac{\pi^2(2k-1)^2 y}{2} \right\}.$$

It turns out that $H(y) \sim 1/\sqrt{8\pi y}$ as $y \downarrow 0$, so for every $\rho > 0$ and $s \in \mathbb{R}$,

$$G(\rho, s) := \frac{s}{H(\rho)} \int_0^1 \mathrm{e}^{sv} \big(H(v\rho) - H(\rho) \big) \mathrm{d}v,$$

is finite. For fixed $\rho > 0$, $s \mapsto G(\rho, s)$ is strictly increasing for $s \in \mathbb{R}$, and $G(\rho, 0) = 0$. For $\rho > 0$, define $s_\rho := \sup\{ s > 0 : G(\rho, s) < 1 \}$, and then set

$$\phi_\rho(s) := \frac{1}{1 - G(\rho, s)}, \text{ for } s < s_\rho.$$

In particular, $\mathbb{E}\, \xi_\rho = \mu(\rho)$, where

$$\mu(\rho) := \frac{1}{\rho H(\rho)} \int_0^\rho H(y) \mathrm{d}y - 1.$$

The critical case

- The function H is a sort of theta function, arising from reflection-principle type arguments for one-dimensional Brownian motion on an interval, see e.g. FELLER's book.
- But we have not located any previous appearance of the distribution ϕ_ρ in the literature.
- One can show $\rho \to 0$ and $\rho \to \infty$ asymptotics for $\mu(\rho)$ that are consistent with the meagre-capacity and confined-space regimes.
- As for the other regimes, one could extend the result to more general initial conditions, but the first excursion would contribute to the limit and lead to a rather more complicated statement.

Outline

Ideas of the proofs

Basic ingredients are:

- Renewal structure provided each time the walk visits the boundary.
- Results on excursions of simple symmetric random walk.
- In particular, some quite delicate estimates on excursions conditioned to be short that we could not find elsewhere.

The overall framework is as follows:

- Consider an immortal walker (i.e., ignore the energy constraint). Let $s_1 < s_2 < \cdots$ be successive times of visits to ∂I_N by the immortal walker, and let $\nu_k = s_k - s_{k-1}$ be duration of the kth excursion.
- The energy-constrained walker starts $\kappa = \inf\{k \in \mathbb{N} : \nu_k > M\}$ excursions. At each one, energy is topped up to level M.
- At the start of each excursion, there is a probability $\theta(N, M) = \mathbf{P}_1(\tau_{0,N} > M)$ that it will be the last, where $\tau_{0,N}$ is the hitting time of ∂I_N, and $\mathbf{P}_1$ means start from 1 (equivalently, $N - 1$).
- Renewal structure shows κ has a geometric distribution with parameter given by the extinction probability $\theta(N, M)$.

Ideas of the proofs: Meagre capacity

Suppose first that $N = \infty$, the simplest case of the meagre-capacity limit.

$\textsc{Darling}$ (1952): Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $\mathbb{R}_+$-valued random variables in the domain of attraction of a (positive) stable law with index $\alpha \in (0, 1)$, and let $S_n := \sum_{i=1}^n Z_i$ and $T_n := \max_{1 \le i \le n} Z_i$. Then $S_n / T_n \xrightarrow{\mathrm{d}} 1 + \xi_\alpha$, as $n \to \infty$, where $\xi_\alpha \sim \mathrm{DM}(\alpha)$.

In the case where $N = \infty$, the durations $\nu_1, \nu_2, \ldots$ of excursions of simple symmetric random walk on $\mathbb{Z}_+$ away from 0 satisfy the $\alpha = 1/2$ case of Darling's result, so that $T_n := \sum_{i=1}^n \nu_i$ and $M_n := \max_{1 \le i \le n} \nu_i$ satisfy $T_n / M_n \xrightarrow{\mathrm{d}} 1 + \xi$ where $\xi \sim \mathrm{DM}(1/2)$. Replacing n by κ, the number of excursions up to extinction, for which $\kappa \to \infty$ in probability as $M, N \to \infty$, it is plausible that $T_\kappa / M_\kappa \xrightarrow{\mathrm{d}} 1 + \xi$ also. But T_κ is essentially λ, while M_κ will be close to M, the upper bound on ν_i, $i < \kappa$.

This argument is not far from a proof in the case $N = \infty$. For $M \ll N^2$, it is unlikely that any excursion will "see" the opposite end of the boundary from which it started.

Ideas of the proofs: Confined space

In the confined-space regime, where $M \gg N^2$, it is very likely that the random walk will traverse the whole of I_N many times before it runs out of energy. The key random walk estimates are as follows.

Proposition.

