Nonparametric Predictive Inference for Multiple Comparisons

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Dedicated

To my mother for her unlimited love and care

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Abstract

This thesis presents Nonparametric Predictive Inference (NPI) for several multiple comparisons problems. We introduce NPI for comparison of multiple groups of data including right-censored observations. Different right-censoring schemes discussed are early termination of an experiment, progressive censoring and competing risks. Several selection events of interest are considered including selecting the best group, the subset of best groups, and the subset including the best group. The proposed methods use lower and upper probabilities for some events of interest formulated in terms of the next future observation per group. For each of these problems the required assumptions are Hill's assumption $A_{(n)}$ and the generalized assumption rc-$A_{(n)}$ for right-censored data.

Attention is also given to the situation where only a part of the data range is considered relevant for the inference, where in addition the numbers of observations to the left and to the right of this range are known. Throughout this thesis, our methods are illustrated and discussed via examples with data from the literature.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, UK. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all the author's original work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

1.1 Overview

This thesis presents Nonparametric Predictive Inference (NPI) for several comparisons problems. Mainly, we introduce NPI for multiple comparisons in situations with right-censored observations. Such data typically occur in reliability or survival analysis, due to several reasons. For example, when interest is in a specific failure mode for a technical unit, it may fail due to a different failure cause. If multiple failure modes are of interest, and failure will be due to only a single failure mode, then this situation is known as 'competing risks', where an observed failure time is actually a right-censoring time with regard to all failure modes that did not cause the failure. Another reason for right-censoring may be removal of units from a lifetime experiment, normally to save time or reduce costs, but this also occurs if, at some point, one wishes to study in more detail units which have not yet failed in an experiment. If right-censoring is due to an experiment being terminated before all units have failed, comparison of different groups of units based on such data is known as 'precedence testing'. If non-failing units are removed from the experiment at several possible stages it is known as 'progressive censoring'.

In this thesis, we develop NPI for multiple comparisons for precedence testing, progressive censoring, and competing risks. It should be emphasized that, throughout the thesis, unspecified reasons for right-censoring are assumed to be based on processes that are independent of the residual lifetimes of the censored units. We also present NPI for situations where the information available consists of precise measurements of real-valued data only within a specific range, with in addition the numbers of observations to the left and to the right of this range are known.

Section 1.2 provides a brief overview of some basic aspects of imprecise probability and the underlying assumption behind NPI, Hill's assumption $A_{(n)}$. In Section 1.3 we review briefly the main idea of NPI and discuss some applications which we will refer to later in the thesis. This includes the generalisation of the $A_{(n)}$ assumption needed to accommodate lifetime data, the so-called assumption rc-$A_{(n)}$. Finally, the outline of this thesis is given in Section 1.4.

1.2 Assumption $A_{(n)}$ and imprecise probability

In this section we briefly overview some basic aspects of imprecise probability and the underlying assumption behind NPI, Hill's assumption $A_{(n)}$ [40]. To introduce $A_{(n)}$ we first need to introduce some notation. Suppose that $X_1, \ldots, X_n, X_{n+1}$ are real-valued absolutely continuous and exchangeable random quantities. Let the ordered observed values of $X_1, X_2, \ldots, X_n$ be denoted by $x_1 < x_2 < \ldots < x_n$, and let $x_0 = -\infty$ and $x_{n+1} = \infty$ for ease of notation. We assume that no ties occur, the results can be generalised to allow ties [42], see also Subsection 1.3.5. Based on n observations, the assumption $A_{(n)}$ is that the probability that the next future observation X_{n+1} falls in the open interval $I_j = (x_j, x_{j+1})$ is $1/(n+1)$, for each $j = 0, 1, \ldots, n$ [40].

$A_{(n)}$ does not assume anything else, and can be considered to be a post-data assumption related to exchangeability [31]. Hill [41] discusses $A_{(n)}$ in detail. $A_{(n)}$ is not sufficient to derive precise probabilities for many events of interest, but it provides bounds for probabilities via the 'fundamental theorem of probability' [31], which are lower and upper probabilities in interval probability theory [76, 79].

Lower and upper probabilities generalise classical probabilities, and a lower (upper) probability for event A, denoted by $\underline{P}(A)$ ($\overline{P}(A)$), can be interpreted in several ways [21]: as supremum buying (infimum selling) price for a gamble on the event A, or as the maximum lower (minimum upper) bound for the probability of A that follows from the assumptions made. Informally, $\underline{P}(A)$ ($\overline{P}(A)$) can be considered to reflect the evidence in favour of (against) event A.

Interval probabilities, also know as imprecise probabilities, have been suggested in various areas of statistics. Recently increasing attention has been given to this topic area resulting in a series of conferences and a project website (The Society for Imprecise Probability: Theories and Applications - www.sipta.org). Walley [76, 77] extended the traditional subjective probability theory via buying and selling prices for gambles, whereas Weichselberger [78, 79] generalised Kolmogorov's axioms without imposing an interpretation.

Below we briefly present some elements of theory of interval probability as relevant to $A_{(n)}$-based inference. According to Weichselberger [78, 79], an axiomization of interval probability can be achieved by supplementing Kolmogorov's axioms as follows:

For a measurable space $(\Omega, \mathcal{A})$, a set function $p(.)$ on $\mathcal{A}$ satisfying Kolmogorov's axioms is called a *classical probability*. Let $\mathcal{K}(\Omega, \mathcal{A})$ be the set of all classical probabilities on $(\Omega, \mathcal{A})$. A function $[\underline{P}(.); \overline{P}(.)]$ on $\mathcal{A}$ is called an *F-probability* with structure $\mathcal{M}$, if

i) $P : \mathcal{A} \to \{[\underline{P}; \overline{P}] | 0 \leq \underline{P} \leq \overline{P} \leq 1\}$ and $A \mapsto [\underline{P}(A); \overline{P}(A)]$,

ii) $\mathcal{M} := \{p(.) \in \mathcal{K}(\Omega, \mathcal{A}) | \ \underline{P}(A) \leq p(A) \leq \overline{P}(A), \forall A \in \mathcal{A}\} \neq \emptyset$,

iii) For all $A \in \mathcal{A}$, $\inf_{p(.) \in \mathcal{M}} p(A) = \underline{P}(A)$ and $\sup_{p(.) \in \mathcal{M}} p(A) = \overline{P}(A)$.

For every F-probability, $\underline{P}(A)$ and $\overline{P}(A)$ are conjugated, i.e. $\underline{P}(A) = 1 - \overline{P}(A^c)$, where A^c is the complement of A.

1.3 Nonparametric Predictive Inference (NPI)

Inferences based on $A_{(n)}$ are predictive and nonparametric, and can be considered suitable if there is hardly any knowledge about the random quantity of interest, other than the n observations, or if one does not want to use such information, e.g. to study effects of additional assumptions underlying other statistical methods. Nonparametric Predictive Inference (NPI) is a statistical method based on Hill's assumption $A_{(n)}$ [40], which gives direct probabilities for a future observable random quantity, given observed values of related random quantities [1, 21]. NPI has been developed in recent years, mainly by Frank Coolen and Pauline Coolen-Schrijner and their collaborators and students, for different applications in statistics, reliability and operational research.

In NPI uncertainty is quantified by lower and upper probabilities for events of interest. Augustin and Coolen [1] introduced predictive lower and upper probabilities based on $A_{(n)}$ as follows:

Let $\mathcal{B}$ be the Borel σ-field over $\mathbb{R}$. For any element $B \in \mathcal{B}$, lower probability $\underline{P}(.)$ and upper probability $\overline{P}(.)$ for the event $X_{n+1} \in B$, based on the intervals $I_j = (x_j, x_{j+1})$ $(j = 0, 1, \ldots, n)$ created by n real-valued non-tied observations, and the assumption $A_{(n)}$, are

$$\underline{P}(X_{n+1} \in B) = \frac{1}{n+1}|\{j : I_j \subseteq B\}|$$
$$\overline{P}(X_{n+1} \in B) = \frac{1}{n+1}|\{j : I_j \cap B \neq \emptyset\}|$$

where $|A|$ is the cardinality of a set A, i.e. the number of elements contained in A. In other words, the lower probability $\underline{P}(X_{n+1} \in B)$ is achieved by taking only probability mass into account that is necessarily within B, which is only the case for the probability mass $\frac{1}{n+1}$ per interval I_j if this interval is completely contained within B. The upper probability $\overline{P}(X_{n+1} \in B)$ is achieved by taking all the probability mass into account that could possibly be within B, which is the case for the probability mass $\frac{1}{n+1}$, per interval I_j, if the intersection of I_j and B is non-empty.

Augustin and Coolen [1] showed that these bounds fit nicely into the framework of interval probability [78, 79]. They proved that, without adding any further assumptions, these $A_{(n)}$-based lower and upper probabilities are F-probability with structure

$$\mathcal{M} := \{p(.) \in \mathcal{K}(\mathbb{R}, \mathcal{B})|\, p(X_{n+1} \in I_j) = \frac{1}{n+1}, \forall j = 0, 1, \ldots, n\}.$$

By the nature of $A_{(n)}$, NPI is a frequentist statistical methodology [1, 40, 41], which however can be interpreted in a way similar to Bayesian statistics [21, 42]. An important advantage over more established frequentist methods is that NPI does not depend on counterfactuals, that is data which were not actually observed but could have been observed. For example, these are important in hypothesis testing, which has led to a large literature on frequentist methods for related problems considering slightly varying experimental procedures. In NPI, as in Bayesian statistics, the inferences only involve the actual data observed, although a warning is needed about the fact that, quite obviously, to apply NPI one must be happy with the exchangeability assumption on the data and future observation(s), which may be non-trivial depending on the experimental set-up.

1.3.1 NPI for multiple comparisons

For complete data, Coolen [19] introduced NPI for comparing two independent groups, say X and Y. In classical statistics these tend to be referred to as 'populations'. Throughout this thesis, we avoid the term 'populations' in NPI as we only consider one future observation and do not make use of any population distribution, even no assumptions about existence of such a distribution or about a meaningful population are made. Suppose that $X_1, \ldots, X_{n_x}, X_{n_x+1}$ and $Y_1, \ldots, Y_{n_y}, Y_{n_y+1}$ are real-valued absolutely continuous and exchangeable random quantities from X and Y, respectively. Let their ordered observed values be $x_1 < x_2 < \ldots < x_{n_x}$ and $y_1 < y_2 < \ldots < y_{n_y}$, and let $x_0 = y_0 = -\infty$ and $x_{n_x+1} = y_{n_y+1} = \infty$. Again we assume that no ties occur, the results can be generalised to allow ties [42].

Such comparisons focus on the next future observation from each group. The NPI lower and upper probabilities for the event that a future observation, X_{n_x+1}, of group X is less than a future observation, Y_{n_y+1}, of group Y (i.e. $X_{n_x+1} < Y_{n_y+1}$), based on n_x and n_y observations of group X and Y, and the assumptions $A_{(n_x)}$ for X_{n_x+1} and $A_{(n_y)}$ for Y_{n_y+1}, are

$$\underline{P}(X_{n_x+1} < Y_{n_y+1}) = \frac{1}{(n_x+1)(n_y+1)} \sum_{j=1}^{n_y} \sum_{i=1}^{n_x} \mathbf{1}\{x_i < y_j\} \tag{1.1}$$

$$\overline{P}(X_{n_x+1} < Y_{n_y+1}) = \frac{1}{(n_x+1)(n_y+1)} \left\{ \sum_{j=1}^{n_y} \sum_{i=1}^{n_x} \mathbf{1}\{x_i < y_j\} + n_x + n_y + 1 \right\} \tag{1.2}$$

where $\mathbf{1}\{E\}$ is an indicator function which is equal to 1 if event E occurs and 0 else. For these lower and upper probabilities the conjugacy property holds, that is for an event E and its complementary event E^c, $\underline{P}(E) = 1 - \overline{P}(E^c)$.

Throughout we assume that information on units from one group does not hold any information about units from the other group, so X_{n_x+1} and Y_{n_y+1} are independent and data from group X contain no information on Y_{n_y+1} and vice versa. We call this 'complete independence' of the groups.

Coolen and van der Laan [25] extended this to compare $k \geq 2$ groups with different events of interest including selection of the best group, the subset of best groups, and the subset that includes the best group.

Suppose we have $k \geq 2$ groups and $n_j + 1$ random quantities from group j, denoted by X_{j,i_j} where $i_j = 1, 2, \ldots, n_j, n_j + 1$, $j = 1, 2, \ldots, k$, and let for each group j $(j = 1, 2, \ldots, k)$ $x_{j,1} < x_{j,2} < \ldots < x_{j,n_j}$ be the ordered observed values and $x_{j,0} = -\infty$ and $x_{j,n_j+1} = \infty$. The inference depends on Hill's assumption $A_{(n_j)}$ [40] for each group j, as described before.

Coolen and van der Laan [25] presented the following NPI lower and upper probabilities for the event that a specific X_{l,n_l+1} is the maximum for all next observations X_{j,n_j+1}, $j = 1, \ldots, k$.

$$\underline{P}\left(X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}\right) = \frac{1}{\prod_{j=1}^{k}(n_j+1)}\left[\sum_{i_l=1}^{n_l}\prod_{\substack{j=1\\ j\neq l}}^{k}\sum_{i_j=1}^{n_j}\mathbf{1}\{x_{j,i_j} < x_{l,i_l}\}\right]$$

$$\overline{P}\left(X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}\right) = \frac{1}{\prod_{j=1}^{k}(n_j+1)}\left[\sum_{i_l=1}^{n_l}\prod_{\substack{j=1\\ j\neq l}}^{k}\left(1+\sum_{i_j=1}^{n_j}\mathbf{1}\{x_{j,i_j} < x_{l,i_l}\}\right)\right] + \frac{1}{n_l+1}$$

They also considered selection of a subset of groups such that all the groups in this subset are 'better' than all not selected groups, that is the next observation of each group in the subset is greater than the next observation of all groups not in the subset. Let $S = \{l_1, l_2, ..., l_m\}$ be a subset of m groups $(1 \leq m \leq k-1)$ from k independent groups and let NS be the complement set of S containing the remaining $k - m$ groups. Then the NPI lower and upper probabilities for the event that the next observation of each group in S is greater than the next observation of each group in NS, i.e. $\min_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}$, are [25]

$$\underline{P}\left(\min_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right) = \frac{1}{\prod_{j=1}^{k}(n_j+1)} \sum_{\substack{i_l=1\\ l\in S}}^{n_l} \prod_{j\in NS}\left[\sum_{i_j=1}^{n_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{x_{l,i_l}\}\}\right]$$

$$\overline{P}\left(\min_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right) = \frac{1}{\prod_{j=1}^{k}(n_j+1)} \sum_{\substack{i_l=1\\ l\in S}}^{n_l+1} \prod_{j\in NS}\left[1+\sum_{i_j=1}^{n_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{x_{l,i_l}\}\}\right]$$

where the notation $\sum_{\substack{i_l=a\\ l\in S}}^{b_l}$ is used for the m sums $\sum_{i_{l_1}=a}^{b_{l_1}} \ldots \sum_{i_{l_m}=a}^{b_{l_m}}$.

Using the same definitions of subsets S and NS, we can also be interested in selecting the subset S that contains the best group. Then the NPI lower and upper probabilities for the event that the next observation from (at least) one of the selected groups in S is greater than the next observation from each group in NS, i.e. $\max_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}$, are [25]

$$\underline{P}\left(\max_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right) = \frac{1}{\prod_{j=1}^{k}(n_j+1)} \sum_{\substack{i_l=0\\ l\in S}}^{n_l} \prod_{j\in NS}\left[\sum_{i_j=1}^{n_j} \mathbf{1}\{x_{j,i_j} < \max_{l\in S}\{x_{l,i_l}\}\}\right]$$

$$\overline{P}\left(\max_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right) = \frac{1}{\prod_{j=1}^{k}(n_j+1)} \sum_{\substack{i_l=1\\ l\in S}}^{n_l+1} \prod_{j\in NS}\left[1+\sum_{i_j=1}^{n_j} \mathbf{1}\{x_{j,i_j} < \max_{l\in S}\{x_{l,i_l}\}\}\right]$$

1.3.2 NPI for right-censored data

In reliability and survival analysis, data on event times, often called lifetime, are often affected by right-censoring, where for a specific unit or individual it is only known that the event has not yet taken place at a specific time. An observation for a unit or an individual is said to be right-censored at c when its lifetime is only known to be greater than c [48].

The assumption $A_{(n)}$ requires fully observed data, and cannot deal directly with right-censored data. Coolen and Yan [27] presented a generalisation of $A_{(n)}$, called right-censoring $A_{(n)}$ or rc-$A_{(n)}$, which is suitable for right-censored data. In comparison to $A_{(n)}$, rc-$A_{(n)}$ uses the extra assumption that, at the moment of censoring, the residual lifetime of a right-censored unit is exchangeable with the residual lifetimes of all other units that have not yet failed or been censored. Further details of rc-$A_{(n)}$ are given in [27]. To formulate the required form of rc-$A_{(n)}$, we need notation for probability mass assigned to intervals without further restrictions on the spread within the intervals. Such a partial specification of a probability distribution is called an M-function [27] which is given by the following definition.

Definition 1.1. A partial specification of a probability distribution for a real-valued random quantity X can be provided via probability masses assigned to intervals, without any further restriction on the spread of the probability mass within each interval. A probability mass assigned, in such a way, to an interval (a, b) is denoted by $M_X(a, b)$, and referred to as M-function value for X on (a, b).

Clearly, each M-function value should be in [0,1] and all M-function values for X on all intervals should sum up to one. The concept of M-function is similar to that of Shafer's 'basic probability assignment' [72].

Let $X_1, \ldots, X_n, X_{n+1}$ be positive, continuous and exchangeable random quantities representing lifetimes. Suppose that there are n observations of group X consisting of u event times, $x_1 < x_2 < \ldots < x_u$, and $\upsilon(= n - u)$ right-censored observations, $c_1 < c_2 < \ldots < c_\upsilon$. Let $x_0 = 0$ and $x_{u+1} = \infty$. Suppose further that there are s_i right-censored observations in the interval (x_i, x_{i+1}), denoted by $c_1^i < c_2^i < \ldots < c_{s_i}^i$, so $\sum_{i=0}^{u} s_i = \upsilon$. The assumption rc-$A_{(n)}$ partially specifies the NPI-based probability distribution for X_{n+1} by the following M-function values, where the random quantity X_{n+1} represents the failure time of one future unit [27].

Definition 1.2. Right-censoring $A_{(n)}$ (rc-$A_{(n)}$) partially specifies the probability distribution for the next observation X_{n+1} by the following M-function values,

$$M_i^X = M_{X_{n+1}}(x_i, x_{i+1}) = \frac{1}{n+1} \prod_{\{r: c_r < x_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}}, \tag{1.3}$$

$$M_{i,i^*}^X = M_{X_{n+1}}(c_{i^*}^i, x_{i+1}) = \frac{1}{(n+1)\tilde{n}_{c_{i^*}^i}} \prod_{\{r: c_r < c_{i^*}^i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}}, \tag{1.4}$$

where $i = 0, 1, \ldots, u$ and $i^* = 1, 2, \ldots, s_i$.

These M-function values can also be written as (for $i = 0, 1, \dots, u$ and $i^* = 0, 1, \dots, s_i$)

$$M_{X_{n+1}}(t^i_{i^*}, x_{i+1}) = \frac{1}{n+1}(\tilde{n}_{t^i_{i^*}})^{\delta^i_{i^*}-1} \prod_{\{r: c_r < t^i_{i^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \tag{1.5}$$

where

$$\delta^i_{i^*} = \begin{cases} 1 & \text{if } i^* = 0 \quad \text{i.e. } t^i_0 = x_i \quad \text{(failure time or time 0)} \\ 0 & \text{if } i^* = 1, \dots, s_i \quad \text{i.e. } t^i_{i^*} = c^i_{i^*} \quad \text{(censoring time)} \end{cases}$$

and $\tilde{n}_{c_r}$ and $\tilde{n}_{t^i_{i^*}}$ are the numbers of units in the risk sets (still functioning or alive and uncensored) just prior to time c_r and $t^i_{i^*}$, respectively. For consistency of notation, the further definition $\tilde{n}_0 = n + 1$ is used throughout. Only intervals of this form have positive M-function values, and these sum up to one over all these intervals. Summing up all M-function values assigned to such intervals with the same x_{i+1} as right end point gives the probability

$$P_i = P(X_{n+1} \in (x_i, x_{i+1})) = \frac{1}{n+1} \prod_{\{r: c_r < x_{i+1}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \tag{1.6}$$

where x_i and x_{i+1} are two sequential failure times (and $x_0 = 0$, $x_{u+1} = \infty$). It should be noted that, throughout this thesis, the product taken over an empty set is defined to be equal to one. To get more insight in rc-$A_{(n)}$, we provide an illustrative example in Appendix A.

Below two useful equalities are given which will be used later in the thesis, these were presented and proven in [27, p. 51].

Lemma 1.1. The following two equalities hold, for all $i = 0, 1, \dots, u$,

(a) $$\sum_{i^*=2}^{s_i} \frac{1}{\tilde{n}_{c^i_{i^*}} \tilde{n}_{c^i_{i^*-1}}} = \frac{1}{\tilde{n}_{c^i_{s_i}}} - \frac{1}{\tilde{n}_{c^i_1}} \quad \text{for } s_i \geq 2$$

(b) $$1 + \sum_{i^*=1}^{s_i} \frac{1}{\tilde{n}_{c^i_{i^*}}} \prod_{\{r: x_i < c_r < c^i_{i^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} = \prod_{\{r: x_i < c_r < x_{i+1}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \quad \text{for } s_i \geq 1.$$

1.3.3 NPI for survival function

A commonly used method for summarizing lifetime data is the survival function, $S(t)$, which specifies the probability that the time to event is greater than t. In a sample of size n, suppose that there are q ($q \leq n$) distinct event times $x_1 < x_2 < \cdots < x_q$. Let h_i be the number of events that occur at time x_i, and $\tilde{n}_{x_i}$ the number of units in the risk set just prior to time x_i. The product-limit estimator of the survival function, first proposed by Kaplan and Meier (KM) [44], is

$$\hat{S}(t) = \prod_{i:x_i \leq t} \left(\frac{\tilde{n}_{x_i} - h_i}{\tilde{n}_{x_i}} \right) \tag{1.7}$$

The product-limit estimator is also the nonparametric maximum likelihood estimator of $S(t)$. In the case where there is no censoring, the product-limit estimator is identical to the empirical survival function, which is obtained by calculating the proportion of units that have not yet experienced the event by time t.

