Imprecise probability inference on masked multicomponent system

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Abstract. Outside of controlled experiment scope, we have only limited information available to carry out desired inferences. One such scenario is when we wish to infer the topology of a system given only data representing system lifetimes without information about states of components in time of system failure, and only limited information about lifetimes of the components of which the system is composed. This scenario, masked system inference, has been studied before for systems with only one component type, with interest of inferring both system topology and lifetime distribution of component composing it. In this paper we study similar scenario in which we consider systems consisting of multiple types of components. We assume that distribution of component lifetimes is known to belong to a prior-specified set of distributions and our intention is to reflect this information via a set of likelihood functions which will be used to obtain an imprecise posterior on the set of considered system topologies.

Keywords: System reliability · Masked system · Topology inference · Survival signatures · Imprecise likelihood

1 Introduction

Masked system inference concerns about carrying out inferences about the underlying system model from system failure time observations, rather than the more commonly studied situations where life test data is available on components. Our inference may concern lifetime distributions of system components and/or structure of the system. Also, the prior information may be available in various forms and sometimes prevents us from constructing suitable prior distributions for Bayesian inference.

We will study here a scenario in which we wish to infer unknown structure of the system from masked system lifetimes given prior distribution on system structure and a set of credible component lifetime distributions. System structures will be specified by survival signatures (introduced in [4]) and we will use theory of imprecise probabilities (IP; more in [3]) to describe and obtain inference results.

System reliability inferences with survival signatures based on component failure observations were described in [2] and further extended for IP framework in [5]. Masked system structural inference in Bayesian framework for single component type systems were studied by Aslett in [1], where further elaboration of the nature of inference on masked system with uncertain structure and its numerical solution by Monte Carlo algorithms is presented.

2 Problem setting

2.1 Masked system inference

Let Ω_S be a set of considered systems. We model underlying distribution of component lifetimes with a parametric model and we index collection of component lifetime distributions by multi-parameter $\theta \in \Omega_\Theta$. For each combination of system s and set of distributions indexed by θ we assume that we can calculate the system survival function $R(t|s,\theta) = Pr(T_{sys} > t)$.

We further assume that the observables, D, are distributed according to system lifetime distribution (elements d_i represent observations of system failure times, r.v. T_{sys}). With additional assumptions about dependency among observations (e.g. i.i.d.), we can construct the observation model $f(d|\theta,s) \triangleq \mathcal{L}(\theta,s;d)$, for inference purposes:

$$\mathcal{L}(s,\theta;D) = \prod_i f(d_i|s,\theta) = \prod_i \left(- \left[\frac{\partial}{\partial_t} R_{sys}(t|s,\theta) \right]_{t=d_i} \right), \tag{1}$$

where specific form of $f(d_i|\theta,s)$ depends on our system model and shall be given by equation 5.
The system design is considered unknown and is therefore included in the likelihood, which then enables joint inference about the reliability and the topology of the system.

2.2 Imprecise probability inference of masked systems

In IP inference we operate with set of models (set of priors, set of likelihoods). For each of singular model of this set, we can carry out standard inference and analyse the collection partial results. If our aim is to infer probability of some event of interest, in IP scenario we can calculate the bounds for coherent inferences - lower and upper probabilities, where lower probability is minimal inferred probability over the models in the set, and similarly for the upper probability.

In system inference with uncertainty about both component lifetime distributions and system structure, we can choose different uncertainty models for these respective variables. By imprecision we model situations in which we know only possible domain of random variable and are unable to specify prior distribution for Bayesian inference. Such case will lead to IP inference, where each particular value of imprecise random variable defines a stochastic model on which standard Bayesian inference can be performed and the results integrated.

In our case, we assume that we can construct prior distribution on system structures and know only set in which component lifetime distribution parameter θ lies. Lower bound on posterior predictive survival function can be obtained as:

$$\underline{Pr}(T_{sys} > t|D = d) = \min_{\theta \in \Omega_\Theta} \int_s R_{sys}(t|s, \theta) \frac{\mathcal{L}(s, \theta; d)}{Z(\theta, d)} f(s|\theta) ds, \tag{2}$$

where R_{sys} is the system lifetime survival function, $\mathcal{L}$ is the likelihood function described in equation 1, f is prior density for Bayesian inference and factor Z is for posterior distribution normalization, i.e. $Z(\theta, d) = \int_s \mathcal{L}(s, \theta; d) f(s|\theta) ds$. Upper bound is obtained via maximization of the same expression.

Similarly we can also introduce the lower posterior distribution on system structure as:

$$\underline{f}(S = s|D = d) = \min_{\theta \in \Omega_\Theta} \frac{\mathcal{L}(s, \theta; d)}{Z(\theta, d)} f(s|\theta), \tag{3}$$

with respective maximizations in case of upper bound.

