Adaptive (opportunity-based) age replacement strategies

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Abstract. In this paper, we summarize some recent results for age replacement based on nonparametric predictive inference, the details of which are presented elsewhere. We also consider opportunity-based age replacement, where preventive replacement can only take place at randomly occurring opportunities. The method is fully adaptive to available failure data, providing an alternative to the classical approach where the probability distribution of a unit's time to failure is assumed to be known.

1. Introduction

In this paper we summarize recent results from our development of age replacement strategies which are fully adaptive to failure data, as based on nonparametric predictive inference (NPI), see Coolen, *et al.* (2002) for an introduction to NPI in reliability. We use the classical age replacement problem formulation based on renewal theory, see e.g. Barlow and Proschan (1965), but instead of assuming a known probability distribution for the time to failure of a unit, we use imprecise predictive survival functions for the time to failure of the next unit, based on failure times of n previous units.

In Section 2, we briefly introduce the aspects of NPI as needed in this paper. In Section 3 we present key results for age replacement, taken from Coolen-Schrijner and Coolen (2004a). In Section 4, we consider opportunity-based age replacement, where preventive replacement of a unit is only possible at random moments which occur as a Poisson process. These results are taken from Coolen-Schrijner, *et al.* (2004), the classical theory for opportunity-based age replacement was presented by Dekker and Dijkstra (1992). Our results are illustrated via a short example in Section 5. Finally, in Section 6 we discuss some further aspects, including some comments on the use of this method and on other related recent research results.

2. Nonparametric Predictive Inference

Nonparametric predictive inference (NPI) is suitable for probabilistic predictions in case one wishes to add very little extra information to observed data. See Coolen, *et al.* (2002) for further details, discussion, and historical background of NPI. Denoting n ordered observed failure times by $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$, direct probabilities for a future lifetime, X_{n+1}, are specified by

$$P(X_{n+1} \in (x_{(j)}, x_{(j+1)})) = \frac{1}{n+1},$$

for $j = 0, \ldots, n$, where, for ease of notation, $x_{(0)} = 0$ and $x_{(n+1)} = \infty$, or $x_{(n+1)} = r$ if we can safely assume a finite upper bound r for the support of X_{n+1}.

Using NPI, one typically does not derive precise probabilities for events of interest. However, it does provide optimal bounds for all probabilities of interest involving X_{n+1}. Such bounds are lower and upper probabilities within the theory of imprecise probability (Coolen, 2004). In this paper, NPI-based lower and upper predictive survival functions are used to provide adaptive age replacement strategies. At previous observations these lower and upper predictive survival functions are equal, as NPI provides precise probabilities for events $X_{n+1} > x_{(j)}$ for $j = 1, \ldots, n$, and have the value

$$\underline{S}_{X_{n+1}}(x_{(j)}) = \overline{S}_{X_{n+1}}(x_{(j)}) = \frac{n+1-j}{n+1}.$$

At other times, the NPI-based lower predictive survival function for the failure time X_{n+1} of the next unit is

$$\underline{S}_{X_{n+1}}(x) = S_{X_{n+1}}(x_{(j+1)}) = \frac{n-j}{n+1} \text{ for } x \in (x_{(j)}, x_{(j+1)}),\ j = 0, \ldots, n,$$

and the corresponding upper predictive survival function is

$$\overline{S}_{X_{n+1}}(x) = S_{X_{n+1}}(x_{(j)}) = \frac{n+1-j}{n+1} \text{ for } x \in (x_{(j)}, x_{(j+1)}),\ j = 0, \ldots, n.$$