The NPI lower and upper survival functions based on the rc-$A_{(n)}$ assumption for right-censored data can be considered as predictive alternatives to the Kaplan-Meier estimator [44], see [27] for detailed discussion and examples.

Below we present new formulae for the NPI lower and upper survival functions, $\underline{S}_{X_{n+1}}(t)$ and $\overline{S}_{X_{n+1}}(t)$, respectively, as first introduced by Coolen *et al.* [23]. These formulae are the simplest closed-form expressions for these lower and upper survival functions presented in the literature thus far, and as such are likely to be useful in many applications of NPI in reliability and survival analysis. In this thesis, they are explicitly used in Chapter 6 and they also presented by Maturi *et al.* [59].

Before introducing the new simple formulae of the NPI lower and upper survival functions, the following lemma is needed, for which some further notation is introduced. Let t_a, $a = 1, \ldots, n$, be n different ordered observations, each either a failure time ($\delta_a = 1$) or a right-censoring time ($\delta_a = 0$), and define $\delta_0 = 1$ corresponding to the definitions $t_0 = 0$ and $\tilde{n}_{t_0} = \tilde{n}_0 = n + 1$.

Lemma 1.2. For all t_a, $a = 0, 1, \ldots, n$,

$$(\tilde{n}_{t_a})^{\delta_a - 1} + \sum_{i=a+1}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r:t_a \leq c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} = \tilde{n}_{t_a} \tag{1.8}$$

Proof. The lemma is proven by induction. First, for $t_a = t_n$ equation (1.8) is easily verified both if t_n is a failure time or a censoring time. Next, for $m = 0, 1, \ldots, n-2$, let $a = n - m$ and suppose that equation (1.8) holds for $t_a = t_{n-m}$,

$$(\tilde{n}_{t_{n-m}})^{\delta_{n-m}-1} + \sum_{i=n-m+1}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_{n-m} \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} = \tilde{n}_{t_{n-m}} \tag{1.9}$$

This implies that equation (1.8) also holds for $t_{a-1} = t_{n-m-1} = t_{n-(m+1)}$, which is shown now. The equality that needs to be proven is

$$(\tilde{n}_{t_{n-m-1}})^{\delta_{n-m-1}-1} + \sum_{i=n-m}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_{n-m-1} \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} = \tilde{n}_{t_{n-m-1}} \tag{1.10}$$

The left hand side of (1.10) can be written as

$$\begin{aligned}
&\left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} + \left(\frac{\tilde{n}_{t_{n-m-1}}+1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} \left((\tilde{n}_{t_{n-m}})^{\delta_{n-m}-1} + \sum_{i=n-m+1}^{n} (\tilde{n}_{t_i})^{\delta_i-1} \prod_{\{r: t_{n-m} \le c_r < t_i\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}}\right) \\
&= \left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} + \left(\frac{\tilde{n}_{t_{n-m-1}}+1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} \tilde{n}_{t_{n-m}} \\
&= \left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} + \left(\frac{\tilde{n}_{t_{n-m-1}}+1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} (\tilde{n}_{t_{n-m-1}} - 1) \\
&= \left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} \left\{1 + \left(\tilde{n}_{t_{n-m-1}} + 1\right)^{1-\delta_{n-m-1}} (\tilde{n}_{t_{n-m-1}} - 1)\right\}
\end{aligned}$$

where the first equality follows from (1.9). Both if t_{n-m-1} is a failure time ($\delta_{n-m-1} = 1$) or a censoring time ($\delta_{n-m-1} = 0$), it follows straightforwardly that this expression is equal to $\tilde{n}_{t_{n-m-1}}$. Hence, by this induction argument equation (1.8) is proven to hold for all $a = 1, \ldots, n$. Finally, for $a = 0$, so $t_0 = 0$ for which $\delta_0 = 1$ and $\tilde{n}_{t_0} = \tilde{n}_0 = n + 1$ were defined, equation (1.8) follows directly by

$$\begin{aligned}
(\tilde{n}_0)^{\delta_0-1} + \sum_{i=1}^{n} (\tilde{n}_{t_i})^{\delta_i-1} \prod_{\{r: t_0 \le c_r < t_i\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} &= 1 + (\tilde{n}_{t_1})^{\delta_1-1} + \sum_{i=2}^{n} (\tilde{n}_{t_i})^{\delta_i-1} \prod_{\{r: t_1 \le c_r < t_i\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \\
&= 1 + \tilde{n}_{t_1} = 1 + n = \tilde{n}_{t_0}
\end{aligned}$$

□

The simple closed-form expressions for the NPI lower and upper survival functions are given by Theorem 1.3. In addition to notation introduced above, let $t^i_{s_i+1} = t^{i+1}_0 = x_{i+1}$ for $i = 0, 1, \ldots, u-1$.

Theorem 1.3. The NPI lower survival function [23] can be expressed as follows, for $t \in [t^i_a, t^i_{a+1})$ with $i = 0, 1, \ldots, u$ and $a = 0, 1, \ldots, s_i$,

$$\underline{S}_{X_{n+1}}(t) = \frac{1}{n+1}\, \tilde{n}_{t^i_a} \prod_{\{r: c_r < t^i_a\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \tag{1.11}$$

and the corresponding NPI upper survival function [23] can be written as follows, for $t \in [x_i, x_{i+1})$ with $i = 0, 1, \ldots, u$,

$$\overline{S}_{X_{n+1}}(t) = \frac{1}{n+1}\, \tilde{n}_{x_i} \prod_{\{r: c_r < x_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \tag{1.12}$$

Proof. For $t \in [t^i_a, t^i_{a+1})$, the lower survival function, as given in [23], is equal to

$$\begin{aligned}
\underline{S}_{X_{n+1}}(t) = \underline{S}_{X_{n+1}}(t^i_a) &= M_{X_{n+1}}(t^i_a, x_{i+1}) + \sum_{C(i,i^*,t^i_a)} M_{X_{n+1}}(t^i_{i^*}, x_{i+1}) \\
&= \frac{1}{n+1}\left\{ (\tilde{n}_{t^i_a})^{\delta^i_a - 1} \prod_{\{r: c_r < t^i_a\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} + \sum_{C(i,i^*,t^i_a)} (\tilde{n}_{t^i_{i^*}})^{\delta^i_{i^*}-1} \prod_{\{r: c_r < t^i_{i^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \prod_{\{r: c_r < t^i_a\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \left\{ (\tilde{n}_{t^i_a})^{\delta^i_a - 1} + \sum_{C(i,i^*,t^i_a)} (\tilde{n}_{t^i_{i^*}})^{\delta^i_{i^*}-1} \prod_{\{r: t^i_a \le c_r < t^i_{i^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \prod_{\{r: c_r < t^i_a\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}}\, \tilde{n}_{t^i_a}
\end{aligned}$$

where $\sum_{C(i,i^*,t^i_a)}$ denotes the sums over all i from 0 to u and over all i^* from 0 to s_i such that $t^i_{i^*} > t^i_a$. Again, t^i_a can be a failure time ($\delta^i_a = 1$) or a censoring time ($\delta^i_a = 0$). The final equality follows from Lemma 1.2.

Lemma 1.2 is also used to prove formula (1.12) for the NPI upper survival function [23], which, for $t \in [x_i, x_{i+1})$, is equal to

$$\begin{aligned}
\overline{S}_{X_{n+1}}(t) &= M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{C(i,i^*,x_i)} M_{X_{n+1}}(t^i_{i^*}, x_{i+1}) \\
&= \frac{1}{n+1} \prod_{\{r: c_r < x_i\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \left\{ 1 + \sum_{C(i,i^*,x_i)} (\tilde{n}_{t^i_{i^*}})^{\delta^i_{i^*}-1} \prod_{\{r: x_i \le c_r < t^i_{i^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \prod_{\{r: c_r < x_i\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}}\, \tilde{n}_{x_i}
\end{aligned}$$

where $\sum_{C(i,i^*,x_i)}$ denotes the sums over all i from 0 to u and over all i^* from 0 to s_i such that $t^i_{i^*} > x_i$. □

Lemma 1.2, and indeed the NPI lower and upper survival functions (1.11) and (1.12), can also be interpreted along the same lines as the probability redistribution algorithm for right-censored data as introduced by Efron [35] and also discussed by Coolen and Yan [27].

1.3.4 NPI for comparing two groups of lifetime data

Coolen and Yan [26] introduced NPI for comparing two independent groups of lifetime data, say X and Y, including right-censored observations. This comparison is in terms of lower and upper probabilities for the event that a future observation X_{n_x+1} of group X is less than a future observation Y_{n_y+1} of group Y, based on n_x and n_y observations of group X and Y, and the assumptions rc-$A_{(n_x)}$ and rc-$A_{(n_y)}$. Suppose that we have observed u_x event times from group X, denoted by $x_1 < x_2 < \ldots < x_{u_x}$, and $\upsilon_x (= n_x - u_x)$ right-censored observations $c_{x,1} < c_{x,2} < \ldots < c_{x,\upsilon_x}$. Let $x_0 = 0$, $x_{u_x+1} = \infty$, and let $s_{x,i}$ be the right-censored observations in the interval (x_i, x_{i+1}), $x_i < c^i_{x,1} < c^i_{x,2} < \ldots < c^i_{x,s_{x,i}} < x_{i+1}$, so $\sum_{i=0}^{u_x} s_{x,i} = \upsilon_x$. Similarly, suppose that there are u_y event times from group Y denoted by $y_1 < y_2 < \ldots < y_{u_y}$ and let $y_0 = 0$ and $y_{u_y+1} = \infty$, and that there are $\upsilon_y (= n_y - u_y)$ right-censored observations $c_{y,1} < c_{y,2} < \ldots < c_{y,\upsilon_y}$ and $s_{y,j}$ right-censored observations in the interval (y_j, y_{j+1}), $y_j < c^j_{y,1} < c^j_{y,2} < \ldots < c^j_{y,s_{y,j}} < y_{j+1}$, so $\sum_{j=0}^{u_y} s_{y,j} = \upsilon_y$. Then the NPI lower and upper probabilities for the event $X_{n_x+1} < Y_{n_y+1}$ are

$$\underline{P}(X_{n_x+1}<Y_{n_y+1})=\sum_{i=0}^{u_x}\sum_{j=0}^{u_y} P_i^X \left\{ \mathbf{1}\{x_{i+1}< y_j\} M_j^Y + \sum_{i_y^*=1}^{s_{y,j}} \mathbf{1}\{x_{i+1}< c^j_{y,i_y^*}\} M^Y_{j,i_y^*} \right\} \quad (1.13)$$

$$\overline{P}(X_{n_x+1}<Y_{n_y+1})=\sum_{i=0}^{u_x}\sum_{j=0}^{u_y} P_j^Y \left\{ \mathbf{1}\{x_i< y_{j+1}\} M_i^X + \sum_{i_x^*=1}^{s_{x,i}} \mathbf{1}\{c^i_{x,i_x^*}< y_{j+1}\} M^X_{i,i_x^*} \right\} \quad (1.14)$$

where the quantities M_i^X (M_j^Y), M^X_{i,i_x^*} (M^Y_{j,i_y^*}) and P_i^X (P_j^Y) are given by (1.3), (1.4) and (1.6), respectively. Coolen and Yan [26] derived these lower and upper probabilities by use of the following lemma, given and proven in [26, 81], which is also used later in the thesis.

Lemma 1.4. For $s \geq 2$, let $J_l = (j_l, r)$, with $j_1 < j_2 < \ldots < j_s < r$, so we have nested intervals $J_1 \supset J_2 \supset \ldots \supset J_s$ with the same right end-point r (which may be infinity). We consider two independent real-valued random quantities, say X and Y. Let the probability distribution for X be partially specified via M-function values, with all probability mass $P(X \in J_1)$ described by the s M-function values $M_X(J_l)$, $l = 1, \ldots, s$, so $\sum_{l=1}^{s} M_X(J_l) = P(X \in J_1)$. Then, without additional assumptions,

$$\sum_{l=1}^{s} P(Y < j_l) M_X(J_l) \leq P(Y < X, X \in J_1) \leq P(Y < r) P(X \in J_1)$$

provides the maximum lower and minimum upper bounds.

1.3.5 Treatment of ties

In NPI it is quite straightforward to deal with tied observations, by assuming that tied observations differ by small amounts which tend to zero [41]. If such a tie would occur among different groups, then one can break it similarly in two ways, different for upper and lower probabilities in such a way that these are maximal and minimal, respectively, over the possible ways of breaking such ties, without changing the order of these observations with respect to all other observations [26]. If ties occur between event time and right-censoring time, then as is common in the literature, the right-censoring time is assumed to be just beyond the event time [44]. Throughout this thesis we deal with tied observations in this way, for more details we refer to [26, 27, 81].

1.4 Outline of Thesis

The thesis is organized such that each chapter addresses one main inference problem, and is related to a paper that has been published in an academic journal or which is in submission. Each chapter is self-contained, with the main problem and the notation introduced before the core results are presented. The same notation may be used for different quantities in different chapters, notation introduced in Chapter 1 may also be used.

In Chapter 2 we introduce NPI for precedence testing for two groups [29]. We extend that in Chapter 3 to $k \geq 2$ groups with focus on different selection problems [60]. Further extension allowing right-censoring to occur before the experiment is ended is presented in Chapter 4 [58]. Chapter 5 presents a comparison of two groups under different progressive censoring schemes [57]. In Chapter 6 we introduce NPI for competing risks, which is an important topic in reliability [59]. Chapter 7 presents NPI for comparison of two groups with only a part of the data available [55]. In the appendix, we enclose the R commands that have been used for calculations. Despite in some chapters, due to notational complexity we only consider pairwise comparisons, the R commands provided in Appendix A.1 can be used for more general events of interest similar to those presented in Chapter 3.

Furthermore, some parts of this thesis have been presented in several conferences and short papers have appeared in related conference proceedings. For example, Chapter 2 has been presented at the 5th International Mathematical Methods in Reliability Conference (Glasgow, UK 2007) [28]. A part of Chapter 3 was presented at the International Workshop on Applied Probability (Compiegne, France 2008) [52]. Chapter 4 was presented at the International Seminar on Nonparametric Inference (Vigo, Spain 2008) [53]. Part of Chapter 6 was presented at the 18th Advances in Risk and Reliability Technology Symposium (Loughborough, UK 2009) [56]. A comprehensive overview of the main parts of this thesis was presented at the 6th International Symposium on Imprecise Probability: Theories and Applications: ISIPTA'09 (Durham, UK 2009) [54].

Chapter 2

Comparison of two groups with early termination

2.1 Introduction

Comparison of lifetimes of units from different groups is a common problem. In this chapter, we consider the situation where units from two groups are simultaneously placed on a life-testing experiment, and decisions may be needed before all units have failed due to cost or time considerations, so the data consist of both observed lifetimes and observations which are right-censored at the moment the experiment was terminated.

In classical precedence testing, the experiment is terminated at a certain time or after a certain number of failures (for a particular group). Epstein [36] first presented precedence testing, Nelson [63] proposed it as an efficient life-test procedure that enables decisions after relatively few lifetimes are observed. Balakrishnan and Ng [5] present an excellent overview, and describe several nonparametric precedence tests based on the hypothesis of equal lifetime distributions.

As an alternative, we propose Nonparametric Predictive Inference (NPI) for precedence testing for two groups, with lower and upper probabilities for the event that the future lifetime of a unit from one group is less than the future lifetime of a unit from the other group. In Section 2.2, we briefly review some classical nonparametric precedence tests. Our method is introduced and justified in Sections

2.3 and 2.4 including some special cases and properties. Finally we illustrate and compare our method with these classical precedence tests via examples in Section 2.5.

2.2 Classical precedence testing

In precedence testing for two groups, units of both groups are placed simultaneously on a life-testing experiment, and failures are observed as they arise during the experiment, which is terminated as soon as a certain stop criterion has been reached, so the lifetimes of some units are typically right-censored.

In this section we briefly review some classical nonparametric precedence tests in the literature, following notations and definitions of Balakrishnan and Ng [5]. Suppose one is interested in comparing the lifetimes of units from two groups X and Y. Their lifetime distributions are denoted by F_X and F_Y, respectively, and n_x and n_y are the number of units of group X and Y that are placed simultaneously on a life-testing experiment. We assume that the experiment is terminated as soon as the r_y^{th} failure of group Y is observed. It should be noted that for the classical precedence tests, the stop criterion used is relevant due to the nature of frequentist hypothesis testing, as it influences the sampling distribution of the test statistic, which is not the case in the NPI approach presented in Section 2.3.

The *classical precedence test* was introduced by Nelson [63]. One is interested in testing the null hypothesis H_0 that $F_X(x) = F_Y(x)$ for all $x \geq 0$. Let D_1 be the random quantity representing the number of observed lifetimes of group X that are less than the first observed lifetime of group Y, and let d_1 be its observed value. Similar, let D_i be the random quantity representing the number of observed lifetimes of group X that are between the $(i-1)^{th}$ and i^{th} observed lifetime of group Y, for $i = 2, \ldots, r_y$, and denote their observed values by d_i. The precedence test statistic $Q_{(r_y)}$ is the number of lifetimes of group X that precede the r_y^{th} lifetime from group Y, i.e. $Q_{(r_y)} = \sum_{i=1}^{r_y} D_i$. Under H_0, the distribution of $Q_{(r_y)}$ is

$$P(Q_{(r_y)}= j|H_0) = \binom{j+r_y-1}{j}\binom{n_x+n_y-j-r_y}{n_x-j}\binom{n_x+n_y}{n_y}^{-1}, \; j = 0, \ldots, n_x \quad (2.1)$$

The classical precedence test may suffer from the masking effect problem, which is that the null hypothesis may not be rejected for a certain value of r_y whilst there may exist a value less than this r_y for which the null hypothesis would be rejected at the same level of significance. To avoid this problem Balakrishnan and Frattina [4] proposed the *maximal precedence test*. The test statistic $U_{(r_y)}$ is simply defined as the maximum of the D_i's defined above, for $i = 1, \ldots, r_y$, i.e. $U_{(r_y)} = \max_{i=1,\ldots,r_y} D_i$. Under H_0, the cumulative distribution function of $U_{(r_y)}$ is given by

$$P(U_{(r_y)} \leq d|H_0) = P(D_1 \leq d, \ldots, D_{r_y} \leq d|H_0) = \sum_{C_U} \binom{n_x+n_y-\sum_{i=1}^{r_y} d_i - r_y}{n_y - r_y} \binom{n_x+n_y}{n_y}^{-1} \quad (2.2)$$

where C_U is the set of all possible combinations of d_i's ($i = 1, \ldots, r_y$) with $d_i \in \{0, 1, \ldots, d\}$ and $\sum_{i=1}^{r_y} d_i \leq n_x$.

The standard Wilcoxon's rank-sum statistic (the sum of the ranks of the X failures among all failures) is generalised by Ng and Balakrishnan [66]. They introduced three Wilcoxon-type rank-sum precedence test statistics, namely; *the minimal, maximal* and *expected Wilcoxon's rank-sum precedence tests.* Let R_{r_y} ($R^*_{r_y}$) be the rank-sum of the observed lifetimes of group X that occurred before (after) the r_y^{th} observed lifetime of group Y. Wilcoxon's rank-sum precedence test statistic W_{r_y} is the sum of R_{r_y} and $R^*_{r_y}$. As the exact lifetimes of group X that occurred after the r_y^{th} observed lifetime of group Y are unknown, so $R^*_{r_y}$ is unknown, the minimal (maximal) value of $R^*_{r_y}$ and consequently the minimal (maximal) value of Wilcoxon's rank-sum precedence test statistic W_{r_y} is as follows: when all remaining ($n_x - \sum_{i=1}^{r_y} d_i$) observations of group X occur between the r_y^{th} and $(r_y + 1)^{th}$ observation of group Y, then Wilcoxon's test statistic will be minimal. The test statistic in this case, called the minimal rank-sum statistic, is

$$W_{\min,r_y} = R_{r_y} + (r_y + \sum_{i=1}^{r_y} D_i + 1) + (r_y + \sum_{i=1}^{r_y} D_i + 2) + \ldots + (r_y + n_x) \quad (2.3)$$

Let w_{min,r_y} be the observed value of the test statistic W_{min,r_y}. Under the null hypothesis that the lifetime distributions of groups X and Y are the same, the distribution of $W_{\min,r_y}$ is given by

$$P(W_{min,r_y} = w|H_0) = \sum_{C_W} \binom{n_x + n_y - \sum_{i=1}^{r_y} d_i - r_y}{n_y - r_y} \binom{n_x + n_y}{n_y}^{-1} \quad (2.4)$$

where C_W is the set of all possible combinations of d_i's $(i = 1, \ldots, r_y)$ with $d_i \in \{0, 1, \ldots, n_x\}$ and $\sum_{i=1}^{r_y} d_i \leq n_x$, for which also $w_{min,r_y} = w$ holds.

If the $n_x - \sum_{i=1}^{r_y} d_i$ remaining observations of group X occur after the n_y^{th} observation of group Y, Wilcoxon's test statistic is maximal. The test statistic in this case, called the maximal rank-sum statistic, is

$$W_{\max,r_y} = R_{r_y} + (n_y + \sum_{i=1}^{r_y} D_i + 1) + (n_y + \sum_{i=1}^{r_y} D_i + 2) + \ldots + (n_y + n_x) \qquad (2.5)$$

The Wilcoxon's expected rank-sum precedence test statistic, W_{E,r_y}, is simply the average of $W_{\min,r_y}$ and $W_{\max,r_y}$. Similar to $W_{\min,r_y}$, the distributions of $W_{\max,r_y}$ and W_{E,r_y}, under H_0, can be obtained [5]. The distributions of all mentioned test statistics under the null-hypothesis will be used to obtain the p-values of these tests later in Example 2.2. For the classical precedence test, this implies that the p-value of the observed test statistic is $P(Q_{(r_y)} \geq \sum_{i=1}^{r_y} d_i | H_0)$ where the distribution of $Q_{(r_y)}$ is given by (2.1). For the maximal precedence test, the p-value of the observed test statistic is given by $P(U_{(r_y)} \geq d | H_0)$, where d is the observed value of $U_{(r_y)}$ and the cumulative distribution of $U_{(r_y)}$ is given by (2.2). For the Wilcoxon's minimal, maximal and expected rank-sum precedence tests, the p-values of the test statistics are given by $P(W_{a,r_y} \leq w_a | H_0)$ where $a = \min, \max, E$ and w_a is the observed value of the test statistic, with the distributions of the test statistics are given under H_0 [5]. For more details of these more established methods we refer to [5].