2.3 Survival signatures for system state modelling

Via component state space decomposition, we can express the system survival function for systems consisting of K distinct types of components, with M_k components of type k with i.i.d. lifetimes for each component type k, as:

$$\begin{aligned} R_{sys}(t|s, \theta) &= \sum_{\boldsymbol{l}} Pr(T_{sys} > t|\boldsymbol{C}(t) = \boldsymbol{l}, s, \theta) Pr(\boldsymbol{C}(t) = \boldsymbol{l}|s, \theta) \\ &= \sum_{\boldsymbol{l}} \phi_s(\boldsymbol{l}) \prod_{k=1}^{K} \binom{M_k}{l_k} R_{\theta,k}^{l_k}(t) F_{\theta,k}^{M_k - l_k}(t), \end{aligned} \tag{4}$$

where $\phi_s(\boldsymbol{l}) = Pr(T_{sys} > t|\boldsymbol{C}(t) = \boldsymbol{l}, s)$ is called survival signature of system s, random vector $\boldsymbol{C}(t)$ represents number of functioning components of each respective type at time t (i.e. C_i is number of functioning components of type i), summation is over all possible combinations $\boldsymbol{l}$ of numbers of functioning component of each type. Survival functions $R_{\theta,k}$ and cumulative distribution functions (CDFs) $F_{\theta,k}$ indexed by component type k and (multi-)parameter θ denote respective lifetime distribution characteristics for distinct component types.

The single observation density for systems described by survival signatures is therefore given by:

$$f(d_i|s,\theta) = -\left[\frac{\partial}{\partial_t} R_{sys}(t|s,\theta)\right]_{t=d_i} =$$

$$-\sum_{\boldsymbol{l}} \phi_s(\boldsymbol{l}) \sum_{k=1}^{K} \left\{ \binom{M_k}{l_k} \left[\frac{\partial}{\partial_t}\left(R_{\theta,k}^{l_k}(t) F_{\theta,k}^{M_k-l_k}(t)\right)\right]_{t=d_i} \prod_{k\neq j=1}^{K} \binom{M_j}{l_j} R_{\theta,j}^{l_j}(d_i) F_{\theta,j}^{M_j-l_j}(d_i) \right\} \tag{5}$$

$$= \sum_{\boldsymbol{l}} \left\{ \phi_s(\boldsymbol{l}) \prod_{k=1}^{K} \left[\binom{M_k}{l_k} F_{\theta,k}^{M_k-l_k}(d_i) R_{\theta,k}^{l_k}(d_i)\right] \sum_{k=1}^{K} \left[\left(\frac{l_k}{R_{\theta,k}(d_i)} - \frac{M_k - l_k}{F_{\theta,k}(d_i)}\right) f_{\theta,k}(d_i)\right] \right\},$$

where $f_{\theta,k}(.)$ is probability density function of kth component type lifetime.

We have derived everything necessary to be able to compute both the imprecise posterior and posterior predictive distributions in the setting where only masked system lifetime data are available and when the system design may be unknown. This allows us to perform joint inference on the component lifetime parameters and the topology of the system using imprecise probability.
In the remainder of the paper we will demonstrate the method for inference of system structure and predictive system lifetime. Inference on λ_2 is not further considered in this paper.

3 Examples

In the experiments, we shall assume that the real system structure is one of those described by survival signatures in Table 1 (those are all simply connected systems of order 4, as defined and listed in [1], each with a random component type assignment). These systems consists of $K = 2$ types components, 2 components of each type ($M_1 = M_2 = 2$). Underlying component type lifetime distributions are assumed to be exponential with rates $\lambda_1 = 0.45$ and $\lambda_2 \in [0.06, 1.12]$ ($\Omega_\Theta = \Omega_{\Lambda_1} \otimes \Omega_{\Lambda_2}$). Prior distribution on systems ($f(s|\theta)$ in equations 2 and 3) is chosen to be uniform for all choices of λ_2 .
The data, observed system failures, for experiments are simulated from system labeled as 6, which will be hereon referred to as the "ground truth" system. Ground truth hazard rate for components of type 2 is chosen to be $\lambda_2 = 0.32$.