3. The age replacement model

In this section, we summarize some key results from Coolen-Schrijner and Coolen (2004a), to which we refer for detailed justifications and discussion. In a basic age replacement model (AR), as e.g. presented by Barlow and Proschan (1965), a unit (e.g. a system in a production process) is correctively replaced upon failure, at a cost c_f, or preventively upon reaching age T, at a cost $c_p < c_f$, whichever occurs first. In the classical setting, a unit's time to failure is represented by a random quantity, say X, with an assumed known probability distribution, with survival function $S_X(x) = P(X > x)$. Let $C(T)$ be the long-run average cost per unit time under this policy, $R(T)$ the cost per cycle (the random period between two replacements), and $L(T)$ the length of a cycle, then the renewal reward theorem gives

$$C(T) = \frac{E[R(T)]}{E[L(T)]} = \frac{c_f - (c_f - c_p)S_X(T)}{\int_0^T S_X(x)dx}.$$

We replace the assumed known survival function for the time to failure by the NPI-based lower and upper survival functions, which fully adapt to available failure data. Using NPI, the optimal lower bound $\underline{C}_{X_{n+1}}(T)$ for the cost function for the age replacement decision for the next unit, based on n observed failure times, is obtained by replacing $S_X(\cdot)$ by the NPI-based upper survival function $\overline{S}_{X_{n+1}}(\cdot)$, and the optimal upper bound $\overline{C}_{X_{n+1}}(T)$ is obtained by using $\underline{S}_{X_{n+1}}(\cdot)$, leading to

$$\underline{C}_{X_{n+1}}(T) = \frac{jc_f + (n+1-j)c_p}{(n+1-j)T + \sum_{l=1}^{j} x_{(l)}}, \quad \text{for } T \in [x_{(j)}, x_{(j+1)}),\ j = 0, \ldots, n,$$

and

$$\overline{C}_{X_{n+1}}(T) = \frac{(j+1)c_f + (n-j)c_p}{(n-j)T + \sum_{l=1}^{j} x_{(l)}}, \quad \text{for } T \in (x_{(j)}, x_{(j+1)}],\ j = 0, \ldots, n.$$

These $\underline{C}_{X_{n+1}}(\cdot)$ and $\overline{C}_{X_{n+1}}(\cdot)$ are both discontinuous at the observed failure times $x_{(j)}$, but in between these failure times they are continuous and strictly decreasing. At the $x_{(j)}$, $\underline{C}_{X_{n+1}}(\cdot)$ is continuous from the right and $\overline{C}_{X_{n+1}}(\cdot)$ is continuous from the left, so the global minimum of $\underline{C}_{X_{n+1}}(\cdot)$ on $(0, x_{(n)}]$ is assumed in one of the $x_{(j)}^-$, $j = 1, \ldots, n$, while the global minimum of $\overline{C}_{X_{n+1}}(\cdot)$ is assumed in one of the points $x_{(j)}$, $j = \ldots, n$. Here $x_{(j)}^-$ is to be interpreted as 'just before $x_{(j)}$', such that the adherent probability mass (see De Finetti (1974)) to the left of $x_{(j)}$ is considered to be to the right of $x_{(j)}^-$ in the extreme situation related to the location of the probability masses corresponding to $\overline{S}_{X_{n+1}}(\cdot)$. The $\overline{C}_{X_{n+1}}(x_{(j)})$ are given above, and

$$\underline{C}_{X_{n+1}}(x_{(j)}^-) = \frac{(j-1)c_f + (n+2-j)c_p}{(n+2-j)x_{(j)} + \sum_{l=1}^{j-1} x_{(l)}}.$$

If we assume a known upper bound r for the support of X_{n+1}, $\underline{C}_{X_{n+1}}(\cdot)$ is also strictly decreasing on $(x_{(n)}, r)$, so we must also consider $\underline{C}_{X_{n+1}}(r^-)$ for finding the global minimum on $(0, r]$. These cost functions, and corresponding optimal age replacement strategies, are illustrated via an example in Section 5. For a detailed study, via simulations, of the adaptive nature of such NPI-based age replacement strategies, we refer to Coolen-Schrijner and Coolen (2004a), the main conclusion of that study is that this method works quite well in most cases for fairly small data sets ($n = 10$), with of course increasingly good performance for larger n.