2.3 NPI for precedence testing

To introduce NPI for precedence testing we need first to introduce some notation. Suppose that $X_1, \ldots, X_n, X_{n+1}$ are positive, continuous and exchangeable random quantities representing lifetimes. We assume that no ties occur, the results can be generalised to allow ties [42], see also Subsection 1.3.5.

In precedence testing the experiment is terminated as soon as a certain stop criterion has been reached. We assume that this stop criterion is expressed in terms of a stopping time T_0, but if instead a number of failures were used as stop criterion then this would not affect our method, as it is of no relevance in NPI how T_0 is determined, as long as T_0 contains no further information on values beyond T_0. When considering a single group of units, let r denote the number of observations of $X_1, \ldots, X_n$ that occur before the stopping time T_0, so $n - r$ observations are right-censored at T_0. Let $x_1 < x_2 < \ldots < x_r$ be the ordered observed values before T_0, and let $x_0 = 0$ for ease of notation. In this case, all right-censored observations are the same which simplifies the use of rc-$A_{(n)}$ [27]. For ease of notation, we will assume that there are no ties between the observed failure times, this 'tied right-censoring time' does not provide any complications, in fact it simplifies the matter when compared to the general case of varying right-censoring times for which rc-$A_{(n)}$ provides an inferential approach. The next theorem provides the M-functions required for precedence testing, which follows from rc-$A_{(n)}$.

Theorem 2.1. For nonparametric predictive precedence testing with stopping time T_0, the assumption rc-$A_{(n)}$ implies that the probability distribution for a nonnegative random quantity X_{n+1} on the basis of data including r real and $n-r$ right-censored observations, is partially specified by the following M-function values:

$$M_{X_{n+1}}(x_{i-1}, x_i) = \frac{1}{n+1}, \quad i = 1, \ldots, r,$$
$$M_{X_{n+1}}(x_r, \infty) = \frac{1}{n+1} \quad \text{and} \quad M_{X_{n+1}}(T_0, \infty) = \frac{n-r}{n+1}$$

Proof. Since there are no censored data before T_0, this follows immediately from Definition 1.2 for $M_{X_{n+1}}(x_{i-1}, x_i)$ and $M_{X_{n+1}}(x_r, \infty)$. Suppose the $n-r$ right-censored observations (beyond T_0) are $c_1 < c_2 < \ldots < c_{n-r}$, then from (1.4)

$$\begin{aligned}
M_{X_{n+1}}(T_0, \infty) &= \sum_{i^*=1}^{n-r} M_{X_{n+1}}(c_{i^*}, \infty) = \sum_{i^*=1}^{n-r} \frac{1}{(n+1)\tilde{n}_{c_{i^*}}} \prod_{\{r: c_r < c_{i^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \\
&= \frac{1}{n+1} \prod_{\{r: c_r < T_0\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \left\{ \sum_{i^*=1}^{n-r} \frac{1}{\tilde{n}_{c_{i^*}}} \prod_{\{r: T_0 < c_r < c_{i^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \left\{ -1 + \prod_{\{r: T_0 < c_r < \infty\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \left\{ \tilde{n}_{c_1} + 1 - 1 \right\} = \frac{n-r}{n+1}
\end{aligned}$$

The fourth equality follows from the fact that the first product is over an empty set, and by using Lemma 1.1 (b). □

2.4 NPI for comparing two groups with early termination

To compare two completely independent groups of lifetime data by the NPI approach for precedence testing, we use the notation as introduced above, but we add an index x or y corresponding to the groups X and Y. So, n_x and n_y units of groups X and Y are placed simultaneously on a life-testing experiment, and r_x and r_y lifetimes of groups X and Y are observed before the experiment is terminated at time T_0. So $n_x - r_x$ and $n_y - r_y$ lifetimes of groups X and Y are right-censored at T_0. Let $x_1 < x_2 < \ldots < x_{r_x}$ and $y_1 < y_2 < \ldots < y_{r_y}$ be the ordered observed values before T_0 from groups X and Y, respectively. And let $x_0 = y_0 = 0$ for ease of notation.

In this section we derive the NPI lower and upper probabilities for the event that a future observation X_{n_x+1} of group X is less than a future observation Y_{n_y+1} of group Y. Optimal bounds for the probability of $X_{n_x+1} < Y_{n_y+1}$, given the data, stopping time T_0 and based on rc-$A_{(n_x)}$ and rc-$A_{(n_y)}$, are presented in Theorem 2.2.

Theorem 2.2. For the above scenario, the NPI lower and upper probabilities for the event $X_{n_x+1} < Y_{n_y+1}$ are

$$\underline{P}(X_{n_x+1} < Y_{n_y+1}) = A\left\{\sum_{j=1}^{r_y}\sum_{i=1}^{r_x} \mathbf{1}\{x_i < y_j\} + r_x(n_y - r_y)\right\} \tag{2.6}$$

$$\overline{P}(X_{n_x+1} < Y_{n_y+1}) = A\left\{\sum_{j=1}^{r_y}\sum_{i=1}^{r_x} \mathbf{1}\{x_i < y_j\} + r_y + (n_x+1)(n_y - r_y + 1)\right\} \tag{2.7}$$

where $A = \dfrac{1}{(n_x+1)(n_y+1)}$.

Proof. The NPI lower probability for the event $X_{n_x+1} < Y_{n_y+1}$ given the data and T_0, i.e. $P = P(X_{n_x+1} < Y_{n_y+1})$, is derived as follows:

$$\begin{aligned}
P &= \sum_{j=1}^{r_y} P(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (y_{j-1}, y_j)) + P(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (y_{r_y}, \infty)) \\
&\geq \sum_{j=1}^{r_y} P(X_{n_x+1} < y_{j-1})\ M_{Y_{n_y+1}}(y_{j-1}, y_j) + P(X_{n_x+1} < y_{r_y})\ M_{Y_{n_y+1}}(y_{r_y}, \infty) + \\
&\qquad P(X_{n_x+1} < T_0)\ M_{Y_{n_y+1}}(T_0, \infty)
\end{aligned}$$

$$= \frac{1}{n_y+1}\sum_{j=1}^{r_y} P(X_{n_x+1} < y_{j-1}) + \frac{1}{n_y+1}P(X_{n_x+1} < y_{r_y}) + \frac{n_y-r_y}{n_y+1}P(X_{n_x+1} < T_0)$$
$$\geq A\left[\sum_{j=1}^{r_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_i < y_{j-1}\} + \sum_{i=1}^{r_x}\mathbf{1}\{x_i < y_{r_y}\} + (n_y-r_y)\sum_{i=1}^{r_x}\mathbf{1}\{x_i < T_0\}\right]$$
$$= A\left[\sum_{j=1}^{r_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_i < y_j\} + r_x(n_y-r_y)\right].$$

The first inequality follows by putting all probability masses for Y_{n_y+1} corresponding to the intervals (y_{j-1}, y_j) $(j = 1, \ldots, r_y)$, (y_{r_y}, ∞) and (T_0, ∞) to the left end points of these intervals, and by using Lemma 1.4 for the nested intervals (y_{r_y}, ∞) and (T_0, ∞). The second inequality follows by putting all probability masses for X_{n_x+1} corresponding to the intervals (x_{i-1}, x_i) $(i = 1, \ldots, r_x)$, (x_{r_x}, ∞) and (T_0, ∞) to the right end points of these intervals.

The derivation of the corresponding NPI upper probability for the event $X_{n_x+1} < Y_{n_y+1}$ is given below. The first inequality follows by putting all probability masses for Y_{n_y+1} corresponding to the intervals (y_{j-1}, y_j) $(j = 1, \ldots, r_y)$, (y_{r_y}, ∞) and (T_0, ∞) to the right end points of these intervals, using Lemma 1.4 for the nested intervals (y_{r_y}, ∞) and (T_0, ∞). The second inequality follows by putting all probability masses for X_{n_x+1} corresponding to the intervals (x_{i-1}, x_i) $(i = 1, \ldots, r_x)$, (x_{r_x}, ∞) and (T_0, ∞) to the left end points of these intervals.

$$P = \sum_{j=1}^{r_y} P(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (y_{j-1}, y_j)) + P(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (y_{r_y}, \infty))$$
$$\leq \sum_{j=1}^{r_y} P(X_{n_x+1} < y_j) M_{Y_{n_y+1}}(y_{j-1}, y_j) + P(X_{n_x+1} < \infty) M_{Y_{n_y+1}}(y_{r_y}, \infty) +$$
$$P(X_{n_x+1} < \infty) M_{Y_{n_y+1}}(T_0, \infty)$$
$$= \frac{1}{n_y+1}\sum_{j=1}^{r_y} P(X_{n_x+1} < y_j) + \frac{1}{n_y+1}P(X_{n_x+1} < \infty) + \frac{n_y-r_y}{n_y+1}P(X_{n_x+1} < \infty)$$
$$\leq A\sum_{j=1}^{r_y}\sum_{i=1}^{r_x+1}\mathbf{1}\{x_{i-1} < y_j\} + \frac{1}{n_y+1} + \frac{n_y-r_y}{n_y+1}$$
$$= A\left[\sum_{j=1}^{r_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_i < y_j\} + r_y + (n_y - r_y + 1)(n_x + 1)\right]$$

□

These NPI lower and upper probabilities are based only on x_i ($i = 1 \ldots, r_x$), y_j ($j = 1, \ldots, r_y$) and T_0, further information on location as contained in the observations is not used. As such, this approach can be regarded as a fully predictive alternative to standard rank-based methods [50]. It is also easy to show that, for these lower and upper probabilities the conjugacy property holds.

If the stopping time T_0 in the precedence tests, as considered above, does not affect the experiment, in the sense that all units tested actually fail during the test, then the results in this chapter are identical to those of NPI for pairwise comparisons presented by Coolen [19], which is a special case of NPI for multiple comparisons presented by Coolen and van der Laan [25], see also Subsection 1.3.1.

2.4.1 Special cases

From Theorem 2.2 it follows that if $r_x = 0$ and $r_y \in \{0, 1, \ldots, n_y\}$, that is, the experiment is terminated before the first observation of group X, we have

$$\underline{P}(X_{n_x+1} < Y_{n_y+1}) = 0 \quad \text{and} \quad \overline{P}(X_{n_x+1} < Y_{n_y+1}) = 1 - A\, n_x r_y \tag{2.8}$$

This lower probability is zero, reflecting that on the basis of the data one cannot exclude the possibility that the X observations will always exceed all Y observations. If $r_y = 0$ and $r_x \in \{0, 1, \ldots, n_x\}$, that is, the experiment is terminated before the first observation of group Y, we have

$$\underline{P}(X_{n_x+1} < Y_{n_y+1}) = A\, r_x n_y \quad \text{and} \quad \overline{P}(X_{n_x+1} < Y_{n_y+1}) = 1 \tag{2.9}$$

This upper probability is one, reflecting that one cannot exclude the possibility that the X observations will always be less than all Y observations. The lower and upper probabilities in (2.9) can also be obtained from (2.8) using the conjugacy property.

If all units of group Y are observed before the first observation of group X, that is $y_{n_y} < x_1$, and the experiment is terminated after the last unit of group Y is observed ($T_0 > y_{n_y}$), i.e. $r_y = n_y$, then, independent of the number of units of group X observed, we have

$$\underline{P}(X_{n_x+1} < Y_{n_y+1}) = 0 \quad \text{and} \quad \overline{P}(X_{n_x+1} < Y_{n_y+1}) = 1 - A\, n_x n_y \tag{2.10}$$

Similarly, if all units of group X are observed before the first observation of group Y, that is $x_{n_x} < y_1$, and the experiment is terminated after the last unit of group X is observed ($T_0 > x_{n_x}$) then, independent of the number of units of group Y observed, we have

$$\underline{P}(X_{n_x+1} < Y_{n_y+1}) = A\, n_x n_y \quad \text{and} \quad \overline{P}(X_{n_x+1} < Y_{n_y+1}) = 1 \tag{2.11}$$

2.4.2 Some properties

We now analyze some properties of the NPI-based lower and upper probabilities derived in Theorem 2.2. Suppose that the stopping time is increased from T_0 to T_0^*, and denote by r_x^* and r_y^* the number of lifetimes of group X and Y, respectively, observed before T_0^*. The lower and upper probabilities for the event $X_{n_x+1}<Y_{n_y+1}$, based on the data, T_0, rc-$A_{(n_x)}$ and rc-$A_{(n_y)}$, are denoted by $\underline{P}(X_{n_x+1}<Y_{n_y+1})$ and $\overline{P}(X_{n_x+1}<Y_{n_y+1})$, while the corresponding lower and upper probabilities for T_0^* are denoted by $\underline{P}^*(X_{n_x+1} < Y_{n_y+1})$ and $\overline{P}^*(X_{n_x+1} < Y_{n_y+1})$. We can write $r_x^* = r_x+a$ and $r_y^* = r_y+b$ with a, b nonnegative integers. Using (2.6) the lower probability $\underline{P}^*(X_{n_x+1}<Y_{n_y+1})$ can be written as:

$$\begin{aligned}\underline{P}^*(X_{n_x+1} < Y_{n_y+1}) &= A\left[\sum_{j=1}^{r_y+b}\sum_{i=1}^{r_x+a} \mathbf{1}\{x_i < y_j\} + (r_x+a)(n_y - r_y - b)\right] \\ &= \underline{P}(X_{n_x+1} < Y_{n_y+1}) + A\left[\sum_{j=1}^{r_y}\sum_{i=r_x+1}^{r_x+a} \mathbf{1}\{x_i < y_j\} + \right. \\ &\qquad \left. \sum_{j=r_y+1}^{r_y+b}\sum_{i=1}^{r_x+a} \mathbf{1}\{x_i < y_j\} + a(n_y - r_y - b) - br_x\right]\end{aligned} \tag{2.12}$$

Similarly, using (2.7) the upper probability $\overline{P}^*(X_{n_x+1} < Y_{n_y+1})$ can be written as:

$$\begin{aligned}\overline{P}^*(X_{n_x+1} < Y_{n_y+1}) &= A\left[\sum_{j=1}^{r_y+b}\sum_{i=1}^{r_x+a} \mathbf{1}\{x_i < y_j\} + r_y + b + (n_x+1)(n_y - r_y - b + 1)\right] \\ &= \overline{P}(X_{n_x+1} < Y_{n_y+1}) + A\left[\sum_{j=1}^{r_y}\sum_{i=r_x+1}^{r_x+a} \mathbf{1}\{x_i < y_j\} + \right. \\ &\qquad \left. \sum_{j=r_y+1}^{r_y+b}\sum_{i=1}^{r_x+a} \mathbf{1}\{x_i < y_j\} - b n_x\right]\end{aligned} \tag{2.13}$$

Theorem 2.3 follows from (2.12) and (2.13).

Theorem 2.3.

(1) Consider the situation that, for a given data set but with increased stopping time T_0, r_x has increased while r_y is unchanged. Then **(i)** the lower probability $\underline{P}(X_{n_x+1} < Y_{n_y+1})$ is strictly increasing in r_x, except if $x_{r_x+1} > y_{n_y}$ in which case the lower probability remains constant, and **(ii)** the upper probability $\overline{P}(X_{n_x+1} < Y_{n_y+1})$ remains constant.

(2) Similarly, consider the situation that r_y has increased while r_x is unchanged. Then **(i)** the lower probability $\underline{P}(X_{n_x+1} < Y_{n_y+1})$ remains constant, and **(ii)** the upper probability $\overline{P}(X_{n_x+1} < Y_{n_y+1})$ is strictly decreasing in r_y, except if $x_{n_x} < y_{r_y+1}$ in which case the upper probability remains constant.

Proof. We prove part (1), the proof of part (2) is similar. To prove **(i)**, increasing r_x while keeping r_y constant implies that a is a positive integer and $b = 0$. Substituting $b = 0$ into (2.12) yields

$$\underline{P}^*(X_{n_x+1}{<}Y_{n_y+1}){=}\underline{P}(X_{n_x+1}{<}Y_{n_y+1}) + A\left[\sum_{j=1}^{r_y}\sum_{i=r_x+1}^{r_x+a}\mathbf{1}\{x_i < y_j\}{+}\, a(n_y{-}r_y)\right]$$

From this it follows that the lower probability is strictly increasing in r_x unless $n_y = r_y$ and the double sum equals zero, that is, if $n_y = r_y$ and all x_i, $i = r_x{+}1, \ldots, r_x{+}a$, are larger than y_{r_y}. These two conditions hold when $x_{r_x+1} > y_{n_y}$. To prove **(ii)**, substituting $b = 0$ into (2.13) yields

$$\overline{P}^*(X_{n_x+1} < Y_{n_y+1}) = \overline{P}(X_{n_x+1} < Y_{n_y+1}) + A\left[\sum_{j=1}^{r_y}\sum_{i=r_x+1}^{r_x+a}\mathbf{1}\{x_i < y_j\}\right]$$

From this it follows that the upper probability is strictly increasing in r_x unless the double sum equals zero, that is, if $x_{r_x+1} > y_{r_y}$. However, x_{r_x+1} is by definition larger than y_{r_y} and consequently the upper probability always remains constant in this case. □

Theorem 2.3 states that the NPI lower (upper) probability for the event $X_{n_x+1} < Y_{n_y+1}$ never decreases (increases) if T_0 increases. This is in line with intuition, as all possible orderings of all lifetimes which are right-censored at T_0 are taken into account, and also with the general idea behind NPI, which is to explore what can be inferred from data with only few assumptions added.

2.5 Examples

In this section, two examples are given. Example 2.1 has been created to illustrate our method presented in Section 2.4 with focus on the special cases of Theorem 2.3. Example 2.2 presents a comparison of the NPI method with the classical precedence tests reviewed in Section 2.2.

Example 2.1. Six units each of group X and group Y are placed simultaneously on a life-testing experiment and their lifetimes are 1, 2, 3, 10, 11, 12 for X, and 4, 5, 6, 7, 8, 9 for Y, so all 6 observations of group Y are between the 3^{rd} and 4^{th} observations of group X. Suppose now that we would have terminated the experiment at stopping time T_0. We calculate the NPI lower and upper probabilities for the event that the lifetime of a future unit of group X is less than the lifetime of a future unit of group Y, given the observed lifetimes before T_0 for both groups and based on rc-$A_{(6)}$ for both groups. Table 2.1 and Figure 2.1 show the lower probabilities (2.6) and upper probabilities (2.7) when T_0 increases from 0 to ∞. As the NPI lower and upper probabilities may only change when a lifetime of either group is observed, we only have to consider a finite number of time-intervals.

T_0	r_x	r_y	$\underline{P}$	$\overline{P}$	T_0	r_x	r_y	$\underline{P}$	$\overline{P}$
$[0,1)$	0	0	0	1	$[7,8)$	3	4	0.3673	0.7551
$[1,2)$	1	0	0.1224	1	$[8,9)$	3	5	0.3673	0.6939
$[2,3)$	2	0	0.2449	1	$[9,10)$	3	6	0.3673	0.6327
$[3,4)$	3	0	0.3673	1	$[10,11)$	4	6	0.3673	0.6327
$[4,5)$	3	1	0.3673	0.9388	$[11,12)$	5	6	0.3673	0.6327
$[5,6)$	3	2	0.3673	0.8776	$[12,\infty)$	6	6	0.3673	0.6327
$[6,7)$	3	3	0.3673	0.8163					

Table 2.1: NPI lower and upper probabilities for the event $X_7 < Y_7$

From Table 2.1 and Figure 2.1 we see that, when increasing r_x while keeping r_y constant, the lower probability is stepwise increasing, except for $T_0 \geq 9$ as then $x_{r_x+1} > y_{n_y}$. When increasing r_y while keeping r_x constant, the upper probability is stepwise decreasing. All this is in agreement with Theorem 2.3 and with intuition.

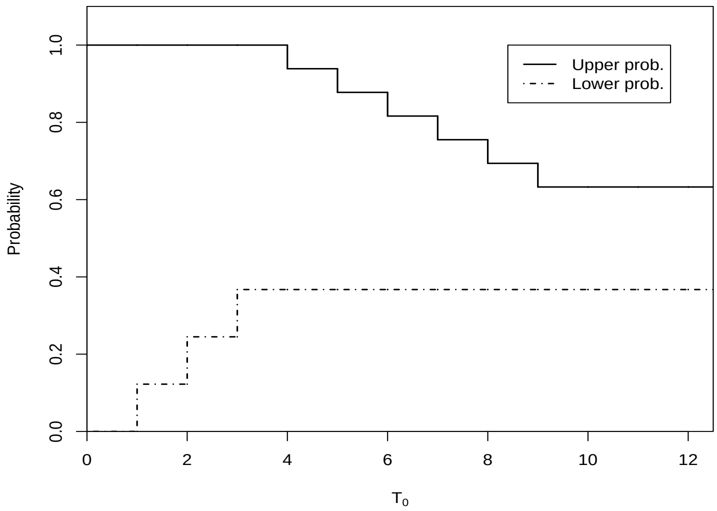


Figure 2.1: NPI lower and upper probabilities for the event $X_7 < Y_7$

For each T_0, $\frac{1}{2} \in [\underline{P}(X_7 < Y_7), \overline{P}(X_7 < Y_7)]$ which can be interpreted as no strong indication for $X_7 < Y_7$, nor for $Y_7 < X_7$ by conjugacy. Table 2.1 also shows that the imprecision decreases as the number of observations (or T_0) increases. The interval $[0.3673, 0.6327]$ is symmetric around $\frac{1}{2}$, due to the fact that our data are 'symmetric' in the order of the observations: first 3 lifetimes of group X, followed by 6 lifetimes of group Y and then again 3 lifetimes of group X. This interval $[0.3673, 0.6327]$ has been reached already at $T_0 = 9$ as at that moment all units of group Y are observed, implying that the 3 remaining lifetimes of group X must be larger than the largest lifetime of group Y. For our method only the order of the observed lifetimes is important, not the magnitude. △

Example 2.2. In this example we compare our NPI approach with the classical precedence tests reviewed in Section 2.2, using a subset of Nelson's dataset [64, p. 462] on breakdown times (in minutes) of an insulating fluid that is subject to high voltage stress. The data are given in Table 2.2.