Suppose that $\lim_{M\to\infty} M/N_M^2 = \infty$. *Then, as* $M \to \infty$,

$$\theta(N_M, M) = \frac{4}{N_M}(1 + o(1)) \cos^M \left(\frac{\pi}{N_M} \right).$$

Moreover, as $M \to \infty$,

$$\mathbb{E}^{N_M,M}[\nu \mid \nu \leq M] \sim N_M, \text{ and } \mathbb{V}\text{ar}^{N_M,M}[\nu \mid \nu \leq M] \sim \frac{N_M^3}{3}.$$

The moment asymptotics for ν are the same as the unconditional asymptotics, since the conditioning is innocuous in this regime; the unconditional variance asymptotics for the gambler's ruin are in Bach (1997) or Anděl & Hudecová (2012).

Ideas of the proofs: Confined space

Roughly speaking, we then have a sum of the form

$$\lambda = \sum_{j=1}^{\kappa_M} \nu_{M,j} + O(M),$$

where κ_M is geometric with parameter $\theta(N_M, M)$ with the given asymptotics, and $\nu_{M,j}$ have given mean and variance asymptotics. The proof is completed by exponential convergence of geometric sums, e.g. Theorem 3.2.4 of Kalashnikov's book:

Lemma.

Let $K_M \in \mathbb{Z}_+$ satisfy $\mathbb{P}(K_M = k) = (1 - p_M)^k p_M$ for $k \in \mathbb{Z}_+$, where $\lim_{M\to\infty} p_M = 0$. Suppose also that $Y_M, Y_{M,1}, Y_{M,2}, \ldots$ are i.i.d., $\mathbb{R}_+$-valued, and independent of K_M, with $\mathbb{E}[Y_M^2] = \sigma_M^2 < \infty$ and $\mathbb{E}\, Y_M = \mu_M > 0$. Let $Z_M := \sum_{i=1}^{K_M} Y_{M,i}$. Assuming that

$$\lim_{M\to\infty} \frac{\sigma_M^2 p_M}{\mu_M^2} = 0,$$

it is the case that, as $M \to \infty$,

$$\frac{p_M Z_M}{\mu_M} \xrightarrow{\mathrm{d}} \mathcal{E}_1.$$

Ideas of the proofs: Critical case

The case that is most delicate is the critical case where $M \sim \rho N^2$. The extinction probability estimate is now:

Proposition.

Suppose that $\lim_{M\to\infty} M/N_M^2 = \rho \in (0,\infty)$. *Then,*

$$\theta(N_M, M) = (4/N_M)(1+o(1))H(\rho), \text{ as } M \to \infty.$$

Moreover, for any $s_0 \in (0,\infty)$, *as* $M \to \infty$, *uniformly for* $s \in (0, s_0]$,

$$\mathbb{E}^{N_M,M}[\mathrm{e}^{s\nu/N_M^2} \mid \nu \le M] = 1 + \frac{4s}{N_M}(1+o(1)) \int_0^\rho \mathrm{e}^{sy}\big(H(y) - H(\rho)\big)\mathrm{d}y.$$

The delicate nature is because, on the critical scale, the two-boundary nature of the problem has an impact (unlike the meagre-capacity regime), while extinction is sufficiently likely that the largest individual excursion fluctuations are on the same scale as the total lifetime (unlike the confined-space regime).

Outline

Random walk models for animal–resource dynamics

Animal movement in ecology is central to phenomena including population structure and dynamics, dispersion patterns, foraging, herding, territoriality, and other aspects of the behaviour of animals and their interactions with and responses to their environment.

For some approaches and overviews, we mention:

- Codling, Plank & Benhamou. Random walk models in biology. *J. R. Soc. Interface* **5** (2008) 813–834.
- Garlick, Powell, Hooten & McFarlane. Homogenization of large-scale movement models in ecology. *Bull. Math. Biol.* **73** (2011) 2088–2108.
- Giuggioli & Bartumeus. Animal movement, search strategies, and behavioural ecology: A cross-disciplinary way forward. *J. Animal Ecology* **79** (2010) 906–909.
- Hooten, Johnson, McClintock & Morales. *Animal Movement: Statistical Models for Telemetry Data*. CRC Press, 2017.
- Prins & Langevelde (eds.) *Resource Ecology.* Springer, 2008.
- Stephens, Brown & Ydenberg. *Foraging: Behavior and Ecology*. University of Chicago Press, 2007.
- Viswanathan, da Luz, Raposo & Stanley. *The Physics of Foraging.* Cambridge University Press, Cambridge, 2011.

Random walk models for animal–resource dynamics

There is a great deal of work to try to incorporate more realistic aspects of animal behaviour, such as interaction with resource, memory and persistence, intermittent rest periods, or anomalous diffusion. For example:

- Berbert & Lewis. Superdiffusivity due to resource depletion in random searches. *Ecological Complexity* **33** (2018) 41–48.
- Tilles, Petrovskii & Natti. A random walk description of individual animal movement accounting for periods of rest. *R. Soc. Open Sci.* **3** (2016) 160566.