4. The opportunity-based age replacement model

Full details and justifications of the results in this section are given in Coolen-Schrijner, *et al.* (2004). Dekker and Dijkstra (1992) introduced the opportunity-based age replacement model (OAR), where it is acknowledged that it may not be possible to carry out preventive replacement at any moment in time. For example, if a unit is continuously in use in a production process, preventive replacement may have to be delayed to moments when the production is interrupted, for example due to other units breaking down. We assume that preventive replacement is only possible at opportunities which occur according to a Poisson process with rate λ.

The OAR rule prescribes preventive replacement at the first opportunity after age T, at a cost c_p, or corrective replacement upon failure, at a cost $c_f > c_p$, whichever occurs first. Let $C_{op}(T)$ be the long-run average cost per unit time under OAR, $R_{op}(T)$ the cost per cycle, and $L_{op}(T)$ the length of a cycle. Let Y denote the time to the next preventive replacement opportunity after time T, with probability density function $f_Y(\cdot)$. Our assumption of these opportunities occurring as a Poisson process implies that Y is independent of T, and has an Exponential distribution with expected value $1/\lambda$. To derive the long-run average cost per unit of time, for OAR of a unit with time to failure X, we use the renewal reward theorem, with

$$E[R_{op}(T)] = c_p E[P(X \geq T+Y)] + c_f E[P(X < T+Y)] = c_f - (c_f - c_p) E[S_X(T+Y)]$$

and

$$\begin{aligned} E[L_{op}(T)] &= E[\min(X, T+Y)] = \int_0^\infty E[\min(X, T+Y)|Y=y] f_Y(y) dy \\ &= \int_{y=0}^{\infty} \int_{x=0}^{T+y} (1 - F_X(x)) f_Y(y) dx dy = E[\min(X,T)] + \int_{y=0}^{\infty} \int_{x=T}^{T+y} S_X(x) f_Y(y) dx dy \\ &= \int_0^T S_X(x) dx + \int_0^\infty S_X(T+x) S_Y(x) dx = \int_0^T S_X(x) dx + E[Y] E[S_X(T+Y)]. \end{aligned}$$

Hence, the long-run average cost per unit time under the OAR rule is (Dekker and Dijkstra (1992))

$$C_{op}(T) = \frac{c_f - (c_f - c_p) E[S_X(T+Y)]}{\int_0^T S_X(x) dx + E[Y]\, E[S_X(T+Y)]}.$$

Substituting the NPI-based upper and lower survival functions of the time to failure of the next unit for $S_X(\cdot)$, we obtain the optimal NPI-based lower and upper bounds for the OAR cost function, which we denote by $\underline{C}_{X_{n+1},op}(T)$ and $\overline{C}_{X_{n+1},op}(T)$, respectively. For $T \in [x_{(j)}, x_{(j+1)})$ and $j = 0, \ldots, n$, this lower bound is

$$\underline{C}_{X_{n+1},op}(T) = \frac{j c_f + (n+1-j) c_p + (c_f - c_p) \sum_{l=j+1}^{n+1} e^{-\lambda(x_{(l)} - T)}}{(n+1-j)(T + E[Y]) + \sum_{l=1}^{j} x_{(l)} - E[Y] \sum_{l=j+1}^{n+1} e^{-\lambda(x_{(l)} - T)}}.$$

For $T \in (x_{(j)}, x_{(j+1)}]$ and $j = 0, \ldots, n$, this upper bound is

$$\overline{C}_{X_{n+1},op}(T) = \frac{(j+1) c_f + (n-j) c_p + (c_f - c_p) \sum_{l=j+1}^{n} e^{-\lambda(x_{(l)} - T)}}{(n-j)(T + E[Y]) + \sum_{l=1}^{j} x_{(l)} - E[Y] \sum_{l=j+1}^{n} e^{-\lambda(x_{(l)} - T)}}.$$