We compare the lifetimes of units from groups X and Y by calculating the NPI lower and upper probabilities for the event $X_{11} < Y_{11}$, given the stopping time T_0, the observed lifetimes of both groups before T_0, and assuming rc-$A_{(10)}$ for both

Group	Lifetimes									
X	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75
Y	1.34	1.49	1.56	2.10	2.12	3.83	3.97	5.13	7.21	8.71

Table 2.2: Lifetimes of two samples of an insulating fluid

groups, with the assumption that both groups are completely independent. These lower and upper probabilities are given in Table 2.3 and Figure 2.2.

T_0	r_x	r_y	$\underline{P}$	$\overline{P}$	T_0	r_x	r_y	$\underline{P}$	$\overline{P}$
$[0, 0.49)$	0	0	0	1	$[2.10, 2.12)$	7	4	0.5289	0.8512
$[0.49, 0.64)$	1	0	0.0826	1	$[2.12, 2.15)$	7	5	0.5289	0.8264
$[0.64, 0.82)$	2	0	0.1653	1	$[2.15, 2.57)$	8	5	0.5702	0.8264
$[0.82, 0.93)$	3	0	0.2479	1	$[2.57, 3.83)$	9	5	0.6116	0.8264
$[0.93, 1.08)$	4	0	0.3306	1	$[3.83, 3.97)$	9	6	0.6116	0.8182
$[1.08, 1.34)$	5	0	0.4132	1	$[3.97, 4.75)$	9	7	0.6116	0.8099
$[1.34, 1.49)$	5	1	0.4132	0.9587	$[4.75, 5.13)$	10	7	0.6364	0.8099
$[1.49, 1.56)$	5	2	0.4132	0.9174	$[5.13, 7.21)$	10	8	0.6364	0.8099
$[1.56, 1.99)$	5	3	0.4132	0.8760	$[7.21, 8.71)$	10	9	0.6364	0.8099
$[1.99, 2.06)$	6	3	0.4711	0.8760	$[8.71, \infty)$	10	10	0.6364	0.8099
$[2.06, 2.10)$	7	3	0.5289	0.8760					

Table 2.3: NPI lower and upper probabilities for the event $X_{11} < Y_{11}$

Table 2.3 and Figure 2.2 show that, for increasing T_0, the lower probability is increasing when r_x increases and remains constant when r_y increases. The upper probability remains constant when r_x increases and is decreasing when r_y increases, except for $r_y \geq 7$ when it remains constant due to $x_{n_x} < y_{r_y+1}$ for such r_y, which illustrates Theorem 2.3. The imprecision is decreasing when more lifetimes are observed. However, if $T_0 \geq 4.75$, increasing r_y while keeping r_x constant does not lead to less imprecision due to the fact that at that time we have observed all lifetimes of group X ($x_{n_x} < y_{r_y+1}$) and consequently increasing r_y will not give us more information about the ordering of the lifetimes of groups X and Y.

One could interpret $\underline{P}(X_{11} < Y_{11}) > \frac{1}{2}$ as a strong indication that indeed $X_{11} < Y_{11}$. From Table 2.3 we see that if we had stopped the experiment at $T_0 = 2.06$ or later, then indeed $\underline{P}(X_{11} < Y_{11}) > \frac{1}{2}$. Had the experiment been stopped earlier,

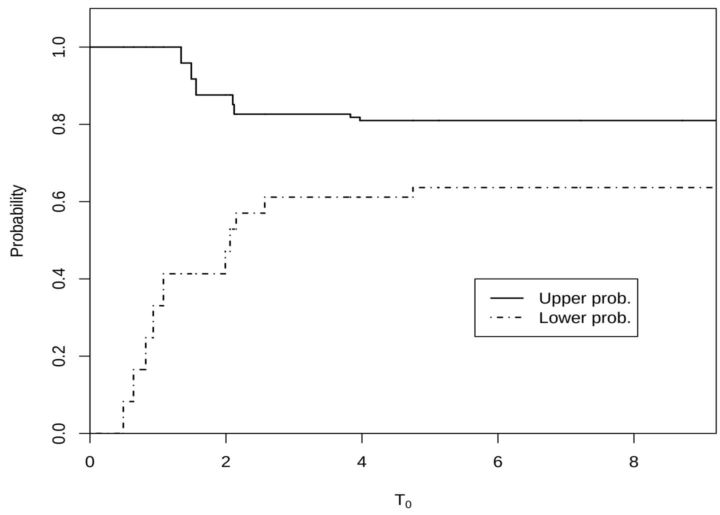


Figure 2.2: NPI lower and upper probabilities for the event $X_{11} < Y_{11}$

then the NPI lower and upper probabilities would not suggest a strong preference between the groups.

To compare this with the classical precedence tests, we test $H_0: \ F_X = F_Y$ against the alternative hypothesis that $F_X(x) \geq F_Y(x)$ for $x \geq 0$, with strict inequality for some x. Table 2.4 gives the values of the test statistics and the corresponding p-values (between brackets) for $r_y = 1, \ldots, 6$, where the restriction to these values of r_y is chosen as these illustrate all relevant issues in the discussion.

r_y	1	2	3	4	5	6
$Q_{(r_y)}$	5 (0.016)	5 (0.070)	5 (0.175)	7 (0.089)	7 (0.185)	9 (0.070)
$U_{(r_y)}$	5 (0.016)	5 (0.033)	5 (0.049)	5 (0.065)	5 (0.081)	5 (0.097)
$W_{\min,r_y}$	60 (0.016)	65 (0.022)	70 (0.033)	73 (0.031)	76 (0.035)	77 (0.029)
W_{E,r_y}	82.5 (0.016)	85 (0.036)	87.5 (0.067)	82 (0.035)	83.5 (0.048)	79 (0.025)
$W_{\max,r_y}$	105 (0.016)	105 (0.036)	105 (0.086)	91 (0.041)	91 (0.065)	81 (0.024)

Table 2.4: Several nonparametric precedence tests

Table 2.4 shows that the classical precedence test will not reject the null hypothesis of equal distributions at 5% significance level except when the experiment is terminated after the first lifetime of group Y. The maximal precedence test will reject the null hypothesis if the experiment is terminated after at most 3 lifetimes of group Y. Intuitively, this is logical as we have first observed 5 lifetimes of group X before the first observation of group Y and no observed lifetimes of group X between the first and third observation of group Y. In this example, Wilcoxon's minimal rank-sum precedence test always rejects the null hypothesis at 5% significance level. However, Wilcoxon's maximal and expected rank-sum precedence tests reject the null hypothesis only for some values of r_y. We saw before that according to our NPI approach there is an indication that $X_{11} < Y_{11}$ when the experiment is terminated after $T_0 = 2.06$. The results of the classical, Wilcoxon's maximal and expected rank-sum precedence tests at this T_0 are not in agreement with this but the maximal and Wilcoxon's minimal rank-sum precedence tests are. As the NPI approach is fundamentally different to these hypothesis tests, studying the results of both might provide useful insights for practical problems. $\triangle$

2.6 Concluding remarks

The lower and upper probabilities for predictive precedence testing for two groups, presented in this chapter, fit in the NPI framework and as such they have strong consistency properties in theory of interval probability [1]. This approach provides an attractive alternative to the more established methods for nonparametric precedence testing [5], as instead of testing a null hypothesis the inference directly considers a comparison of the next observations from the groups considered.

When considering the NPI lower and upper probabilities for the event $X_{n_x+1} < Y_{n_y+1}$ as a function of the stopping time T_0, we showed that these probabilities can only change at observed lifetimes for groups X or Y. In particular, we showed that, except for one special case, the lower probability is strictly increasing in r_x while keeping r_y constant, and the upper probability is strictly decreasing in r_y while keeping r_x constant. As a consequence of this, the imprecision is decreasing as a function of the number of observed lifetimes and hence as a function of time.

An important issue in statistics is guidance on required design of experiments, in this situation the numbers of units to be used for both groups and choice of the stopping time for the experiment. Due to the rather minimal assumptions underlying our NPI approach, with the inferences largely based on observed data, it does not offer a satisfactory solution to this important question. However, once an experiment is underway, one can monitor the lower and upper probabilities as presented in this chapter, and one can stop the experiment if one judges these to indicate a strong enough preference between the two groups. Of course, before any data become available, one can study some design issues, e.g. the minimum required number of observations to possibly get a lower probability greater than a half, but as these would be based on most or least favourable configurations of the not yet observed data, indications from such studies might be of little practical value.

Chapter 3

Multiple comparisons with early termination

3.1 Introduction

In Chapter 2 we introduced NPI for comparison of two groups with early termination of experiments. In this chapter, we consider the situation where units from several groups ($k \geq 2$) are simultaneously placed on a life-testing experiment, and decisions may be needed before all units have failed due to cost or time considerations.

Balakrishnan and Ng [5] described several nonparametric precedence tests based on the hypothesis of equal lifetime distributions. In Section 3.2 we briefly describe some of these classical methods. In Sections 3.3, 3.4 and 3.5 we present NPI for precedence testing for $k \geq 2$ groups in order to select the best group, the subset of best groups and the subset including the best group, respectively. Examples are provided throughout to illustrate our method and to compare it with the classical methods. Section 3.6 contains some concluding remarks.

3.2 Classical methods

When the null hypothesis of the equality (homogeneity) of two (or more) populations (e.g. processes, treatments) is rejected, one may want to identify which of these populations is the best. Balakrishnan and Ng [5] introduced several nonparametric tests for this selection problem when an early decision is required (called precedence testing). Below we briefly describe these precedence selection methods using notation and definitions from Balakrishnan and Ng [5].

Suppose that we have independent random samples from $k \geq 2$ different populations. Let X_{j,i_j} $(i_j = 1, \ldots, n_j)$ be the lifetime of the i_jth component of a random sample from population π_j with distribution function F_j $(j = 1, \ldots, k)$. We have $N = \sum_{j=1}^{k} n_j$ units placed simultaneously on a lifetime testing experiment. The question of interest is to test whether these populations are homogeneous, i.e. $H_0 : F_1 = F_2 = \ldots = F_k$ against the alternative that population π_i is the best (longer life), that is $H_{Ai} : F_i < F_j$ for all $j \neq i$ and $j = 1, \ldots, k$. That is, it can be concluded that X_i, a random quantity representing the lifetime of a unit of population i, is stochastically larger than X_j (i.e. $X_i \succ_{st} X_j$) if and only if $F_i(x) \leq F_j(x)$ for all $x \geq 0$ with strict inequality for at least one x, consequently $F_i < F_j$.

In precedence testing the aim is to reach a decision before all units have failed. So the experiment is terminated as soon as the $\bar{r}_i$th failure from group i is observed, where $\bar{r}_i = \lfloor n_i q \rfloor$ for $i = 1, \ldots, k$ and $0 < q < 1$, where $\lfloor a \rfloor$ is the largest integer not greater than a. Consequently, the stopping time T_0 can be defined as $T_0 = \min_{1 \leq i \leq k} X_{i,(\bar{r}_i)}$, where $X_{i,(\bar{r}_i)}$ is the $\bar{r}_i$th order statistic of sample i.

Suppose now that the experiment is terminated at sample i, i.e. $T_0 = X_{i,(\bar{r}_i)}$, then the *ordinary precedence statistic* [6] is defined as

$$Q^{*(i)} = \min_{\substack{1 \leq j \leq k \\ j \neq i}} \left(Q_j^{(i)} / n_j \right) \tag{3.1}$$

where $Q_j^{(i)}$ is the number of failures observed before $X_{i,(\bar{r})}$ from the sample j $(j = 1, \ldots, k,\ j \neq i)$. Small values of $Q^{*(i)}$ will lead to rejection of the null hypothesis H_0. In this case one can choose H_{Aj} (π_j is the best) if and only if $(Q_j^{(i)}/n_j) = Q^{*(i)}$ for $j \neq i$ and $T_0 = X_{i,(\bar{r}_i)}$. If for two or more samples the statistic $(Q_j^{(i)}/n_j)$ is equal to $Q^{*(i)}$ then one of the corresponding populations is randomly selected as the best.

Now let $D_{j,s}^{(i)}$ be the number of failures of sample j that occur between the $(s-1)$th and sth failure of group i, $s = 2, \ldots, \bar{r}_i$ and let $D_{j,1}^{(i)}$ be the number of failures of sample j that occur before the first failure of group i. Let $W_j^{(i)}$ $(j = 1, \ldots, k,\ j \neq i)$

be a random quantity defined by

$$W_j^{(i)} = \frac{1}{2} n_j(n_j + 2\bar{r}_i + 1) - (\bar{r}_i + 1) \sum_{s=1}^{\bar{r}_i} D_{j,s}^{(i)} + \sum_{s=1}^{\bar{r}_i} s \; D_{j,s}^{(i)}$$

Then the *minimal Wilcoxon rank-sum statistic* [68] is given by

$$W^{*(i)} = \max_{\substack{1 \le j \le k \\ j \ne i}} \left(\frac{W_j^{(i)} - E[W_j^{(i)}|H_0]}{\sqrt{Var[W_j^{(i)}|H_0]}} \right) \tag{3.2}$$

where $E[W_j^{(i)}|H_0]$ and $Var[W_j^{(i)}|H_0]$ are the expected value and the variance of the statistic $W_j^{(i)}$ under H_0. Large values of $W^{*(i)}$ will lead to rejection of the null hypothesis, in which case one can choose the alternative hypothesis H_{Aj} (π_j is the best) if and only if $T_0 = X_{i,(\bar{r}_i)}$ and $(W_j^{(i)} - E[W_j^{(i)}|H_0])(Var[W_j^{(i)}|H_0])^{-1/2} = W^{*(i)}$, for $j \ne i$.

In this chapter we will focus on the balanced-sample case only ($n_j = n$ for all j) when we compare our method with the classical precedence selection procedures, since these classical procedures may not be effective when the sample sizes vary much [5, pp. 226, 265]. For the balanced-sample case, the statistics in (3.1) and (3.2) reduce to $Q^{*(i)} = \min Q_j^{(i)}$ and $W^{*(i)} = \max W_j^{(i)}$, over all $j = 1, \ldots, k$ and $j \ne i$, respectively. For more details we refer to Balakrishnan and Ng [5]. It should be emphasized that the NPI method presented in this chapter is equally straightforward to implement for balanced-sample and unbalanced-sample cases.

A somewhat separate, yet strongly related, branch of statistical research is so-called 'selection methods', which also have the explicit target to select a single 'best' group or population or a subset of groups, along the same lines considered in this chapter. The two main classical approaches in this field are indifference zone selection [11, 12] and subset selection [38], which were combined by Verheijen *et al.* [75]. Coolen and van der Laan [25] presented NPI methods for selection, in this chapter we follow the same approach with the generalization to allow early termination of the experiments, hence linking to the classical concepts of precedence testing.

3.3 Selecting the best group

In precedence testing, units of all groups are placed simultaneously on a life-testing experiment, and failures are observed as they arise during the experiment. The experiment is terminated as soon as a certain stop criterion has been reached, so the lifetimes of some units are typically right-censored. We assume that this stop criterion is expressed in terms of a stopping time T_0, but if instead a number of failures were used as stop criterion then this would not affect our method, as it is of no relevance in NPI how T_0 is determined as long as T_0 contains no further information on the residual event times beyond T_0 for right-censored units. In Chapter 2 we introduced NPI for precedence testing for two groups. In this chapter we extend NPI to precedence testing for $k \geq 2$ groups in order to select the best group, the subset of best groups, and the subset including the best group. Again we use the assumption rc-$A_{(n)}$ required for precedence testing which is given in Theorem 2.1.

Suppose we have $k \geq 2$ groups and $n_j + 1$ random quantities from group j, denoted by X_{j,i_j} where $i_j = 1, 2, \ldots, n_j, n_j + 1$, $j = 1, 2, \ldots, k$. For each group j, n_j units are put on a lifetime experiment and we are interested in the behaviour of the future random variable X_{j,n_j+1}. Therefore, we have $N = \sum_{j=1}^{k} n_j$ units on the lifetime experiment and one may want to terminate the experiment at certain time T_0. Let $0 = x_{j,0} < x_{j,1} < x_{j,2} < \ldots < x_{j,r_j} \leq T_0 < \infty$ be the ordered observed values (failures) from group j, $j = 1, \ldots, k$.

These observed values from group j produce $r_j + 2$ intervals, where the first r_j intervals are defined by $I^j_{i_j} = (x_{j,i_j-1}, x_{j,i_j})$, $i_j = 1, \ldots, r_j$, $j = 1, \ldots, k$, and the remaining intervals are defined by $I^j_{r_j+1} = (x_{j,r_j}, \infty)$, $I^j_{r_j+2} = (T_0, \infty)$, notice that these are overlapping. Let $L(I^j_{i_j})$ and $U(I^j_{i_j})$ be the lower and the upper bounds for the interval $I^j_{i_j}$, $i_j = 1, \ldots, r_j + 2$, $j = 1, \ldots, k$. That is, $L(I^j_{i_j}) = x_{j,i_j-1}$ for $i_j = 1, \ldots, r_j + 1$ and $L(I^j_{r_j+2}) = T_0$. Similar for the upper bound, $U(I^j_{i_j}) = x_{j,i_j}$ for $i_j = 1, \ldots, r_j$, and $U(I^j_{r_j+1}) = U(I^j_{r_j+2}) = \infty$. Here the intervals $I^j_{i_j}$ are open intervals, but in future when we mention the (left or right) end points we actually mean the limit end points which are not included in these open intervals.

For our NPI approach we assume rc-$A_{(n_j)}$ for each group [27]. Beyond the data, our method requires the exchangeability assumptions of the random variables per group to be met. We also assume that groups are completely independent. We will specify partially the probability distribution for a future quantity, X_{j,n_j+1}, $j = 1,\ldots,k$, using M-functions presented in Theorem 2.1.

Theorem 3.1 gives the NPI lower and upper probability for the event that the lifetime of the next observation from one group, say l, is greater than the lifetime of the next observation from each other group, that is

$$\underline{P}^{(l)} = \underline{P}\left(X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}\right) \quad \text{and} \quad \overline{P}^{(l)} = \overline{P}\left(X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}\right)$$

Theorem 3.1. The NPI lower and upper probabilities for the event that the lifetime of the next observation from group l is greater than the lifetime of the next observation from each other group are

$$\underline{P}^{(l)} = \frac{1}{\prod_{j=1}^{k}(n_j+1)} \left\{ \sum_{i_l=1}^{r_l} \prod_{\substack{j=1\\ j\neq l}}^{k} \sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < x_{l,i_l}\} + (n_l - r_l) \prod_{\substack{j=1\\ j\neq l}}^{k} r_j \right\} \tag{3.3}$$

$$\overline{P}^{(l)} = \frac{1}{\prod_{j=1}^{k}(n_j+1)} \sum_{i_l=1}^{r_l} \prod_{\substack{j=1\\ j\neq l}}^{k} \left(1 + \sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < x_{l,i_l}\}\right) + \frac{n_l - r_l + 1}{n_l + 1} \tag{3.4}$$

Proof. The proof is a special case of the proof of Theorem 3.3, with S (in Theorem 3.3) now only containing group l. □

3.3.1 Special cases

In this part, we discuss some special cases of these lower and upper probabilities, which are easily verified from (3.3) and (3.4).

1. If $r_l \geq 0$ and there exists at least one $j \neq l$ for which $r_j = 0$, then the NPI lower probability is $\underline{P}^{(l)} = 0$, since we have not seen any failure from group $j \neq l$. Hence, we cannot exclude the possibility that we would never see a failure from group(s) $j \neq l$. Further, if $r_l = 0$ then the upper probability $\overline{P}^{(l)}$ is equal to one.

2. If $r_l = 0$ and $r_j > 0$ for all $j \neq l$, then the NPI upper probability $\overline{P}^{(l)}$ is one, as we cannot exclude the possibility that we would never see a failure of group l. The corresponding lower probability is

$$\underline{P}^{(l)} = \frac{n_l}{\prod_{j=1}^{k}(n_j+1)} \prod_{\substack{j=1 \\ j\neq l}}^{k} r_j$$

   Further, if $r_l = 0$ and $r_j = n_j$ for all $j \neq l$, that is we have observed all units from each group $j \neq l$ and the experiment is ended before we observe any failure from group l, then the NPI lower probability is

$$\underline{P}^{(l)} = \prod_{j=1}^{k} \frac{n_j}{n_j+1}$$

3. If $r_l > 0$ and $r_j = 0$ for all $j \neq l$, so we have not seen any failure for all groups $j \neq l$, then we cannot exclude the possibility that we would never see a failure of these groups and consequently $\underline{P}^{(l)} = 0$. The corresponding upper probability is

$$\overline{P}^{(l)} = \frac{r_l}{\prod_{j=1}^{k}(n_j+1)} + \frac{n_l - r_l + 1}{n_l + 1}$$

   Further, if $r_l = n_l$ and $r_j = 0$, that is we have observed all units from group l and the experiment is ended before we observe any failure from all other groups, then the NPI upper probability is

$$\overline{P}^{(l)} = \frac{n_l}{\prod_{j=1}^{k}(n_j+1)} + \frac{1}{n_l + 1}$$

4. If $r_l > 0$, $r_j > 0$ and $x_{j,r_j} < x_{l,1}$ for all $j \neq l$, then the NPI lower and upper probabilities are

$$\underline{P}^{(l)} = \frac{n_l}{\prod_{j=1}^{k}(n_j+1)} \prod_{\substack{j=1 \\ j\neq l}}^{k} r_j, \quad \overline{P}^{(l)} = \frac{r_l}{\prod_{j=1}^{k}(n_j+1)} \prod_{\substack{j=1 \\ j\neq l}}^{k} (r_j+1) + \frac{n_l - r_l + 1}{n_l + 1}$$

   But if $x_{j,1} > x_{l,r_l}$, for all $j \neq l$, then the NPI lower and upper probabilities are

$$\underline{P}^{(l)} = \frac{(n_l - r_l)}{\prod_{j=1}^{k}(n_j+1)} \prod_{\substack{j=1 \\ j\neq l}}^{k} r_j, \quad \overline{P}^{(l)} = \frac{r_l}{\prod_{j=1}^{k}(n_j+1)} + \frac{n_l - r_l + 1}{n_l + 1}$$

3.3.2 Some properties

Now, we study the effect upon the NPI lower and upper probabilities when the stopping time is increased from T_0 to $T_0 + \epsilon$, for small $\epsilon > 0$, such that there is only one extra failure from one group occurs.

Theorem 3.2. (i) If a failure occurs from group l then the NPI lower probability $\underline{P}^{(l)}$ remains constant. However the NPI upper probability $\overline{P}^{(l)}$ decreases by

$$\frac{1}{n_l + 1} + \frac{1}{\prod_{j=1}^{k}(n_j + 1)} \prod_{\substack{j=1 \\ j \neq l}}^{k} (r_j + 1)$$

except when $r_j = n_j$, for all $j \neq l$, in which case the upper probability remains constant.