C1	C2	1	2	3	4	5	**6**	7	8	9	10	11
0	1	0.00	0.00	0.00	0.00	0.00	**0.00**	0.00	0.50	0.50	0.50	1.00
0	2	0.00	0.00	0.00	1.00	0.00	**0.00**	1.00	1.00	1.00	1.00	1.00
1	0	0.00	0.00	0.00	0.00	0.00	**0.00**	0.00	0.00	0.00	0.50	1.00
1	1	0.00	0.00	0.25	0.50	0.50	**0.75**	0.50	0.50	0.75	1.00	1.00
1	2	0.00	0.50	1.00	1.00	1.00	**1.00**	1.00	1.00	1.00	1.00	1.00
2	0	0.00	0.00	0.00	0.00	0.00	**0.00**	1.00	0.00	1.00	1.00	1.00
2	1	0.00	0.50	0.50	0.50	1.00	**1.00**	1.00	1.00	1.00	1.00	1.00
2	2	1.00	1.00	1.00	1.00	1.00	**1.00**	1.00	1.00	1.00	1.00	1.00

Table 1. Survival signatures of systems in Ω_S. Zero row is being omitted ($\phi_s(\mathbf{0}) = 0$).

3.1 Survival predictions are not monotonic, nor convex in λ_2

Since the predictions are defined by their bounds, it is necessary to acquire them by optimization. Optimization problems are greatly simplified for monotonic functions (we only need to investigate bounds of the set) and/or convex/concave functions (where efficient gradient based algorithms may be employed). Although the survival function predictions are monotone in case of know system structure ($|\Omega_s| = 1$), neither of these desired properties could be proven analytically in general for predictions with unknown system structure. Conducted experiment (see Figure 1) provides a counterexample for monotonicity, convexity and concavity of posterior predictions in case of unknown system structure. Furthermore, Figure 3 provides a counterexample for the same in case of system structure posterior inference.

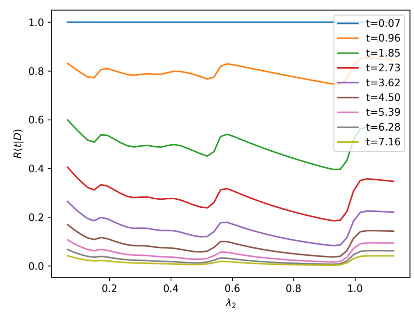

Fig. 1. Dependency of posterior survival function predictions for various selected times on imprecise λ_2 obtained by analysing 250 data samples

3.2 Imprecise structure posterior and system identification

Two basic inferences of our interest are for the system lifetime survival function (via equation 2), and for posterior system distribution (via equation 3). An example of predictive and structure inferences are shown in Figure 2. On the left side, the intervals for each system represents lower and upper bounds for posterior on the set Ω_Θ. On the right picture, one set of prediction bounds for prior distribution on system structures (before updating by observations) and another for posterior obtained via Bayesian updating are compared with the Kaplan-Meier estimate and the ground truth survival function.

The system identification, which would be done by comparing system posterior probabilities in Bayesian decision making, has to be done in IP setting. As can be seen in Figure 2, left, upper probabilities for multiple systems approach 1 in this experiment whilst the lower remain near 0. Therefore, there are several systems for which we are indecisive. The explanation of this wide range is illuminated in Figure 3, where we plot system posterior distributions obtained for various fixed λ_2 by standard Bayesian inference (i.e. inner function which is optimized in equation 3). In different regions of Ω_Θ, one system becomes dominant over others and this effect is further increased with increasing sample size.

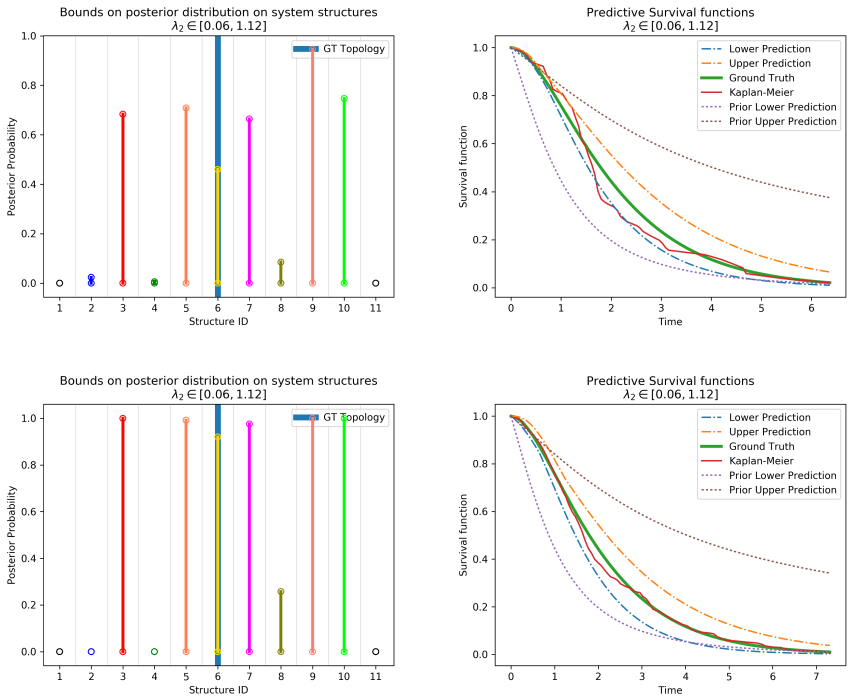

Fig. 2. Inference results with imprecise distribution parameter λ_2 for sample sizes 50 (top) and 250 (bottom). Left: imprecise posterior distribution on systems. Right: predictions of system lifetime survival function.