Let $\underline{T}^j_{n+1,op}$ denote the minimum of $\underline{C}_{X_{n+1},op}(T)$ over the interval $[x_{(j)}, x_{(j+1)})$, and $\overline{T}^j_{n+1,op}$ the minimum of $\overline{C}_{X_{n+1},op}(T)$ over the interval $(x_{(j)}, x_{(j+1)}]$. Then the optimal NPI-based OAR strategies are

$$\underline{T}^*_{n+1,op} = \arg\{\min_{0\leq j\leq n} \underline{C}_{X_{n+1},op}(\underline{T}^j_{n+1,op})\} \text{ and } \overline{T}^*_{n+1,op} = \arg\{\min_{0\leq j\leq n} \overline{C}_{X_{n+1},op}(\overline{T}^j_{n+1,op})\}.$$

The values of $\underline{T}^j_{n+1,op}$ (and $\overline{T}^j_{n+1,op}$) can be obtained by use of the fact that these optima are the unique values in the relevant intervals where the NPI-based lower (upper) OAR cost function is equal to the corresponding NPI-based lower (upper) AR cost function, as presented in Section 3 (if there exists no such point, then one of the end points of the interval is the OAR minimum). This result from Coolen-Schrijner, *et al.* (2004), is similar to a result by Dekker and Dijkstra (1992) in the classical setting. This result implies that, in our NPI setting and restricting attention to an interval between two consecutive observed failure times, the local OAR minimum (both for the lower and upper cost functions) cannot exceed the local AR minimum. This does, however, not imply that the global minima are similarly related, which is illustrated in the example in Section 5. In the classical setting, if a probability distribution with increasing hazard rate, with sufficiently large limiting value, is assumed for the unit's time to failure, Dekker and Dijkstra (1992) show that the OAR optimal preventive replacement time is less than the corresponding AR optimum.

5. Example

We briefly illustrate the results of Sections 3 and 4 via an example (see also Coolen-Schrijner and Coolen (2004a) and Coolen-Schrijner, *et al.* (2004)). Suppose that we have five observed failure times: 4, 6, 10, 11 and 15, and that preventive replacement costs $c_p = 1$ and corrective replacement costs $c_f = 10$. The times that minimise the NPI-based lower and upper AR cost functions for the next unit are 4^- and 4, respectively, with $\underline{C}_{X_{n+1}}(4^-) = 0.2500$ and $\overline{C}_{X_{n+1}}(4) = 0.7500$.

For the opportunity-based age replacement problem, we assume that the opportunities occur according to a Poisson process with rate $\lambda = 2$. The times that minimise the NPI-based lower and upper OAR cost functions are 2.900 and 8.961, respectively, with $\underline{C}_{X_{n+1},op}(2.900) = 0.3449$ and $\overline{C}_{X_{n+1},op}(8.961) = 0.8947$. The fact that this upper OAR cost function is minimised at 8.961, whereas the corresponding AR upper cost function is minimised at 4, illustrates the comment at the end of Section 4, with the apparent disagreement with the result by Dekker and Dijkstra (1992) for the classical situation due to the adaptiveness of our method to the available data, where in this example the data do not strongly indicate a probability distribution for the time to failure with a hazard rate that increases everywhere. Figure 1 shows the NPI-based lower and upper AR and OAR cost functions for this example.

If the rate λ, at which the preventive replacement opportunities occur, increases, then the OAR lower and upper cost functions get closer to the corresponding AR cost functions. Also the optimal opportunity-based age replacement times converge to the corresponding optimal age replacement times, for large enough λ. It is clear that the AR situation occurs as the limit for OAR with $\lambda \to \infty$. For example, if $\lambda = 5$ then the optimal lower and upper OAR times are 3.35 and 3.52, respectively, with corresponding costs 0.2982 and 0.8512, while for $\lambda = 15$ these optimal OAR times are 3.70 and 3.76, respectively, with corresponding costs 0.2699 and 0.7969.

Concluding remarks

It is important to consider how the results presented in this paper can be used in practice. We do not advocate the use of our NPI-based strategies instead of optimal strategies corresponding to the classical approach. However, we do feel that using these together may provide valuable insights, if the optima differ substantially then it is most likely that the failure data available do not support an assumed probability distribution in the classical approach. Of course, our approach requires available failure data from the same process, and for similar units, which might often not be realistic. Hence, in situations without such data, one has to rely on distributional assumptions for the unit's time to failure.