(ii) If a failure occurs for group j^*, where $j^* \in \{1, \ldots, k\} \setminus \{l\}$, then the NPI upper probability $\overline{P}^{(l)}$ remains constant. However, the NPI lower probability increases by

$$\frac{n_l - r_l}{\prod\limits_{j=1}^{k}(n_j + 1)} \prod_{\substack{j=1 \\ j \neq \{l, j^*\}}}^{k} r_j$$

except when $r_l = n_l$, or when at least one $r_j = 0$ for a $j \neq \{j^*, l\}$, in which cases the lower probability remains constant.

Proof. For case i (ii), replace r_l (r_{j^*}) by $r_l + 1$ ($r_{j^*} + 1$) in formula (3.3) and (3.4), then this follows by basic analysis of the lower and upper probabilities of Theorem 3.1. This is similar to the proof of Theorem 3.4. □

Theorem 3.2 is in line with the intuition that the lower probability for a certain event quantifies the amount of information in favour of the event while the upper probability quantifies the amount of information against the event. If r_l is increased while leaving all other r_j the same, then, when considering the event $X_{l,n_l+1} = \max\limits_{1 \leq j \leq k} X_{j,n_j+1}$, the amount of information in favour of this event remains the same but the amount of information against this event increases, except when $r_j = n_j$ for all $j \neq l$. Consequently, $\underline{P}^{(l)}$ does not change but $\overline{P}^{(l)}$ may decrease. For the same event, when r_j for a $j \neq l$ increases while all other r_i, $i \neq j$, remain constant, the amount of information in favour of the event increases, except when $r_l = n_l$ or when there exists a $j \neq \{l, j^*\}$ for which $r_j = 0$, while the amount of information against the event remains the same. Consequently, $\underline{P}^{(l)}$ may increase but $\overline{P}^{(l)}$ does not change.

At any value of T_0, we can state that the data provide a strong indication that group l is the best if $\underline{P}^{(l)} > \overline{P}^{(j)}$ for all $j \neq l$. Of course, this may not occur, and we may be happy to have data providing a weak indication that group l is the best. It might seem attractive to state that, if $\underline{P}^{(l)} > \underline{P}^{(j)}$ and $\overline{P}^{(l)} > \overline{P}^{(j)}$ for all $j \neq l$, there would be a weak indication that group l is the best. Indeed, if one has to select one group and there is a group for which such a weak indication of being best holds, then that is the natural candidate. However, such a weak indication can be very weak indeed, in particular as it can already occur for relatively small T_0, with $\underline{P}^{(l)}$ positive but very small. If such a weak indication holds for one group, and in addition one judges the lower probability of this group being best to be sufficiently high, then it seems a reasonable basis for the choice of this group as being the best. In all these considerations, it is an advantage that the the difference between corresponding lower and upper probabilities $(\overline{P}^{(l)} - \underline{P}^{(l)})$ reflects the amount of information available, and it decreases if more relevant information becomes available. If one judges this difference to be too large, or if one judges each lower probability of a group being best too small to base a choice on the information available, then clearly one must either get more information, e.g. by continuing the experiment or try to repeat the experiment with more units, or one could explore the use of other statistical approaches with more modeling assumptions.

In discussions in the examples in this chapter, we will call one group 'better' than another, or 'best', if the first of these conditions is satisfied, of course the use of 'better' and 'best' must be interpreted with care as these judgments are just based on direct comparison of one next observation for each group according to the NPI method.

Below two examples are given to illustrate our method for selecting the best group and to compare it with the classical methods reviewed in Section 3.2.

Example 3.1. To illustrate our method for selecting the best group among k other groups, we use the data from Coolen and van der Laan [25] as presented in Table 3.1.

Group										
1	5.01	5.04	5.60	5.78	6.43	6.53	6.96	7.00	7.21	7.58
	8.12	8.26	8.27	8.34	8.62	8.66	8.91	8.94	9.05	9.16
2	4.50	4.86	5.10	5.15	5.17	5.34	5.99	6.18	6.72	7.39
	7.44	7.46	7.47	7.76	8.38	8.42	8.52	8.81		
3	6.84	6.91	7.22	7.24	7.25	7.35	7.55	7.62	7.69	7.98
	7.99	8.04	8.08	8.18	8.97					
4	4.71	8.20	9.03							

Table 3.1: Data set, Example 3.1

This data set consists of four groups and is used by Coolen and van der Laan [25] in order to demonstrate the NPI method for selection of the best source and a subset to include the best source for complete data, so without censoring. We interpret this data set as the lifetimes of units from 4 different groups. The size of the groups are $n_1 = 20$, $n_2 = 18$, $n_3 = 15$ and $n_4 = 3$, and X_{j,i_j} $(i_j = 1, \ldots, n_j)$ represents the lifetime of unit i_j in group j.

A group is considered as the 'best' when the lifetime of a future unit from this group is larger than the lifetime of a future unit from all other groups. Our inference depends on the data, the rc-$A_{(n_j)}$ assumptions $(j = 1, 2, 3, 4)$ for each group, and stopping time T_0. Table 3.2 presents the NPI lower and upper probabilities for the event that the lifetime of a future unit of group l $(l = 1, 2, 3, 4)$ is larger than the lifetimes of a future unit of all other groups, as given by (3.3) and (3.4), for stopping time T_0 in several intervals. We denote these lower and upper probabilities by $\underline{P}^{(l)}$ and $\overline{P}^{(l)}$, respectively.

Let us consider the situation when we terminate the experiment at $T_0 = 5$. Until this point we observed only two failures from group 2 and one failure from group 4, and we have not yet observed any failures from groups 1 and 3. Here all lower probabilities are equal to zero since for each l, there exists a group $j \neq l$ for which we have not observed a failure yet. Moreover, while the upper probabilities for the

T_0	r_1	r_2	r_3	r_4	$\underline{P}^{(1)}$	$\overline{P}^{(1)}$	$\underline{P}^{(2)}$	$\overline{P}^{(2)}$	$\underline{P}^{(3)}$	$\overline{P}^{(3)}$	$\underline{P}^{(4)}$	$\overline{P}^{(4)}$
[4.86, 5.01)	0	2	0	1	0	1	0	0.895	0	1	0	0.750
[5.99, 6.18)	4	7	0	1	0	0.811	0	0.633	0.016	1	0	0.750
[7.00, 7.21)	8	9	2	1	0.010	0.627	0.006	0.529	0.041	0.886	0.011	0.750
[7.21, 7.22)	9	9	3	1	0.014	0.582	0.010	0.529	0.046	0.831	0.019	0.750
[7.22, 7.24)	9	9	4	1	0.018	0.582	0.013	0.529	0.046	0.777	0.025	0.750
[7.46, 7.47)	9	12	6	1	0.033	0.582	0.019	0.387	0.055	0.667	0.051	0.750
[7.47, 7.55)	9	13	6	1	0.036	0.582	0.019	0.340	0.058	0.667	0.055	0.750
[7.55, 7.58)	9	13	7	1	0.041	0.582	0.021	0.340	0.058	0.616	0.064	0.750
[7.99, 8.04)	10	14	11	1	0.066	0.543	0.029	0.296	0.065	0.416	0.121	0.750
[8.42, 8.52)	14	16	14	2	0.164	0.448	0.073	0.244	0.077	0.268	0.207	0.606
[8.62, 8.66)	15	17	14	2	0.171	0.432	0.075	0.218	0.080	0.268	0.224	0.606
[8.66, 8.81)	16	17	14	2	0.171	0.416	0.076	0.218	0.081	0.268	0.234	0.606
[8.81, 8.91)	16	18	14	2	0.175	0.416	0.076	0.195	0.082	0.268	0.242	0.606
[8.91, 8.94)	17	18	14	2	0.175	0.402	0.076	0.195	0.084	0.268	0.252	0.606
[8.94, 8.97)	18	18	15	2	0.178	0.388	0.076	0.195	0.085	0.248	0.275	0.606
[8.97, 9.03)	18	18	15	2	0.178	0.388	0.076	0.195	0.085	0.248	0.275	0.606
[9.03, 9.05)	18	18	15	3	0.199	0.388	0.076	0.195	0.085	0.248	0.275	0.582
[9.05, 9.16)	19	18	15	3	0.199	0.388	0.076	0.195	0.085	0.248	0.275	0.582
[9.16, ∞)	20	18	15	3	0.199	0.388	0.076	0.195	0.085	0.248	0.275	0.582

Table 3.2: The best group: NPI lower and upper probabilities

first and third groups are equal to 1, those for groups 2 and 4 are less than 1, being equal 0.895 and 0.750, respectively. As no failure is observed from groups 1 and 3, we cannot exclude the possibility that we will never observe any failures from these groups and consequently $\overline{P}^{(1)} = \overline{P}^{(3)} = 1$.

At $T_0 = 6$, we still have not observed any failure from group 3, so we cannot exclude the possibility that we will never observe any failure from this group and consequently $\overline{P}^{(3)} = 1$. However, the lower probability for this group is now positive as there is no other group for which we have not observed a failure. For all other groups the lower probability is still zero as we have not seen any failure yet from group 3.

From Theorem 3.2 we know that the lower probability never decreases and the upper probability never increases. For example, consider the situation where the stopping time T_0 is increased from 7.50 to 7.55. At $T_0 = 7.55$, a failure of group 3 occurs. We want to calculate the lower and upper probabilities for the event $X_{3,n_3+1} = \max_{1\leq j\leq 4} X_{j,n_j+1}$. For this case, the lower probability remains constant ($\underline{P}^{(3)} = 0.058$), but the upper probability decreases from 0.667 at $T_0 = 7.50$ to 0.616 at

$T_0 = 7.55$, which illustrates Theorem 3.2(i). However, for the event $X_{1,n_1+1} = \max_{1 \leq j \leq 4} X_{j,n_j+1}$ the upper probability remains constant ($\overline{P}^{(1)} = 0.582$) but the lower probability increases from 0.036 at $T_0 = 7.50$ to 0.041 at $T_0 = 7.55$, which illustrates Theorem 3.2(ii).

There are some special cases when all lower and upper probabilities remain constant when a failure occurs from any group. For example, at $T_0 = 9.03$ we have observed all units from all groups except the first group which still has two units which have not failed. Let $l = 1$ and assume we will allow for an extra failure to occur. Here of course the failure must be from the first group ($T_0 = 9.05$). In this case all lower and upper probabilities remain as they were at $T_0 = 9.03$, as the amount of information in favour and against the event does not change. In fact, all lower and upper probabilities do not change anymore after 9.03.

At $T_0 = 8.81$, we have observed failure times of all units from the second group. Now consider $l = 2$ and let the stopping time increase to 8.91, so that we observe an extra failure of group 1. In this case the lower and upper probabilities remain constant ($\underline{P}^{(2)} = 0.076$ and $\overline{P}^{(2)} = 0.195$). In fact, any failure from other groups after we have observed failures of all units from group 2 will not affect the lower and upper probabilities $\underline{P}^{(2)}$ and $\overline{P}^{(2)}$.

From $T_0 = 7.55$ on, the fourth group has the greatest lower and upper probabilities. However, from the beginning of the experiment till $T_0 = 7.22$ the third group has the greatest lower and upper probabilities. Which means that at $T_0 \leq 7.22$, there is a weak indication that group 3 is best, since $\underline{P}^{(j)} < \underline{P}^{(3)} < \overline{P}^{(j)} < \overline{P}^{(3)}$ for $j = 1, 2, 4$, however, there is a weak indication that group 4 is best for $T_0 \geq 7.55$, since $\underline{P}^{(j)} < \underline{P}^{(4)} < \overline{P}^{(j)} < \overline{P}^{(4)}$ for $j = 1, 2, 3$. Also we can note that $\overline{P}^{(3)}$ remains equal to 1 for quite a long time, since the first failure from the third group occurs relatively late.

The imprecision decreases as the stopping time T_0 increases, which reflects the amount of information we have (Table 3.2). For example, we can see that the fourth group has larger imprecision as there are only a few observations in this group.

A crucial question is how to make decisions using these NPI lower and upper probabilities. If we observe all units from groups 2, 3 and 4, so for $T_0 \geq 9.03$, we see that $\overline{P}^{(2)} < \underline{P}^{(1)}$ and $\overline{P}^{(2)} < \underline{P}^{(4)}$ implying that group 1 and 4 are certainly better than group 2. Also $\overline{P}^{(3)} < \underline{P}^{(4)}$ implying that group 4 is better than group 3. It is still difficult to distinguish between groups 1 and 4. As $\underline{P}^{(1)} < \underline{P}^{(4)} < \overline{P}^{(1)} < \overline{P}^{(4)}$ there is a weak preference for group 4. For $T_0 \geq 8.62$, group 4 is better than group 2 ($\overline{P}^{(2)} < \underline{P}^{(4)}$), and for $T_0 \geq 8.97$, group 4 is better than groups 2 and 3 ($\overline{P}^{(2)} < \underline{P}^{(4)}$ and $\overline{P}^{(3)} < \underline{P}^{(4)}$). However, we have to be careful as group 4 only has 3 observations and its imprecision is large. Therefore, we will now exclude the fourth group from the comparison and we will recompute the NPI lower and upper probabilities to study the effect of the fourth group on the comparison.

T_0	r_1	r_2	r_3	$\underline{P}^{(1)}$	$\overline{P}^{(1)}$	$\underline{P}^{(2)}$	$\overline{P}^{(2)}$	$\underline{P}^{(3)}$	$\overline{P}^{(3)}$
$[4.86, 5.01)$	0	2	0	0	1	0	0.895	0	1
$[5.99, 6.18)$	4	7	0	0	0.813	0	0.635	0.066	1
$[7.00, 7.21)$	8	9	2	0.040	0.634	0.023	0.531	0.164	0.897
$[7.47, 7.55)$	9	13	6	0.143	0.592	0.076	0.365	0.233	0.710
$[7.55, 7.58)$	9	13	7	0.165	0.592	0.083	0.365	0.233	0.669
$[7.99, 8.04)$	10	14	11	0.264	0.561	0.117	0.329	0.258	0.519
$[8.42, 8.52)$	14	16	14	0.354	0.510	0.171	0.294	0.274	0.411
$[8.62, 8.66)$	15	17	14	0.367	0.504	0.173	0.277	0.279	0.411
$[8.66, 8.81)$	16	17	14	0.367	0.499	0.175	0.277	0.281	0.411
$[8.81, 8.91)$	16	18	14	0.376	0.499	0.175	0.264	0.284	0.411
$[8.91, 8.94)$	17	18	14	0.376	0.496	0.175	0.264	0.287	0.411
$[8.94, 8.97)$	18	18	14	0.376	0.493	0.175	0.264	0.289	0.411
$[8.97, 9.05)$	18	18	15	0.381	0.493	0.175	0.264	0.289	0.405
$[9.05, 9.16)$	19	18	15	0.381	0.493	0.175	0.264	0.289	0.405
$[9.16, \infty)$	20	18	15	0.381	0.493	0.175	0.264	0.289	0.405

Table 3.3: The best group: NPI lower and upper probabilities (without group 4)

Table 3.3 presents NPI lower and upper probabilities (3.3) and (3.4) after we have excluded the fourth group from this comparison, to study the effect of this group on our inferences. For example, at $T_0 = 8.42$ we observed 14, 16 and 14 failures from groups 1, 2 and 3, respectively. Here $\overline{P}^{(2)} < \underline{P}^{(1)}$ which indicates that the first group is better than the second group. The second group would be the worst group for $T_0 \geq 8.62$ since then $\overline{P}^{(2)} < \underline{P}^{(1)}$ and $\overline{P}^{(2)} < \underline{P}^{(3)}$. When observing all units from all groups, there exists a weak preference for group 1 compared to group 3 as $\underline{P}^{(3)} < \underline{P}^{(1)} < \overline{P}^{(3)} < \overline{P}^{(1)}$. In addition, the imprecision is slightly larger for group 3 than for group 1.

Furthermore, as we can see from Tables 3.2 and 3.3, dropping group 4 leads to substantial increases in the NPI lower and upper probabilities for both the first group and the third group, with slight increases in the lower and upper probabilities for the second group. However, it is still not possible to make a clear decision on which group will have the largest next observation. Removing the fourth group has an influence not only on improving the lower and upper probabilities but also on reducing the imprecision for other groups. △

Example 3.2. In this example, we compare our method with the classical precedence selection methods in order to select the best group. Table 3.4 shows the natural logarithm of times to breakdown of an insulating fluid at three voltage levels (30kv, 35kv and 40kv), as given by Nelson [64, p. 278]. We will refer to these voltage levels as groups $j = 1, 2, 3$, respectively. Here we have a balanced-sample case where $n_1 = n_2 = n_3 = 12$. Let X_{j,i_j} represent the natural logarithm of time to breakdown for the i_jth unit at voltage level j, $i_j = 1, \ldots, 12$ and $j = 1, 2, 3$. Balakrishnan *et al.* [6] and Ng *et al.* [68] considered the last two values at level

Group		Times to breakdown of an insulating fluid					
1	30kv	3.912	4.898	5.231	6.782	7.279	7.293
		7.736	7.983	8.338	9.668	10.282^{+}	11.363^{+}
2	35kv	3.401	3.497	3.715	4.466	4.533	4.585
		4.754	5.553	6.133	7.073	7.208	7.313
3	40kv	0.000	0.000	0.693	1.099	2.485	3.219
		3.829	4.025	4.220	4.691	5.778	6.033

Table 3.4: Times (ln) to breakdown of an insulating fluid

30kv (group) as real failures although they are in fact censored observations. We follow their approach, although as these values are larger than all observations for the other groups, it makes no difference to our approach for any $T_0 < 10.282$.

The classical precedence selection procedures normally test the homogeneity of the lifetime distributions against the alternative that one distribution stochastically dominates the other distributions (so one population is the best) in terms of their reliability (longer life). That means, the classical selection procedures are designed to test $H_0 : F_1 = F_2 = F_3$ in favour of the alternative $H_{Ai} : F_i < F_j$, for all $j \neq i,\ j = 1, 2, 3$.

Here we have $k = 3$ lifetime samples with equal sample sizes. We will stop the experiment as soon as the 8th failure ($\bar{r} = 8$) from any group has occurred following Ng *et al.* [68]. So the experiment is terminated at $T_0 = 4.025$, when the 8th breakdown time of group 3 is observed. The test statistic for the ordinary precedence test, calculated from (3.1), is $Q^{*(3)} = \min\{1, 3\} = 1$ and the p-value of this test is 0.0256. The minimal Wilcoxon rank-sum precedence test statistic (3.2) equals $W^{*(3)} = \max\{173, 168\} = 173$ and the p-value is 0.0066.

In this case, at significance level 5%, we reject the null hypothesis for both test statistics $Q^{*(3)}$ and $W^{*(3)}$, and therefore we will select the first population (30kv) as the best, i.e. we reject H_0 in favour of H_{A1}. We would get a different decision at significance level 1%, for which the minimal Wilcoxon rank-sum precedence selection method leads to rejection of the null hypothesis while the ordinary precedence selection method does not lead to rejection of the null hypothesis. In such a situation it is a good idea to apply our method to the data to see whether our method leads to a 'best' or 'worst' group. Table 3.5 contains the NPI lower and upper probabilities that the lifetime of the next observation of group l ($l = 1, 2, 3$) is larger than the lifetime of the next observation of each other group for certain values of T_0.

T_0	r_1	r_2	r_3	$\underline{P}^{(1)}$	$\overline{P}^{(1)}$	$\underline{P}^{(2)}$	$\overline{P}^{(2)}$	$\underline{P}^{(3)}$	$\overline{P}^{(3)}$
0.693	0	0	3	0	1	0	1	0	0.771
3.401	0	1	6	0.033	1	0	0.926	0	0.541
4.025	1	3	8	0.130	0.938	0.033	0.779	0.007	0.393
4.691	1	6	10	0.310	0.938	0.040	0.575	0.011	0.249
4.898	2	7	10	0.360	0.901	0.062	0.508	0.018	0.249
6.033	3	8	12	0.467	0.864	0.096	0.452	0.027	0.128
6.133	3	9	12	0.516	0.864	0.096	0.398	0.027	0.128
7.293	6	11	12	0.603	0.834	0.123	0.304	0.027	0.128
11.363	12	12	12	0.636	0.834	0.123	0.268	0.027	0.128

Table 3.5: The best group: NPI lower and upper probabilities

Table 3.5 shows that, after we have only observed three failures from group 3 (40kv, $T_0 = 0.693$) we cannot make any reasonable decision ($\underline{P}^{(1)} = \underline{P}^{(2)} = 0$ and $\overline{P}^{(1)} = \overline{P}^{(2)} = 1$) on whether the first or the second group is the best, since we have not yet observed any failures from both groups.

However, at $T_0 = 4.025$, when we have observed the 8th failure from group 3, we have $\underline{P}^{(3)} < \underline{P}^{(2)} < \underline{P}^{(1)}$ and $\overline{P}^{(3)} < \overline{P}^{(2)} < \overline{P}^{(1)}$ but $\overline{P}^{(3)} \nless \underline{P}^{(1)}$ or $\overline{P}^{(3)} \nless \underline{P}^{(2)}$. So, there is no strong indication to select group 3 as the best group, which is in agreement with the ordinary precedence test at significance level 1% that one particular group is the worst. However, there is a weak indication that group 3 is the worst, but this does not follow from the classical methods.

Here, when we have observed all failures, we have a strong indication that group 1 is the best as $\overline{P}^{(2)} < \underline{P}^{(1)}$ and $\overline{P}^{(3)} < \underline{P}^{(1)}$. In fact this holds already at $T_0 = 6.033$. At $T_0 = 4.691$ we can conclude already that group 1 is better than group 3 as from that moment on we have $\overline{P}^{(3)} < \underline{P}^{(1)}$. Then at $T_0 = 6.133$, we also have in addition $\overline{P}^{(2)} < \underline{P}^{(1)}$ and consequently we have a strong indication that group 1 is the best. So at $T_0 = 4.025$, our method leads to a conclusion in the line with the minimal Wilcoxon rank-sum precedence selection method.