Some adjacent models to our energy-constrained walker include "mortal random walks" and "starving random walks":

- Balakrishnan, Abad, Abil & Kozak. First-passage properties of mortal random walks: Ballistic behavior, effective reduction of dimensionality, and scaling functions for hierarchical graphs. *Phys. Rev. E* **99** (2019) 062110.
- Yeakel, Kempes & Redner. Dynamics of starvation and recovery predict extinction risk and both Damuth's law and Cope's rule. *Nature Communications* **9** (2018) 657.

In neither of these does the walker carry an internal energy state.

Random walk models for animal–resource dynamics

Various other models incorporate resource depletion by feeding:

- Bénichou, Bhat, Krapivsky & Redner. Optimally frugal foraging. *Phys. Rev. E* **97** (2018) 022110.
- Bénichou, Chupeau & Redner. Role of depletion on the dynamics of a diffusing forager. *J. Phys. A: Math. Theor.* **49** (2016) 394003.
- Chupeau, Bénichou & Redner. Universality classes of foraging with resource renewal. *Phys. Rev. E* **93** (2016) 032403.
- Grebenkov. Depletion of resources by a population of diffusing species. *Phys. Rev. E* **105** (2022) 054402.

Extension of the energy-constrained walker

We believe that our main results should, in part, extend to a much more general class of domains. Take a fixed smooth, simply-connected domain $\mathcal{D} \subseteq \mathbb{R}^d$, and define $\mathcal{D}_N := (N\mathcal{D}) \cap \mathbb{Z}^d$, the lattice domain corresponding to an N-scale copy of $\mathcal{D}$. The boundary is $\partial\mathcal{D}_N := \{x \in \mathcal{D}_N : \|x - y\| \leq 1 \text{ for some } y \in \mathbb{R}^d \setminus \mathcal{D}_N\}$.

Now take X_n to be reflected simple symmetric random walk on $\mathcal{D}_N$, with energy η_n that decreases in the interior $\mathcal{D}_N \setminus \partial\mathcal{D}_N$ and is replenished at the boundary $\partial\mathcal{D}_N$.

We expect that the regimes $M \ll N^2$ and $M \gg N^2$ display universal behaviour, in the sense that in the meagre-capacity regime λ/M has a $\mathrm{DM}(1/2)$-limit, while in the confined-space regime there is an exponential limit for λ after a suitable scaling. The critical case looks most challenging.

Resource-depletion and self-organized criticality

In our model, the energy supply in inexhaustible. Suppose instead that resource is depleted once consumed by the walker, leading to the domain to expand as the walker must forage further and further to find resources.

A model of this type was introduced by Bénichou, Chupeau & Redner (2016). Take the integer lattice $\mathbb{Z}^d$ ($d = 1$ for sake of comparison). Initially each site of the lattice carries a unit of "food". Once the food is consumed, the site is "depleted".

The walker has energy reserve with capacity M, and uses one unit of energy each time it visits a depleted site. When it visits a site containing food, the energy is restored to level M and the food is consumed.

Now the domain where the walker is in danger of extinction grows with time as the food is consumed. There are two variables of interest: λ (lifetime) and final size of the depleted territory, F, say. Under some conditions, Bénichou, Chupeau & Redner (BCR) give an argument that $\mathbb{E}\,\lambda \approx M$, while $\mathbb{E}\,F \approx M^{1/2}$.

Resource-depletion and self-organized criticality

In some sense the model of BCR is a "self-organized" version of our model; the only parameter is M, and there is no N.

Indeed, every time the walker visits a new food source, it will typically enlarge the depleted domain by size $O(1)$ before embarking on an excursion into the depleted domain.

While the domain remains of size $\ll M^{1/2}$, the walker is very likely to survive the excursion.

Thus the model "self-organizes" to the critical case $F \approx M^{1/2}$. At criticality, distributional behaviour may be non-universal, but at least in the case $d = 1$, one may expect a relative of our $H(\rho)$, perhaps mixed over some distribution for ρ, in the limit for λ.

Outline

Concluding remarks

- Energy-constrained walk in d-dimensional domain is a reflecting random walk (with some degeneracies) in $d + 1$ dimensions.
- Apart from total lifetime, could consider other statistics, e.g., terminal location.
- Several variations in the energy constraint could be considered. E.g., unbounded energy capacity, but each replenishment draws from a given distribution. The simplest version of this process is essentially classical random walk/ruin problem, but other variations can be considered.
- Several other interesting models of random walks interacting with their environment, motivated by ecology.

Thank you!

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