Although the lower and upper cost functions in our approach are indeed optimal bounds for any cost function corresponding to a probability distribution consistent with NPI, this does not imply that

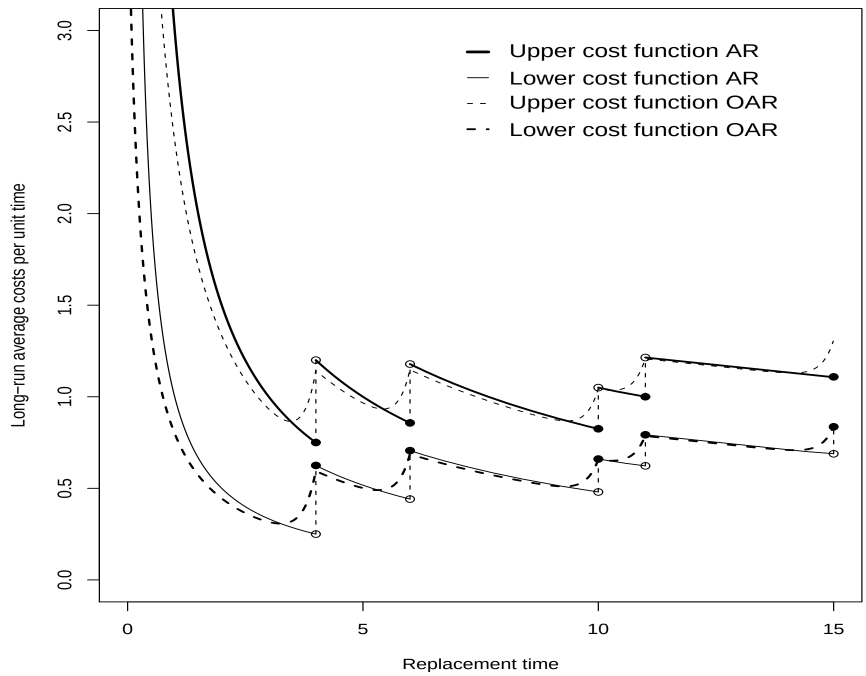

Figure 1: NPI-based lower and upper cost functions for AR and OAR

the optimum of any such cost function is in between the optima of the lower and upper cost functions. However, if the number of observed failure times increases, the NPI lower and upper survival functions will converge to an underlying survival function for a population of such units, and hence our lower and upper cost functions will also converge to each other.

It is interesting to consider the effect on the optimal NPI-based (opportunity-based) age replacement strategies for unit $n+2$, when using an optimal (opportunity-based) age replacement strategy for unit $n+1$, and the resulting information about the time to failure of X_{n+1} when unit $n+1$ is replaced. Under the (opportunity-based) age replacement strategy, the observation for X_{n+1} is either a failure time if the unit is replaced correctively, or a right-censored observation if the unit is replaced preventively. The effect of such further information, in particular the manner in which our NPI-based optimum preventive replacement times adapt to it, is studied via simulations for AR in Coolen-Schrijner and Coolen (2004b) and for OAR in Coolen-Schrijner, *et al.* (2004). The statistical method for NPI to deal with right-censored observations, that is needed to take observed preventive replacement times into account, is described in Coolen, *et al.* (2002), where further references are given.

We are also studying a different formulation of the age replacement problem, explicitly minimising expected costs per unit of time over a single cycle. From theoretical point of view, this may be more appropriate when using NPI-based strategies adapting to new information for each cycle, as the renewal reward theorem implicitly assumes that the same strategy will be used for many cycles. Adaptive replacement strategies based on such a one-cycle criterion have been developed, within the Bayesian framework, by Mazzuchi and Soyer (1996), but because the probability distribution for the time to failure of the unit considered is assumed to belong to a parametric family of distributions, such an approach is less adaptive to failure data than NPI-based strategies.

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