In this example we show that the NPI method and the classical precedence tests do not necessarily lead to the same conclusions, but it is difficult to compare these two due to the different inferential goals and the different basic underlying assumptions. Hence, we do not see these as competing methods for the same problems, but more as complementary methods that can provide further insight into specific applications, and which may be more or less suitable depending on the explicit inferential goal. △

3.4 Selecting the subset of best groups

Suppose that the experiment is terminated at time T_0 and our interest is to select a subset of groups such that all the groups in this subset are 'better' than all not selected groups, that is the lifetime of the next observation of each group in the subset will be greater than the lifetime of the next observation of all groups not in the subset. Let $S = \{l_1, l_2, \ldots, l_m\}$ be a subset of m groups $(1 \le m \le k-1)$ from k independent groups, and let NS be the complementary set of S which contains the remaining $k - m$ groups.

We will derive the NPI lower and upper probabilities for the event that the next observation of each group in S has longer lifetime than the next observation of each group in NS, denoted by

$$\underline{P}^{S} = \underline{P}\left(\min_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right) \text{ and } \overline{P}^{S} = \overline{P}\left(\min_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right)$$

These NPI lower and upper probabilities are given in Theorem 3.3, where the following notation is used

$$\sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} = \sum_{i_{l_1}=1}^{r_{l_1}+2} \cdots \sum_{i_{l_m}=1}^{r_{l_m}+2} \tag{3.5}$$

Theorem 3.3. The NPI lower and upper probabilities for the event that the next observation of each group in S has longer lifetime than the next observation of each group in NS are

$$\underline{P}^{S} = \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \prod_{j\in NS} \left[\frac{\sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{L(I^l_{i_l})\}\}}{n_j+1}\right] \cdot \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l}) \tag{3.6}$$

$$\overline{P}^{S} = \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \prod_{j\in NS} \left[\frac{1+\sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{U(I^l_{i_l})\}\}}{n_j+1} + \frac{(n_j-r_j)\mathbf{1}\{T_0 < \min_{l\in S}\{U(I^l_{i_l})\}\}}{n_j+1}\right] \cdot \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l}) \tag{3.7}$$

Proof. First, we derive the lower probability as follows

$$P\left(\min_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right) = P\left(\bigcap_{j\in NS} \{X_{j,n_j+1} < \min_{l\in S} X_{l,n_l+1}\}\right)$$

$$= \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} P\left(\bigcap_{j\in NS} \{X_{j,n_j+1} < \min_{l\in S} X_{l,n_l+1}\} | X_{l,n_l+1} \in I^l_{i_l}, l\in S\right) \cdot \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

$$\geq \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} P\left(\bigcap_{j\in NS} \{X_{j,n_j+1} < \min_{l\in S}\{L(I^l_{i_l})\}\}\right) \cdot \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

$$\geq \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \prod_{j\in NS} \left[\frac{\sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{L(I^l_{i_l})\}\}}{n_j+1}\right] \cdot \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

The first inequality follows by putting all probability mass for X_{l,n_l+1} $(l \in S)$ assigned to the intervals $I^l_{i_l} = (x_{l,i_l-1}, x_{l,i_l})$ for $i_l = 1, \ldots, r_l$, (x_{l,r_l}, ∞) and (T_0, ∞) in the left end points of these intervals, and by using Lemma 1.4 for the nested intervals (x_{l,r_l}, ∞) and (T_0, ∞). The second inequality follows by putting all probability mass for X_{j,n_j+1} $(j \in NS)$ assigned to the intervals $I^j_{i_j} = (x_{j,i_j-1}, x_{j,i_j})$ for $i_j = 1, \ldots, r_j$, (x_{j,r_j}, ∞) and (T_0, ∞) in the right end points of these intervals. The upper probability is obtained in a similar way, but now all probability masses for the random quantities involved are put at the opposite end points of the respective intervals, which leads to

$$P\left(\min_{l\in S} X_{l,n_l+1} > \max_{j\in NS} X_{j,n_j+1}\right) = P\left(\bigcap_{j\in NS}\{X_{j,n_j+1} < \min_{l\in S} X_{l,n_l+1}\}\right)$$

$$= \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} P\left(\bigcap_{j\in NS}\{X_{j,n_j+1} < \min_{l\in S} X_{l,n_l+1}\} | X_{l,n_l+1} \in I^l_{i_l}, l \in S\right) . \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

$$\leq \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} P\left(\bigcap_{j\in NS}\{X_{j,n_j+1} < \min_{l\in S}\{U(I^l_{i_l})\}\}\right) . \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

$$\leq \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \prod_{j\in NS}\left[\sum_{i_j=1}^{r_j+2} \mathbf{1}\{L(I^j_{i_j}) < \min_{l\in S}\{U(I^l_{i_l})\}\}.\ M_{X_{j,n_j+1}}(I^j_{i_j})\right] \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

$$= \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \prod_{j\in NS}\left[\frac{1+\sum_{i_j=1}^{r_j}\mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{U(I^l_{i_l})\}\} + (n_j - r_j)\mathbf{1}\{T_0 < \min_{l\in S}\{U(I^l_{i_l})\}\}}{n_j+1}\right] \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

□

3.4.1 Special cases

We now present some special cases of the NPI lower and upper probabilities (3.6) and (3.7).

1. If $r_l = 0$ for all $l \in S$, then the NPI lower probability is

$$\underline{P}^S = \prod_{j\in NS} \frac{r_j}{n_j+1} \prod_{l\in S} \frac{n_l}{n_l+1}$$

   and $\underline{P}^S = 0$ if there exists at least one $j \in NS$ for which $r_j = 0$. Since we have not seen any failure from any group in S, this means that we cannot exclude the possibility that we will never see a failure from any group in S, consequently $\overline{P}^S = 1$.

2. If $r_j = 0$ for at least one $j \in NS$ and $r_l \geq 0$ for all $l \in S$, then the NPI lower probability $\underline{P}^S = 0$ since there exists a group in NS for which we have not seen any failure. This means that we cannot exclude the situation that we will never see a failure from this group. Further, if $r_j = 0$ for all $j \in NS$ then the NPI upper probability is

$$\overline{P}^S = \prod_{l \in S} \frac{n_l - r_l + 1}{n_l + 1} \left(1 - \prod_{j \in NS} \frac{1}{n_j + 1}\right) + \prod_{j \in NS} \frac{1}{n_j + 1}$$

3.4.2 Some properties

Now, we study the effect upon the lower and upper probabilities (3.6) and (3.7) when the stopping time is increased from T_0 to $T_0 + \epsilon$ for small $\epsilon > 0$, such that there is only one extra failure from one group occurs.

Theorem 3.4. (i) If a failure from group $l^* \in S$ occurs in the interval $(T_0, T_0 + \epsilon)$, then the NPI lower probability $\underline{P}^S$ remains constant. However, the NPI upper probability $\overline{P}^S$ decreases by

$$\frac{1}{n_{l^*} + 1} \prod_{l \in S \setminus \{l^*\}} \frac{n_l - r_l + 1}{n_l + 1} \left(1 - \prod_{j \in NS} \frac{r_j + 1}{n_j + 1}\right)$$

except when $r_j = n_j$, for all $j \in NS$, in which case the upper probability remains constant.

(ii) If a failure from group $j^* \in NS$ occurs in the interval $(T_0, T_0 + \epsilon)$, then the NPI upper probability $\overline{P}^S$ remains constant. However, the NPI lower probability $\underline{P}^S$ increases by

$$\frac{1}{n_{j^*} + 1} \prod_{j \in NS \setminus \{j^*\}} \frac{r_j}{n_j + 1} \prod_{l \in S} \frac{n_l - r_l}{n_l + 1}$$

except when $r_l = n_l$ for at least one $l \in S$ or when there exists a $j \in NS \setminus \{j^*\}$ for which $r_j = 0$, in which cases the lower probability remains constant.

Proof. For case i (ii), replace r_{l^*} (r_{j^*}) by $\tilde{r}_{l^*} = r_l + 1$ ($\tilde{r}_{j^*} = r_{j^*} + 1$) in formula (3.6) and (3.7), then this follows by basic analysis of the lower and upper probabilities of Theorem 3.3. For the sake of completeness we include the detailed proof below. To reduce notation we use $L^*(I^l_{i_l})=\min_{l\in S\setminus\{l^*\}} L(I^l_{i_l})$ and $U^*(I^l_{i_l})=\min_{l\in S\setminus\{l^*\}} U(I^l_{i_l})$. First to proof case (i),

$$\begin{aligned}
\underline{P}^S_{l^*} = \sum_{\substack{i_l=1\\ l\in S\setminus\{l^*\}}}^{r_l+2} \Bigg[& \sum_{i_{l^*}=1}^{r_{l^*}} \prod_{j\in NS} \sum_{i_j=1}^{r_j} \frac{\mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{L(I^l_{i_l})\}\}}{(n_j+1)(n_{l^*}+1)} + \\
& \prod_{j\in NS} \sum_{i_j=1}^{r_j} \frac{\mathbf{1}\{x_{j,i_j} < \min\{x_{l^*,r_{l^*}}, L^*(I^l_{i_l})\}\}}{(n_j+1)(n_{l^*}+1)} + \prod_{j\in NS} \sum_{i_j=1}^{r_j} \frac{\mathbf{1}\{x_{j,i_j} < \min\{x_{l^*,\tilde{r}_{l^*}}, L^*(I^l_{i_l})\}\}}{(n_j+1)(n_{l^*}+1)} + \\
& \prod_{j\in NS} \sum_{i_j=1}^{r_j} \frac{\mathbf{1}\{x_{j,i_j} < \min\{T_0+\epsilon, L^*(I^l_{i_l})\}\}(n_{l^*}-r_{l^*}-1)}{(n_j+1)(n_{l^*}+1)} \Bigg] \prod_{l\in S\setminus\{l^*\}} M_{X_{l,n_l+1}}(I^l_{i_l}) \\
= \underline{P}^S + \sum_{\substack{i_l=1\\ l\in S\setminus\{l^*\}}}^{r_l+2} \Bigg[& \prod_{j\in NS} \sum_{i_j=1}^{r_j} \frac{\mathbf{1}\{x_{j,i_j} < \min\{x_{l^*,\tilde{r}_{l^*}}, L^*(I^l_{i_l})\}\}}{(n_j+1)(n_{l^*}+1)} + \\
& \prod_{j\in NS} \sum_{i_j=1}^{r_j} \frac{\mathbf{1}\{x_{j,i_j} < \min\{T_0+\epsilon, L^*(I^l_{i_l})\}\}(n_{l^*}-r_{l^*}-1)}{(n_j+1)(n_{l^*}+1)} - \\
& \prod_{j\in NS} \sum_{i_j=1}^{r_j} \frac{\mathbf{1}\{x_{j,i_j} < \min\{T_0, L^*(I^l_{i_l})\}\}(n_{l^*}-r_{l^*})}{(n_j+1)(n_{l^*}+1)} \Bigg] \prod_{l\in S\setminus\{l^*\}} M_{X_{l,n_l+1}}(I^l_{i_l})
\end{aligned}$$

And since for all $l \in S\setminus\{l^*\}$, $0 \le L(I^l_{i_l}) \le T_0 < x_{l^*,\tilde{r}_{l^*}} < T_0+\epsilon$, and consequently

$$\min\{x_{l^*,\tilde{r}_{l^*}}, L^*(I^l_{i_l})\} = \min\{T_0+\epsilon, L^*(I^l_{i_l})\} = \min\{T_0, L^*(I^l_{i_l})\} = L^*(I^l_{i_l})$$

then the last term between square brackets will vanish. And,

$$\begin{aligned}
\overline{P}^S_{l^*} = \sum_{\substack{i_l=1\\ l\in S\setminus\{l^*\}}}^{r_l+2} \Bigg[& \sum_{i_{l^*}=1}^{r_{l^*}} \prod_{j\in NS} \sum_{i_j=1}^{r_j+2} \mathbf{1}\{L(I^j_{i_j}) < \min_{l\in S}\{U(I^l_{i_l})\}\} M_{X_{j,n_j+1}}(I^j_{i_j}) + \\
& \prod_{j\in NS} \sum_{i_j=1}^{r_j+2} \frac{\mathbf{1}\{L(I^j_{i_j}) < \min\{x_{l^*,\tilde{r}_{l^*}}, U^*(I^l_{i_l})\}\}}{n_{l^*}+1} M_{X_{j,n_j+1}}(I^j_{i_j}) + \\
& \prod_{j\in NS} \sum_{i_j=1}^{r_j+2} \frac{\mathbf{1}\{L(I^j_{i_j}) < \min\{U(I^{l^*}_{\tilde{r}_{l^*}+1}), U^*(I^l_{i_l})\}\}}{n_{l^*}+1} M_{X_{j,n_j+1}}(I^j_{i_j}) + \\
& \prod_{j\in NS} \sum_{i_j=1}^{r_j+2} \frac{\mathbf{1}\{L(I^j_{i_j}) < \min\{U(I^{l^*}_{\tilde{r}_{l^*}+2}), U^*(I^l_{i_l})\}\}(n_{l^*}-r_{l^*}-1)}{n_{l^*}+1} M_{X_{j,n_j+1}}(I^j_{i_j}) \Bigg] \\
& \cdot \prod_{l\in S\setminus\{l^*\}} M_{X_{l,n_l+1}}(I^l_{i_l})
\end{aligned}$$

$$= \overline{P}^S + \frac{1}{n_{l^*}+1} \sum_{\substack{i_l=1 \\ l \in S \setminus \{l^*\}}}^{r_l+2} \left[\prod_{j \in NS} \sum_{i_j=1}^{r_j+2} \mathbf{1}\{L(I^j_{i_j}) < \min\{x_{l^*,\tilde{r}_{l^*}}, U^*(I^l_{i_l})\}\} M_{X_{j,n_j+1}}(I^j_{i_j}) - \right.$$

$$\left. \prod_{j \in NS} \sum_{i_j=1}^{r_j+2} \mathbf{1}\{L(I^j_{i_j}) < U^*(I^l_{i_l})\} M_{X_{j,n_j+1}}(I^j_{i_j}) \right] \prod_{l \in S \setminus \{l^*\}} M_{X_{l,n_l+1}}(I^l_{i_l}) \tag{3.8}$$

Now,

$$\sum_{i_j=1}^{r_j+2} \mathbf{1}\{L(I^j_{i_j}) < U^*(I^l_{i_l})\} M_{X_{j,n_j+1}}(I^j_{i_j}) = \frac{1 + \sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < U^*(I^l_{i_l})\} + (n_j - r_j)\mathbf{1}\{T_0 + \epsilon < U^*(I^l_{i_l})\}}{n_j+1} \tag{3.9}$$

and since $\mathbf{1}\{T_0 + \epsilon < \min\{x_{l^*,\tilde{r}_{l^*}}, U^*(I^l_{i_l})\}\} = 0$ for $l \in S \setminus \{l^*\}$ and all i_l, we have

$$\sum_{i_j=1}^{r_j+2} \mathbf{1}\{L(I^j_{i_j}) < \min\{x_{l^*,\tilde{r}_{l^*}}, U^*(I^l_{i_l})\}\} M_{X_{j,n_j+1}}(I^j_{i_j}) = \frac{1 + \sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \min\{x_{l^*,\tilde{r}_{l^*}}, U^*(I^l_{i_l})\}\}}{n_j+1} \tag{3.10}$$

If $U^*(I^l_{i_l}) \neq \infty$, then (3.9) and (3.10) will be equal and consequently the term between square brackets in (3.8) will vanish. However, for $U^*(I^l_{i_l}) = \infty$, the equations (3.9) and (3.10) reduce to 1 and $(r_j + 1)/(n_j + 1)$, respectively. Therefore, we can rewrite (3.8) as

$$\begin{aligned}
\overline{P}^S_{l^*} &= \overline{P}^S + \frac{1}{n_{l^*}+1} \left(\prod_{j \in NS} \frac{r_j+1}{n_j+1} - 1 \right) \sum_{\substack{i_l=r_l+1 \\ l \in S \setminus \{l^*\}}}^{r_l+2} \prod_{l \in S \setminus \{l^*\}} M_{X_{l,n_l+1}}(I^l_{i_l}) \\
&= \overline{P}^S + \frac{1}{n_{l^*}+1} \left(\prod_{j \in NS} \frac{r_j+1}{n_j+1} - 1 \right) \prod_{l \in S \setminus \{l^*\}} \sum_{\substack{i_l=r_l+1 \\ l \in S \setminus \{l^*\}}}^{r_l+2} M_{X_{l,n_l+1}}(I^l_{i_l}) \\
&= \overline{P}^S + \frac{1}{n_{l^*}+1} \left(\prod_{j \in NS} \frac{r_j+1}{n_j+1} - 1 \right) \prod_{l \in S \setminus \{l^*\}} \left(\frac{1}{n_l+1} + \frac{n_l - r_l}{n_l+1} \right)
\end{aligned} \tag{3.11}$$

We used the known identity $(\sum_i a_i)(\sum_j b_j) = \sum_i \sum_j a_i b_j$ to obtain the final form of $\overline{P}^S_{l^*}$, i.e. (3.11). And when $r_j = n_j \; \forall j \in NS$, it follows from (3.11) that $\overline{P}^S_{l^*} = \overline{P}^S$ which completes the proof of part (i).

And for case (ii),

$$\underline{P}^{S}_{j^*} = \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \left[\prod_{j\in NS\setminus\{j^*\}} \frac{\sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{L(I^l_{i_l})\}\}}{n_j+1} \right]$$

$$\cdot \left[\frac{\sum_{i_{j^*}=1}^{r_{j^*}} \mathbf{1}\{x_{j^*,i_{j^*}} < \min_{l\in S}\{L(I^l_{i_l})\}\} + \mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < \min_{l\in S}\{L(I^l_{i_l})\}\}}{n_{j^*}+1} \right] \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l})$$

$$= \underline{P}^{S} + \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \left[\prod_{j\in NS\setminus\{j^*\}} \frac{\sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{L(I^l_{i_l})\}\}}{n_j+1} \right]$$

$$\cdot \left[\frac{\mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < \min_{l\in S}\{L(I^l_{i_l})\}\}}{n_{j^*}+1} \right] \cdot \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l}) \tag{3.12}$$

Since for all $l \in S$, $x_{l,r_l} < x_{j^*,\tilde{r}_{j^*}} < T_0 + \epsilon$, we have $\mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < \min_{l\in S}\{L(I^l_{i_l})\}\} = 0$ except when $i_l = r_l + 2$, $\forall l \in S$, in which case $L(I^l_{i_l}) = T_0 + \epsilon$ and therefore $\mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < T_0 + \epsilon\} = 1$. Recall that when the experiment is terminated at $x_{j^*,\tilde{r}_{j^*}}$ (i.e. $x_{j^*,\tilde{r}_{j^*}} = T_0 + \epsilon$), we assume that this stopping time is beyond $x_{j^*,\tilde{r}_{j^*}}$ by a very small value which tends to zero. We can rewrite (3.12) as

$$\underline{P}^{S}_{j^*} = \underline{P}^{S} + \frac{\mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < T_0+\epsilon\}}{n_{j^*}+1} \prod_{j\in NS\setminus\{j^*\}} \frac{\sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < T_0+\epsilon\}}{n_j+1} \prod_{l\in S} \frac{n_l - r_l}{n_l+1}$$

$$= \underline{P}^{S} + \frac{1}{n_{j^*}+1} \prod_{j\in NS\setminus\{j^*\}} \frac{r_j}{n_j+1} \prod_{l\in S} \frac{n_l - r_l}{n_l+1} \tag{3.13}$$

and indeed we see that the lower probability $\underline{P}^{S}_{j^*}$ is greater than or equal to $\underline{P}^{S}$. From (3.13), it is clear that when there exists at least one $l \in S$ for which $r_l = n_l$ then $\underline{P}^{S}_{j^*} = \underline{P}^{S}$. Also $\underline{P}^{S}_{j^*} = \underline{P}^{S}$ when there exists a $j \in NS \setminus \{j^*\}$ for which $r_j = 0$.

$$\overline{P}^{S}_{j^*} = \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \prod_{j\in NS\setminus\{j^*\}} \left[\frac{1 + \sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{U(I^l_{i_l})\}\} + (n_j - r_j)\mathbf{1}\{T_0+\epsilon < \min_{l\in S}\{U(I^l_{i_l})\}\}}{n_j+1} \right]$$

$$\cdot \left[\frac{1 + \sum_{i_{j^*}=1}^{r^*_j} \mathbf{1}\{x_{j^*,i_{j^*}} < \min_{l\in S}\{U(I^l_{i_l})\}\}}{n_{j^*}+1} + \frac{\mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < \min_{l\in S}\{U(I^l_{i_l})\}\}}{n_{j^*}+1} + \right.$$

$$\left. \frac{(n_{j^*} - r_{j^*} - 1)\mathbf{1}\{T_0+\epsilon < \min_{l\in S}\{U(I^l_{i_l})\}\}}{n_{j^*}+1} \right] \prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l}) \tag{3.14}$$

Since only a failure from group $j^* \in NS$ occurs in the interval $(T_0, T_0+\epsilon)$, we have $\mathbf{1}\{T_0+\epsilon < \min_{l\in S}\{U(I^l_{i_l})\}\} = \mathbf{1}\{T_0 < \min_{l\in S}\{U(I^l_{i_l})\}\}$, hence (3.14) can be rewritten as

$$\overline{P}^{\,S}_{j^*} = \overline{P}^{\,S} + \sum_{\substack{i_l=1\\ l\in S}}^{r_l+2} \prod_{j\in NS\setminus\{j^*\}} \left[\frac{1+\sum_{i_j=1}^{r_j}\mathbf{1}\{x_{j,i_j}<\min_{l\in S}\{U(I^l_{i_l})\}\}+(n_j-r_j)\mathbf{1}\{T_0<\min_{l\in S}\{U(I^l_{i_l})\}\}}{n_j+1}\right]$$
$$\cdot\left[\frac{\mathbf{1}\{x_{j^*,\tilde{r}_{j^*}}<\min_{l\in S}\{U(I^l_{i_l})\}\}}{n_{j^*}+1} - \frac{\mathbf{1}\{T_0<\min_{l\in S}\{U(I^l_{i_l})\}\}}{n_{j^*}+1}\right]\prod_{l\in S} M_{X_{l,n_l+1}}(I^l_{i_l}) \quad (3.15)$$

If $\min_{l\in S}\{U(I^l_{i_l})\} \neq \infty$, then $\mathbf{1}\{T_0 < \min_{l\in S}\{U(I^l_{i_l})\}\} = \mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < \min_{l\in S}\{U(I^l_{i_l})\}\} = 0$, and when $\min_{l\in S}\{U(I^l_{i_l})\} = \infty$, we have $\mathbf{1}\{T_0 < \min_{l\in S}\{U(I^l_{i_l})\}\} = \mathbf{1}\{x_{j^*,\tilde{r}_{j^*}} < \min_{l\in S}\{U(I^l_{i_l})\}\} = 1$. Therefore, the whole last term in (3.15) will vanish, and consequently $\overline{P}^{\,S}_{j^*} = \overline{P}^{\,S}$ which completes the proof of part (ii). □

Example 3.3. We use the data set of Example 3.1 to illustrate our method for selecting the subset of best groups. A subset S is considered as the 'best' when the lifetime of a future unit from each group in S is larger than the lifetime of a future unit from each group outside this set, so in NS. Our inference depends on the data, the rc-$A_{(n_j)}$ assumption for group j ($j = 1, 2, 3, 4$) and stopping time T_0. To begin, we compute the NPI lower and upper probabilities from (3.6) and (3.7) for all possible subsets that contain only two groups, so S equal to $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$ and $\{3,4\}$, the results for several ranges of values of T_0 are presented in Table 3.6.