We can observe that an useful informative inference, we might obtain in IP setting, is that of rejection of several system structures. As is apparent from Figure 2 and also from Figure 4, upper posterior probability for some of the systems tends to approach 0, which indicates their unfitness to observations. Although, further analyses have to be performed to investigate properties of these rejections in IP decision-making framework(s), which is out of the scope of this paper.

3.3 Response to varying the support of λ_2

Next example investigates differences between disjoint choices of underlying support set Ω_Θ. We perform two imprecise inferences separately for λ_2 support divided by the value of (known) λ_1. The resulting imprecise system posteriors are shown in Figure 4.

From Figure 4 it is apparent, that some structures like 3 and 10, which were comparable by the means of inference in original support set (Figure 2, left), exhibit significant differences in case when the support is focused because the likelihood

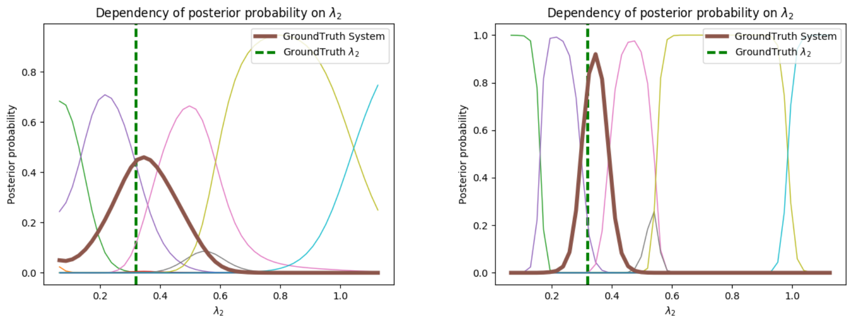

Fig. 3. Dependency of system structure posterior distributions on fixed λ_2 . Each vertical slice at selected λ_2 represents system posterior distribution (i.e. sums to 1). Left image for 50 data samples, right for 250. Thick curve denotes the evolution of posterior distribution of the ground truth system.

of these systems is small in these regions (see Figure 3). Similar behaviour was also observed in case of simply narrowing the λ_2 support where upper posterior probability of many systems approached 0. These results are being omitted here due to space limitations.

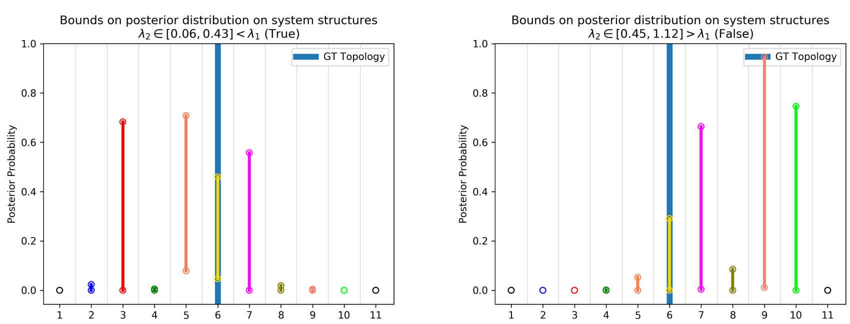

Fig. 4. Influence of choice of the support set for λ_2 on structure posterior distribution. In the left picture, the GT λ_2 lies in investigated set, in the right one it does not.

This scenario might be applicable for purposes of experimental design towards inference about adversarial systems. Proper choice of the support set Ω_{Θ}, and therefore the experimental settings, seems to influence identifiability of underlying unknown system structure.

4 Concluding remarks

We have demonstrated a novel methodology for inference in limited prior knowledge scenario, which allows us to avoid introducing some redundant and possibly unjustified modelling assumptions.

For the described situation, we have shown that the optimized functions of interest are nor monotone nor convex and, so far, have to be solved by general optimization procedures (in section 3.1).
It has also been indicated in section 3.2 and 3.3, that IP inference cannot generally serve for proper system identification, as IP reasoning allows for indecisiveness, but rather as a tool for system rejection in case of low upper posterior probability.

The behaviour which was presented was observed among multiple experiments that were conducted, although no analytical guarantees may be given at this stage of research. A follow-up generalizing study which would take into account even aspects which were only touched here (symmetrical properties of systems and rigorous IP decision theory) is necessary to further understand advantages and limitations of proposed methodology.

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