For example, at $T_0 = 4.50$ we observe the first failure for group 2. Here all lower probabilities are equal to zero since there exists at least one $j \notin S$ for which $r_j = 0$, which means that we cannot exclude the possibility that we will never observe any failure from this group. However, the upper probabilities for these events are not all the same. For example, the upper probability for S equal to $\{1,3\}$ is one, since in this situation we have not observed any failure from any group in S, so there is a possibility that we would never observe any failure from any group in S. However, for all subsets S which include group 2, the NPI upper probabilities are less than one, since in this case a failure from a group belonging to S has occurred which is an indication against the event of interest.

T_0	r_1	r_2	r_3	r_4	$\underline{P}^{\{1,2\}}$	$\overline{P}^{\{1,2\}}$	$\underline{P}^{\{1,3\}}$	$\overline{P}^{\{1,3\}}$	$\underline{P}^{\{1,4\}}$	$\overline{P}^{\{1,4\}}$
$[4.50, 4.71)$	0	1	0	0	0	0.948	0	1	0	1
$[4.71, 4.86)$	0	1	0	1	0	0.948	0.012	1	0	0.752
$[4.86, 5.01)$	0	2	0	1	0	0.897	0.024	1	0	0.752
$[5.60, 5.78)$	3	6	0	1	0	0.599	0.066	0.873	0	0.646
$[6.72, 6.84)$	6	9	0	1	0	0.395	0.093	0.762	0	0.542
$[7.00, 7.21)$	8	9	2	1	0.010	0.349	0.093	0.635	0.020	0.477
$[7.21, 7.22)$	9	9	2	1	0.010	0.327	0.093	0.604	0.020	0.445
$[7.47, 7.55)$	9	13	6	1	0.025	0.233	0.108	0.499	0.071	0.445
$[7.55, 7.58)$	9	13	7	1	0.028	0.233	0.108	0.476	0.083	0.445
$[8.04, 8.08)$	10	14	12	1	0.036	0.202	0.111	0.359	0.143	0.423
$[8.08, 8.12)$	10	14	13	1	0.038	0.202	0.111	0.339	0.154	0.423
$[8.42, 8.52)$	14	16	14	2	0.059	0.174	0.117	0.308	0.170	0.362
$[8.52, 8.62)$	14	17	14	2	0.059	0.168	0.117	0.308	0.174	0.362
$[8.66, 8.81)$	16	17	14	2	0.059	0.166	0.117	0.305	0.174	0.357
$[8.81, 8.91)$	16	18	14	2	0.059	0.162	0.117	0.305	0.176	0.357
$[8.91, 8.94)$	17	18	14	2	0.059	0.161	0.117	0.303	0.176	0.355
$[8.94, 8.97)$	18	18	14	2	0.059	0.160	0.117	0.302	0.176	0.354
$[8.97, 9.03)$	18	18	15	2	0.059	0.160	0.117	0.299	0.177	0.354
$[9.03, 9.05)$	18	18	15	3	0.059	0.160	0.117	0.299	0.177	0.354
$[9.05, 9.16)$	19	18	15	3	0.059	0.160	0.117	0.299	0.177	0.354
$[9.16, \infty)$	20	18	15	3	0.059	0.160	0.117	0.299	0.177	0.354

T_0	r_1	r_2	r_3	r_4	$\underline{P}^{\{2,3\}}$	$\overline{P}^{\{2,3\}}$	$\underline{P}^{\{2,4\}}$	$\overline{P}^{\{2,4\}}$	$\underline{P}^{\{3,4\}}$	$\overline{P}^{\{3,4\}}$
$[4.50, 4.71)$	0	1	0	0	0	0.948	0	0.948	0	1
$[4.71, 4.86)$	0	1	0	1	0	0.948	0	0.711	0	0.751
$[4.86, 5.01)$	0	2	0	1	0	0.897	0	0.672	0	0.751
$[5.60, 5.78)$	3	6	0	1	0.026	0.701	0	0.516	0.021	0.751
$[6.72, 6.84)$	6	9	0	1	0.045	0.565	0	0.399	0.063	0.751
$[7.00, 7.21)$	8	9	2	1	0.054	0.510	0.011	0.399	0.082	0.674
$[7.21, 7.22)$	9	9	2	1	0.058	0.510	0.013	0.399	0.091	0.674
$[7.47, 7.55)$	9	13	6	1	0.058	0.309	0.038	0.274	0.116	0.533
$[7.55, 7.58)$	9	13	7	1	0.058	0.294	0.042	0.274	0.116	0.503
$[8.04, 8.08)$	10	14	12	1	0.060	0.212	0.062	0.248	0.129	0.363
$[8.08, 8.12)$	10	14	13	1	0.060	0.200	0.065	0.248	0.129	0.336
$[8.42, 8.52)$	14	16	14	2	0.063	0.182	0.079	0.200	0.134	0.293
$[8.52, 8.62)$	14	17	14	2	0.063	0.179	0.079	0.191	0.135	0.293
$[8.66, 8.81)$	16	17	14	2	0.063	0.179	0.080	0.191	0.136	0.293
$[8.81, 8.91)$	16	18	14	2	0.063	0.176	0.080	0.185	0.137	0.293
$[8.91, 8.94)$	17	18	14	2	0.063	0.176	0.080	0.185	0.137	0.293
$[8.94, 8.97)$	18	18	14	2	0.063	0.176	0.080	0.185	0.138	0.293
$[8.97, 9.03)$	18	18	15	2	0.063	0.175	0.080	0.185	0.138	0.290
$[9.03, 9.05)$	18	18	15	3	0.063	0.175	0.080	0.183	0.138	0.288
$[9.05, 9.16)$	19	18	15	3	0.063	0.175	0.080	0.183	0.138	0.288
$[9.16, \infty)$	20	18	15	3	0.063	0.175	0.080	0.183	0.138	0.288

Table 3.6: The subset of best groups: NPI lower and upper probabilities

We can also study the behaviour of the NPI lower and upper probabilities when a failure from any group occurs. To this end, assume that we terminate the experiment at $T_0 = 7$. In this situation we have observed 8, 9, 2 and 1 failures from groups 1, 2, 3 and 4, respectively. Suppose that we are interested in the subset $S = \{1, 2\}$. In this case, $\underline{P}^{\{1,2\}} = 0.010$ and $\overline{P}^{\{1,2\}} = 0.349$. Suppose now that the stopping time is increased from 7 to 7.21. In this case a failure occurs from the first group at time 7.21. We see that, while the lower probability remains constant, the upper probability decreases from 0.349 to 0.327, which illustrates Theorem 3.4(i). However, when increasing the stopping time from 8.04 to 8.08, so that an extra failure of group 3 occurs, the upper probability that $S = \{1, 2\}$ is the subset with the best groups remains constant, but the lower probability increases from 0.036 to 0.038, which illustrates Theorem 3.4(ii).

Suppose now that we stop the experiment at $T_0 = 8.97$ and that we are interested in $S = \{2, 3\}$. Then we have observed all units from all groups in S, but there are 3 units that still have not failed in groups in NS. For $T_0 \geq 8.97$, the lower and upper probabilities that $S = \{2, 3\}$ is the subset with the two best groups will not change ($\underline{P}^{\{2,3\}} = 0.063$, $\overline{P}^{\{2,3\}} = 0.175$), which illustrates the special case of Theorem 3.4(ii). If we change attention to $S = \{1, 4\}$, also for $T_0 \geq 8.97$, then the lower and upper probabilities again remain constant ($\underline{P}^{\{1,4\}} = 0.177$, $\overline{P}^{\{1,4\}} = 0.354$) since we have observed all units from NS, which illustrates the special case of Theorem 3.4(i).

To carry out the comparison to select the best subset, we notice that if we terminate the experiment at $T_0 = 8.52$, $\overline{P}^{\{1,2\}} < \underline{P}^{\{1,4\}}$ which provides a strong indicates to exclude $\{1, 2\}$ from being the best. In addition, at $T_0 = 8.97$ we can exclude the set $\{2, 3\}$ from being the subset with the best groups as $\overline{P}^{\{2,3\}} < \underline{P}^{\{1,4\}}$. This may be due to the fact that the second group is included in these sets, since the second group was the worse group as found in Example 3.1. However, this does not hold for $\{2, 4\}$ as $\overline{P}^{\{2,4\}} \not> \underline{P}^{\{1,4\}}$. This happens because this set consists of the best and the worse group (see the results of Example 3.1). So, we only have a strong indication that $\{1, 2\}$ and $\{2, 3\}$ are not the best subsets. As $\underline{P}^{\{2,4\}} < \underline{P}^{\{1,3\}} < \underline{P}^{\{3,4\}} < \underline{P}^{\{1,4\}}$ and $\overline{P}^{\{2,4\}} < \overline{P}^{\{3,4\}} < \overline{P}^{\{1,3\}} < \overline{P}^{\{1,4\}}$ there is a weak indication that $S = \{1, 4\}$ is the best subset of size 2. $\triangle$

3.5 Selecting the subset including the best group

In this section we consider a similar scenario as in Section 3.4, with the experiment terminated at time T_0 but now our objective is to select a subset of groups such that the group that provides the largest future lifetime is included in this subset. As in Section 3.4, let $S = \{l_1, l_2, ..., l_m\}$ be a selected subset of m groups $(1 \leq m \leq k-1)$ from k independent groups and let NS be the complementary set of S which contains the $k-m$ nonselected groups. We will derive the NPI lower and upper probabilities for the event that the next observation from at least one of the selected groups in S is greater than the next observation from each group in NS, denoted by

$$\underline{P}^{\tilde{S}} = \underline{P}\left(\max_{l \in S} X_{l,n_l+1} > \max_{j \in NS} X_{j,n_j+1}\right) \text{ and } \overline{P}^{\tilde{S}} = \overline{P}\left(\max_{l \in S} X_{l,n_l+1} > \max_{j \in NS} X_{j,n_j+1}\right)$$

These lower and upper probabilities are given in Theorem 3.5, using the notation (3.5) as before.

Theorem 3.5. The NPI lower and upper probabilities for the event that the next observation of at least one group in S is greater than the next observation of each group in NS are

$$\underline{P}^{\tilde{S}} = \sum_{\substack{i_l=1 \\ l \in S}}^{r_l+2} \prod_{j \in NS} \left[\frac{\sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \max_{l \in S}\{L(I_{i_l}^l)\}\}}{n_j+1} \right] \cdot \prod_{l \in S} M_{X_{l,n_l+1}}(I_{i_l}^l) \tag{3.16}$$

$$\overline{P}^{\tilde{S}} = \sum_{\substack{i_l=1 \\ l \in S}}^{r_l+2} \prod_{j \in NS} \left[\frac{1 + \sum_{i_j=1}^{r_j} \mathbf{1}\{x_{j,i_j} < \max_{l \in S}\{U(I_{i_l}^l)\}\}}{n_j+1} + \frac{(n_j - r_j)\mathbf{1}\{T_0 < \max_{l \in S}\{U(I_{i_l}^l)\}\}}{n_j+1} \right] \cdot \prod_{l \in S} M_{X_{l,n_l+1}}(I_{i_l}^l) \tag{3.17}$$

Proof. This is similar to the proof of Theorem 3.3, but with 'min' replaced by 'max' in every step. □

3.5.1 Special cases

Below we present some special cases of the NPI lower and upper probabilities (3.16) and (3.17).

1. If $r_l = 0$ for at least one $l \in S$, then the NPI upper probability $\overline{P}^{\tilde{S}} = 1$, since we have not seen any failure from at least one group in S. This means that we cannot exclude the possibility that we would never see a failure from such a group in S. Further, if $r_l = 0$ for all $l \in S$, then the NPI lower probability is

$$\underline{P}^{\tilde{S}} = \prod_{j \in NS} \frac{r_j}{n_j + 1} \left(1 - \prod_{l \in S} \frac{1}{n_l + 1}\right)$$

which is equal to zero if there exists at least one $j \in NS$ for which $r_j = 0$.

2. If $r_j = 0$ for at least one $j \in NS$ and $r_l > 0$ for all $l \in S$, then the NPI lower probability $\underline{P}^{\tilde{S}} = 0$ since there exists a group in NS for which we have not seen any failure. This means that we cannot exclude the possibility that we would never see a failure from this group. Further, if $r_j = 0$ for all $j \in NS$, then the NPI upper probability is

$$\overline{P}^{\tilde{S}} = 1 - \prod_{l \in S} \frac{r_l}{n_l + 1} \left[1 - \prod_{j \in NS} \frac{1}{n_j + 1}\right]$$

so if $r_l = 0$ for at least one $l \in S$, then $\overline{P}^{\tilde{S}} = 1$.

3.5.2 Some properties

Now, we study the effect upon the lower and upper probabilities when the stopping time is increased from T_0 to $T_0 + \epsilon$, for small $\epsilon > 0$ such that only one extra failure from one group occurs.

Theorem 3.6. (i) If a failure from group $l^* \in S$ occurs in the interval $(T_0, T_0 + \epsilon)$, then the NPI lower probability $\underline{P}^{\tilde{S}}$ remains constant, and the NPI upper probability $\overline{P}^{\tilde{S}}$ decreases by

$$\frac{1}{n_{l^*} + 1} \prod_{l \in S \setminus \{l^*\}} \frac{r_l}{n_l + 1} \left(1 - \prod_{j \in NS} \frac{r_j + 1}{n_j + 1}\right)$$

except when $r_j = n_j$ for all $j \in NS$ or when there exists a $l \in S \setminus \{l^*\}$ for which $r_l = 0$, in which cases the upper probability remains constant.

(ii) If a failure from group $j^* \in NS$ occurs in the interval $(T_0, T_0 + \epsilon)$, then the NPI upper probability $\overline{P}^{\tilde{S}}$ remains constant, and the NPI lower probability $\underline{P}^{\tilde{S}}$ increases by

$$\frac{1}{n_{j^*}+1}\prod_{j\in NS\setminus\{j^*\}}\frac{r_j}{n_j+1}\left(1-\prod_{l\in S}\frac{r_l+1}{n_l+1}\right)$$

except when $r_l = n_l$ for all $l \in S$ or when there exists a $j \in NS \setminus \{j^*\}$ for which $r_j = 0$, in which cases the lower probability remains constant.

Proof. For case i (ii), replace r_l (r_{j^*}) by $r_l + 1$ ($r_{j^*} + 1$) in formula (3.16) and (3.17), then this follows by basic analysis of the NPI lower and upper probabilities of Theorem 3.5, see the proof of Theorem 3.4. □

It can easily be shown that the NPI lower and upper probabilities for selecting the subset of best groups, given by (3.6) and (3.7), cannot exceed those for selecting the subset including the best group, given by (3.16) and (3.17). This follows from $\mathbf{1}\{x_{j,i_j} < \min_{l\in S}\{\bullet\}\} \leq \mathbf{1}\{x_{j,i_j} < \max_{l\in S}\{\bullet\}\}$ and $\mathbf{1}\{T_0 < \min_{l\in S}\{\bullet\}\} \leq \mathbf{1}\{T_0 < \max_{l\in S}\{\bullet\}\}$, where '$\bullet$' refers to $L(I^l_{i_l})$ or $U(I^l_{i_l})$.

Example 3.4. Consider again the data set from Example 3.1, which we also used in Example 3.3. The NPI lower and upper probabilities for the event that the lifetime of the next observation from at least one group in S is greater than the lifetime of the next observation of each group in NS, are calculated from (3.16) and (3.17) at different stopping times T_0 for all possible subsets containing 2 groups and are presented in Table 3.7.

At $T_0 = 4.5$, which is the moment when we observe the first failure (group 2), all lower probabilities are zero and all upper probabilities are one, which is different from the case when we select the subset of 2 best groups (Example 3.3), since for that case there were some upper probabilities which are less than one. This is because at $T_0 = 4.5$, whichever subset of 2 groups we consider, this subset will always contain at least one group for which we have not seen any failure, so there is no evidence against the possibility that this subset can still contain the best group.

For example, for $S = \{1, 3\}$, the lower probability at $T_0 = 4.71$ is 0.013, while the corresponding upper probability is one. At $T_0 = 4.71$ we have seen failures from groups 2 and 4. Therefore, we cannot exclude the possibility that we will not observe any failure from any group in S. In fact, the upper probabilities for the sets that

T_0	r_1	r_2	r_3	r_4	$\underline{P}^{\{\widetilde{1,2}\}}$	$\overline{P}^{\{\widetilde{1,2}\}}$	$\underline{P}^{\{\widetilde{1,3}\}}$	$\overline{P}^{\{\widetilde{1,3}\}}$	$\underline{P}^{\{\widetilde{1,4}\}}$	$\overline{P}^{\{\widetilde{1,4}\}}$
$[4.50, 4.71)$	0	1	0	0	0	1	0	1	0	1
$[4.71, 4.86)$	0	1	0	1	0	1	0.013	1	0	1
$[4.86, 5.01)$	0	2	0	1	0	1	0.026	1	0	1
$[5.60, 5.78)$	3	6	0	1	0	0.956	0.078	1	0	0.965
$[6.72, 6.84)$	6	9	0	1	0	0.869	0.117	1	0	0.930
$[6.84, 6.91)$	6	9	1	1	0.013	0.869	0.117	0.987	0.025	0.930
$[7.00, 7.21)$	8	9	2	1	0.026	0.828	0.117	0.965	0.049	0.909
$[7.21, 7.22)$	9	9	2	1	0.026	0.808	0.117	0.961	0.049	0.898
$[7.47, 7.55)$	9	13	6	1	0.073	0.737	0.159	0.882	0.200	0.898
$[7.55, 7.58)$	9	13	7	1	0.083	0.737	0.159	0.865	0.232	0.898
$[8.04, 8.08)$	10	14	12	1	0.129	0.695	0.168	0.760	0.419	0.890
$[8.08, 8.12)$	10	14	13	1	0.139	0.695	0.168	0.742	0.453	0.890
$[8.42, 8.52)$	14	16	14	2	0.267	0.624	0.271	0.648	0.529	0.834
$[8.52, 8.62)$	14	17	14	2	0.267	0.613	0.279	0.648	0.550	0.834
$[8.66, 8.81)$	16	17	14	2	0.267	0.588	0.279	0.624	0.550	0.829
$[8.81, 8.91)$	16	18	14	2	0.267	0.576	0.286	0.624	0.568	0.829
$[8.91, 8.94)$	17	18	14	2	0.267	0.563	0.286	0.613	0.568	0.827
$[8.94, 8.97)$	18	18	14	2	0.267	0.549	0.286	0.603	0.568	0.826
$[8.97, 9.03)$	18	18	15	2	0.270	0.549	0.286	0.589	0.587	0.826
$[9.03, 9.05)$	18	18	15	3	0.293	0.549	0.308	0.589	0.587	0.826
$[9.05, 9.16)$	19	18	15	3	0.293	0.549	0.308	0.589	0.587	0.826
$[9.16, \infty)$	20	18	15	3	0.293	0.549	0.308	0.589	0.587	0.826

T_0	r_1	r_2	r_3	r_4	$\underline{P}^{\{\widetilde{2,3}\}}$	$\overline{P}^{\{\widetilde{2,3}\}}$	$\underline{P}^{\{\widetilde{2,4}\}}$	$\overline{P}^{\{\widetilde{2,4}\}}$	$\underline{P}^{\{\widetilde{3,4}\}}$	$\overline{P}^{\{\widetilde{3,4}\}}$
$[4.50, 4.71)$	0	1	0	0	0	1	0	1	0	1
$[4.71, 4.86)$	0	1	0	1	0	1	0	0.987	0	1
$[4.86, 5.01)$	0	2	0	1	0	1	0	0.974	0	1
$[5.60, 5.78)$	3	6	0	1	0.035	1	0	0.922	0.044	1
$[6.72, 6.84)$	6	9	0	1	0.070	1	0	0.883	0.131	1
$[6.84, 6.91)$	6	9	1	1	0.070	0.975	0.013	0.883	0.131	0.987
$[7.00, 7.21)$	8	9	2	1	0.091	0.951	0.035	0.883	0.172	0.974
$[7.21, 7.22)$	9	9	2	1	0.102	0.951	0.040	0.883	0.192	0.974
$[7.47, 7.55)$	9	13	6	1	0.102	0.800	0.118	0.841	0.263	0.927
$[7.55, 7.58)$	9	13	7	1	0.102	0.768	0.135	0.841	0.263	0.917
$[8.04, 8.08)$	10	14	12	1	0.110	0.581	0.240	0.832	0.305	0.871
$[8.08, 8.12)$	10	14	13	1	0.110	0.547	0.258	0.832	0.305	0.861
$[8.42, 8.52)$	14	16	14	2	0.166	0.471	0.352	0.729	0.376	0.733
$[8.52, 8.62)$	14	17	14	2	0.166	0.450	0.352	0.721	0.387	0.733
$[8.66, 8.81)$	16	17	14	2	0.171	0.450	0.377	0.721	0.412	0.733
$[8.81, 8.91)$	16	18	14	2	0.171	0.432	0.377	0.714	0.424	0.733
$[8.91, 8.94)$	17	18	14	2	0.173	0.432	0.387	0.714	0.437	0.733
$[8.94, 8.97)$	18	18	14	2	0.174	0.432	0.397	0.714	0.451	0.733
$[8.97, 9.03)$	18	18	15	2	0.174	0.413	0.411	0.714	0.451	0.730
$[9.03, 9.05)$	18	18	15	3	0.174	0.413	0.411	0.692	0.451	0.708
$[9.05, 9.16)$	19	18	15	3	0.174	0.413	0.411	0.692	0.451	0.708
$[9.16, \infty)$	20	18	15	3	0.174	0.413	0.411	0.692	0.451	0.708

Table 3.7: The set including the best group: NPI lower and upper probabilities

contain group 3, i.e. $\{1,3\}$, $\{2,3\}$ and $\{3,4\}$, will be one until $T_0 = 6.84$ (i.e. the time at which we observe the first failure from group 3). The lower probabilities for the sets that do not include the third group, i.e. $\{1,2\}$, $\{1,4\}$ and $\{2,4\}$, are zero until $T_0 = 6.84$.

To study the behaviour of these NPI lower and upper probabilities, let us consider the situation when the stopping time T_0 is increased from 7 to 7.21, and let $S = \{1,2\}$ be the set of interest. At time 7.21 a failure of group 1 is observed. Here, the lower probability remains constant as the amount of information in favour of this event remains the same. However, the upper probability decreases from 0.828 to 0.808 as the amount of information against this event has increased (Theorem 3.6(i)). When we consider $S = \{2,3\}$, with T_0 increasing from 7 to 7.21, then the upper probability remains constant, but, the lower probability increases from 0.091 to 0.102, as the amount of information in favour of this event now increases (Theorem 3.6(ii)).

At $T_0 = 8.97$ we have observed failures of all units from groups 2 and 3. If the set of interest is $S = \{2,3\}$, we see that the lower and upper probabilities remain constant for $T_0 \geq 8.97$ since we have observed all units of all groups in S (special case Theorem 3.6(ii)). Also, if the set of interest is $S = \{1,4\}$, the lower and upper probabilities remain constant since we have observed all units from all groups in NS (special case Theorem 3.6(i)).

At the time when we have observed all units from all groups, i.e. $T_0 = 9.16$, we have a strong indication that the set $\{1,4\}$ is better than $\{1,2\}$ and $\{2,3\}$, in the sense that it is more likely that $\{1,4\}$ contains the best group since

$$0.549 = \overline{P}^{\,\widetilde{\{1,2\}}} < \underline{P}^{\,\widetilde{\{1,4\}}} = 0.587 \text{ and } 0.413 = \overline{P}^{\,\widetilde{\{2,3\}}} < \underline{P}^{\,\widetilde{\{1,4\}}} = 0.587$$

In fact the set $\{1,4\}$ has the highest lower and upper probability when all failure times have been observed for all units from all groups. However, we have a weak indication that $\{1,4\}$ is the set which contains the best group since we have

$$\underline{P}^{\,\widetilde{\{1,3\}}} < \underline{P}^{\,\widetilde{\{2,4\}}} < \underline{P}^{\,\widetilde{\{3,4\}}} < \underline{P}^{\,\widetilde{\{1,4\}}} < \overline{P}^{\,\widetilde{\{1,3\}}} < \overline{P}^{\,\widetilde{\{2,4\}}} < \overline{P}^{\,\widetilde{\{3,4\}}} < \overline{P}^{\,\widetilde{\{1,4\}}}$$

We can exclude $\{2,3\}$ from the comparison from $T_0 = 8.42$ onwards, since $\overline{P}^{\,\widetilde{\{2,3\}}} < \underline{P}^{\,\widetilde{\{1,4\}}}$. Also the set $\{1,2\}$ can be excluded from $T_0 = 8.91$ onwards since $\overline{P}^{\,\widetilde{\{1,2\}}} < \underline{P}^{\,\widetilde{\{1,4\}}}$. △

Example 3.5. In this example all NPI procedures that have been introduced in this chapter are illustrated, so we consider selecting the best group (Section 3.3), selecting the subset of best groups (Section 3.4) and selecting the subset including the best group (Section 3.5). We consider subsets that contain three groups as well as subsets that contain one or two groups. Due to space limitations only the stopping times T_0 at which we exclude a subset or a group from the comparison are reported.

Group	breakdown times									
1	7.74	17.05	20.46	21.02	22.66	43.40	47.30	139.07	144.12	175.88	194.90
2	0.27	0.40	0.69	0.79	2.75	3.91	9.88	13.95	15.93	27.80	53.24
	82.85	89.29	100.58	215.10							
3	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50	7.35
	8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89			
4	0.35	0.59	0.96	0.99	1.69	1.97	2.07	2.58	2.71	2.90	3.67
	3.99	5.35	13.77	25.50							
5	0.09	0.39	0.47	0.73	0.74	1.13	1.40	2.38			

Table 3.8: The times to breakdown (in minutes) at five voltage levels

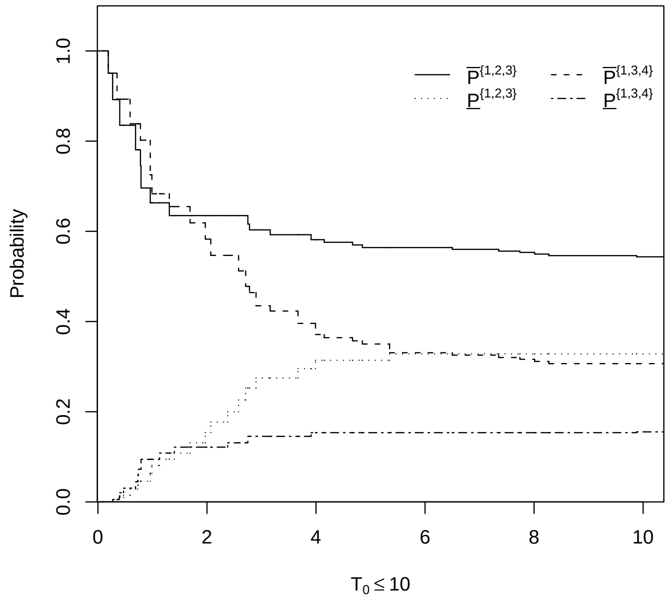
We use a data set also used by Lawless [48, p. 3], which consists of the times to breakdown (in minutes) of electrical insulating fluids at seven voltage levels. These data were originally studied by Nelson [65], who particularly studied the accelerated life testing nature of the data. We do not attempt to model the explicit effect of accelerated life testing but just use these data to illustrate the NPI methods presented in this chapter. We will use only five out of these seven voltage levels in this example, see Table 3.8, to illustrate our method. More precisely, we exclude the first two voltage levels from the original data set since they contain relatively few units compared with the other voltage levels. Again let X_{j,i_j} represent the time to breakdown for unit i_j at voltage level j, which we refer to as group j, with $i_j = 1, \ldots, n_j$ and $j = 1, \ldots, 5$ representing voltage level 30, 32, 34, 36 and 38, respectively. The corresponding sample sizes are 11, 15, 19, 15 and 8, respectively. In this data set, the range of times to breakdown vary from 0.09 at voltage level 5 to 215.10 at voltage level 2.

Let us consider selection of a subset of 3 groups, so we have 10 different possible subsets: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,3,5\}$, $\{2,4,5\}$ and $\{3,4,5\}$. Of course, there are also 10 possible subsets containing 2 different voltage levels: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$ and $\{4,5\}$.

We start considering selection of the subset of 3 best groups. When we have observed all units from all groups we will select the subset $\{1,2,3\}$ as the subset of best groups with NPI lower and upper probabilities $\underline{P}^{\{1,2,3\}} = 0.337$ and $\overline{P}^{\{1,2,3\}} = 0.535$, respectively. Actually we can conclude the same result (i.e. $\{1,2,3\}$ as the best subset) at an early stage. In fact, already if we stopped the experiment at $T_0 = 6.5$ we would select the set $\{1,2,3\}$ as the best subset among all ten subsets, see Table 3.9. Note that, at this point we still have not observed any breakdown from the first group while we have observed already the breakdown times of all units from group 5. Table 3.9 explains how we can establish an early decision from the beginning. For example, at $T_0 = 2.38$, we have observed all breakdown times for units from group 5 and we have not observed any breakdowns from the first group, and therefore we will exclude any set that contains group 5 from being the best. Moreover, at $T_0 = 2.90$ we can exclude 7 of the 10 subsets from comparison of becoming the best subset. At this time, the subsets $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,3,4\}$ remain in the comparison process. Figure 3.1 shows a pairwise comparison between the best subset $\{1,2,3\}$ and the two second best subsets $\{1,2,4\}$ and $\{1,3,4\}$, for different T_0. We can see that at $T_0 = 3.99$, $\overline{P}^{\{1,2,4\}} < \underline{P}^{\{1,2,3\}}$ and at $T_0 = 6.50$, $\overline{P}^{\{1,3,4\}} < \underline{P}^{\{1,2,3\}}$.

Consequently, if we terminate the experiment at $T_0 = 6.5$ we get the same decision as when we would have observed all units from all groups. Doing that would lead to a much shorter testing time, and we can keep 9 units out of 15 from group 2, 9 out of 19 from group 3, 2 out of 15 from group 4 and all units from group 1 to be possibly used for other purposes.

Now we consider selecting a subset of 3 groups that includes the best group. Table 3.10 shows that 4 out of 10 subsets could be excluded from the comparison at $T_0 = 13.77$, and a fifth subsets at $T_0 = 25.5$. Unlike the case of selecting the subset of best groups, there are three sets $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,2,5\}$ which cannot be



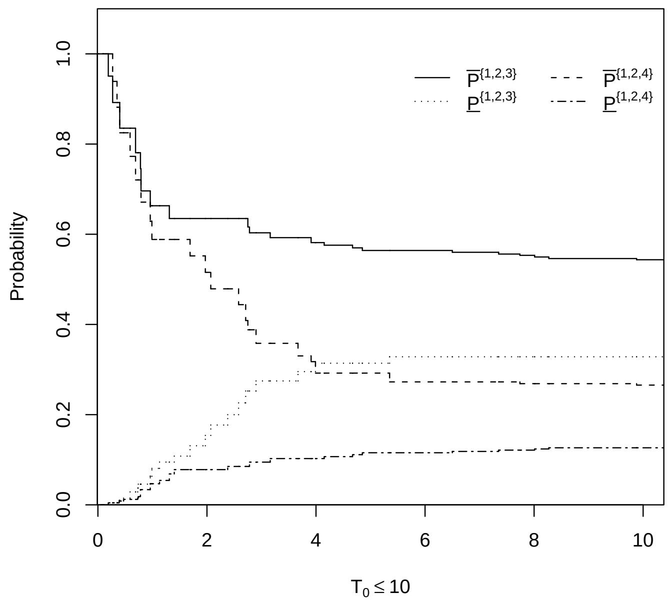


Figure 3.1: The subset of best groups: NPI lower and upper probabilities for $T_0 \leq 10$

T_0	r_1	r_2	r_3	r_4	r_5	Set(s) out	Pairwise comparison with $\{1,2,3\}$
1.69	0	4	4	5	7	$\{2,4,5\}$	$0.123 = \overline{P}^{\{2,4,5\}} < \underline{P}^{\{1,2,3\}} = 0.131$
1.97	0	4	4	6	7	$\{3,4,5\}$	$0.126 = \overline{P}^{\{3,4,5\}} < \underline{P}^{\{1,2,3\}} = 0.154$
						$\{2,3,5\}$	$0.145 = \overline{P}^{\{2,3,5\}} < \underline{P}^{\{1,2,3\}} = 0.154$
2.07	0	4	4	7	7	$\{1,4,5\}$	$0.153 = \overline{P}^{\{1,4,5\}} < \underline{P}^{\{1,2,3\}} = 0.177$
2.38	0	4	4	7	8	$\{1,2,5\}$	$0.114 = \overline{P}^{\{1,2,5\}} < \underline{P}^{\{1,2,3\}} = 0.200$
						$\{1,3,5\}$	$0.139 = \overline{P}^{\{1,3,5\}} < \underline{P}^{\{1,2,3\}} = 0.200$
2.90	0	5	5	10	8	$\{2,3,4\}$	$0.235 = \overline{P}^{\{2,3,4\}} < \underline{P}^{\{1,2,3\}} = 0.275$
3.99	0	6	6	12	8	$\{1,2,4\}$	$0.292 = \overline{P}^{\{1,2,4\}} < \underline{P}^{\{1,2,3\}} = 0.314$
6.50	0	6	10	13	8	$\{1,3,4\}$	$0.326 = \overline{P}^{\{1,3,4\}} < \underline{P}^{\{1,2,3\}} = 0.328$

Table 3.9: The subset of best groups: pairwise comparison with $\{1,2,3\}$

excluded from any time onwards. Consequently, we do not have a strong indication to select one of these three sets as the set that is most likely to include the best group as even when we have observed all units, we have $\overline{P}^{\widetilde{\{1,2,4\}}} \not< \underline{P}^{\widetilde{\{1,2,3\}}}$ and $\overline{P}^{\widetilde{\{1,2,5\}}} \not< \underline{P}^{\widetilde{\{1,2,3\}}}$.

In Table 3.11, the stopping times at which we exclude a group from being the best group are reported, where the NPI lower and upper probabilities are calculated from (3.3) and (3.4). As we can see from Table 3.11, we can exclude group 5 from being the best group already at $T_0 = 6.5$, at which time $0.112 = \overline{P}^{(5)} < \underline{P}^{(1)} = 0.124$. Groups 4 and 3 can be excluded at $T_0 = 12.06$ and $T_0 = 31.75$, respectively, as then we have $0.193 = \overline{P}^{(4)} < \underline{P}^{(1)} = 0.197$ and $0.288 = \overline{P}^{(3)} < \underline{P}^{(1)} = 0.310$. In addition, when we have observed breakdown times of all units from all groups (or even before, i.e. at $T_0 = 82.85$) we can conclude that the first group is the best since $\underline{P}^{(1)} > \overline{P}^{(l)}$ for $l = 2,3,4,5$. At $T_0 = 82.85$ we can exclude the second group from being the best group (where $0.378 = \overline{P}^{(2)} < \underline{P}^{(1)} = 0.391$) which may explain the situation of being $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,2,5\}$ to be the subsets that contains the best group since these contain the best group and the second best group (i.e. group 2).

Suppose, for example, that we terminate the experiment at $T_0 = 25.5$. From Table 3.10 it follows that in this case we can exclude 5 of the 10 subsets from being the subset that includes the best group, therefore, the sets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,5\}$ and $\{1,3,4\}$ will remain under comparison. At the same time (at

$T_0 = 25.5$), see Table 3.11, we can exclude groups 4 and 5 from being the best group, therefore we still have groups 1, 2 and 3 under comparison. This explains why any set that contains two of these groups is still under consideration for being the set that includes the best group. However, this is not the case for $\{2,3,5\}$ and $\{2,3,4\}$ since the first group (the best) is not included in them.

T_0	r_1	r_2	r_3	r_4	r_5	Set(s) out	Pairwise comparison with $\{1,2,3\}$
5.35	0	6	9	13	8	$\{3,4,5\}$	$0.685 = \overline{P}^{\,\widetilde{\{3,4,5\}}} < \underline{P}^{\,\widetilde{\{1,2,3\}}} = 0.717$
9.88	1	7	13	13	8	$\{2,4,5\}$	$0.696 = \overline{P}^{\,\widetilde{\{2,4,5\}}} < \underline{P}^{\,\widetilde{\{1,2,3\}}} = 0.717$
13.77	1	7	14	14	8	$\{2,3,5\}$	$0.751 = \overline{P}^{\,\widetilde{\{2,3,5\}}} < \underline{P}^{\,\widetilde{\{1,2,3\}}} = 0.769$
						$\{2,3,4\}$	$0.762 = \overline{P}^{\,\widetilde{\{2,3,4\}}} < \underline{P}^{\,\widetilde{\{1,2,3\}}} = 0.769$
25.50	5	9	14	15	8	$\{1,4,5\}$	$0.803 = \overline{P}^{\,\widetilde{\{1,4,5\}}} < \underline{P}^{\,\widetilde{\{1,2,3\}}} = 0.811$
72.89	7	11	19	15	8	$\{1,3,5\}$	$0.811 = \overline{P}^{\,\widetilde{\{1,3,5\}}} < \underline{P}^{\,\widetilde{\{1,2,3\}}} = 0.811$
139.07	8	14	19	15	8	$\{1,3,4\}$	$0.809 = \overline{P}^{\,\widetilde{\{1,3,4\}}} < \underline{P}^{\,\widetilde{\{1,2,3\}}} = 0.811$

Table 3.10: The subset including the best group: pairwise comparison with $\{1,2,3\}$

T_0	r_1	r_2	r_3	r_4	r_5	group out	Pairwise comparison with group 1
6.50	0	6	10	13	8	group 5	$0.112 = \overline{P}^{\,(5)} < \underline{P}^{\,(1)} = 0.124$
12.06	1	7	14	13	8	group 4	$0.193 = \overline{P}^{\,(4)} < \underline{P}^{\,(1)} = 0.197$
31.75	5	10	15	15	8	group 3	$0.288 = \overline{P}^{\,(3)} < \underline{P}^{\,(1)} = 0.310$
82.85	7	12	19	15	8	group 2	$0.378 = \overline{P}^{\,(2)} < \underline{P}^{\,(1)} = 0.391$

Table 3.11: The best group: pairwise comparison with group 1

Now let us consider the case of selecting the subset of 2 best groups. From $T_0 = 8.27$ onwards, see Table 3.12, all subsets except $\{1,2\}$ and $\{1,3\}$ are excluded from being the subset of 2 best groups. Here we have a strong indication to exclude these subsets since their corresponding upper probabilities are less than $\underline{P}^{\,\{1,2\}}$, but there is only a weak indication that $\{1,2\}$ is the best subset of 2 best groups since $\underline{P}^{\,\{1,3\}} < \underline{P}^{\,\{1,2\}} < \overline{P}^{\,\{1,3\}} < \overline{P}^{\,\{1,2\}}$.

From the one-group comparison, see Table 3.11, at $T_0 = 8.27$ the fifth group can be excluded (in fact this can be concluded already for $T_0 \geq 6.5$) as $\overline{P}^{\,(5)} < \underline{P}^{\,(1)}$. In addition, we exclude the fourth and the third group at $T_0 = 12.06$ and $T_0 = 31.75$, respectively. That may explain why, when considering subsets consisting of two

T_0	r_1	r_2	r_3	r_4	r_5	Set(s) out	Pairwise comparison with $\{1,2\}$
2.71	0	4	4	9	8	$\{4,5\}$	$0.051 = \overline{P}^{\{4,5\}} < \underline{P}^{\{1,2\}} = 0.064$
2.90	0	5	5	10	8	$\{2,5\}$	$0.079 = \overline{P}^{\{2,5\}} < \underline{P}^{\{1,2\}} = 0.086$
3.16	0	5	6	10	8	$\{3,5\}$	$0.082 = \overline{P}^{\{3,5\}} < \underline{P}^{\{1,2\}} = 0.102$
4.15	0	6	7	12	8	$\{1,5\}$	$0.122 = \overline{P}^{\{1,5\}} < \underline{P}^{\{1,2\}} = 0.137$
4.85	0	6	9	12	8	$\{3,4\}$	$0.155 = \overline{P}^{\{3,4\}} < \underline{P}^{\{1,2\}} = 0.172$
						$\{2,4\}$	$0.168 = \overline{P}^{\{2,4\}} < \underline{P}^{\{1,2\}} = 0.172$
8.27	1	6	13	13	8	$\{1,4\}$	$0.243 = \overline{P}^{\{1,4\}} < \underline{P}^{\{1,2\}} = 0.256$
						$\{2,3\}$	$0.251 = \overline{P}^{\{2,3\}} < \underline{P}^{\{1,2\}} = 0.256$

Table 3.12: The subset of best groups: pairwise comparison with $\{1,2\}$

groups (Table 3.12) the subsets that contain the fifth group can be excluded from $T_0 = 4.15$. However, by the end of the experiment we do not have a strong indication to choose between the subsets $\{1,2\}$ and $\{1,3\}$ for selecting the subset of best groups.

With regard to selection of the subset of 2 groups that includes the best group, from $T_0 = 32.52$ onwards (Table 3.13) the subsets $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$ and $\{4,5\}$ are excluded from being the subset including the best group. Here we have a strong indication that these subsets can be excluded since their corresponding upper probabilities are less than $\underline{P}^{\widetilde{\{1,2\}}}$, but there is only a weak indication to select $\{1,2\}$ as the subset including the best group since $\underline{P}^{\widetilde{\{1,4\}}} < \underline{P}^{\widetilde{\{1,5\}}} < \underline{P}^{\widetilde{\{1,3\}}} < \underline{P}^{\widetilde{\{1,2\}}}$ and $\overline{P}^{\widetilde{\{1,5\}}} < \overline{P}^{\widetilde{\{1,4\}}} < \overline{P}^{\widetilde{\{1,3\}}} < \overline{P}^{\widetilde{\{1,2\}}}$.

T_0	r_1	r_2	r_3	r_4	r_5	Set(s) out	Pairwise comparison with $\{1,2\}$
5.35	0	6	9	13	8	$\{4,5\}$	$0.283 = \overline{P}^{\widetilde{\{4,5\}}} < \underline{P}^{\widetilde{\{1,2\}}} = 0.315$
8.27	1	6	13	13	8	$\{3,5\}$	$0.438 = \overline{P}^{\widetilde{\{3,5\}}} < \underline{P}^{\widetilde{\{1,2\}}} = 0.451$
12.06	1	7	14	13	8	$\{3,4\}$	$0.455 = \overline{P}^{\widetilde{\{3,4\}}} < \underline{P}^{\widetilde{\{1,2\}}} = 0.484$
25.50	5	9	14	15	8	$\{2,4\}$	$0.516 = \overline{P}^{\widetilde{\{2,4\}}} < \underline{P}^{\widetilde{\{1,2\}}} = 0.547$
						$\{2,5\}$	$0.522 = \overline{P}^{\widetilde{\{2,5\}}} < \underline{P}^{\widetilde{\{1,2\}}} = 0.547$
32.52	5	10	16	15	8	$\{2,3\}$	$0.592 = \overline{P}^{\widetilde{\{2,3\}}} < \underline{P}^{\widetilde{\{1,2\}}} = 0.601$

Table 3.13: The subset including the best group: pairwise comparison with $\{1,2\}$

On the other hand, at $T_0 = 8.27$ we can exclude only the subsets $\{3,5\}$ and $\{4,5\}$ from being the subset including the best group. In the one-group comparison (Table 3.11), we exclude group 5 from being the best group already at $T_0 = 6.5$.

However, while $\{2, 5\}$ contains the fifth group, it is still under comparison until $T_0 = 25.5$. With respect to $\{1, 5\}$, and by the end of the experiment, there is no strong indication to exclude this subset from being the subset including the best group although this subset includes the best (group 1) and the worst group (group 5). Note that this subset was excluded at $T_0 = 4.15$ from being the subset of 2 best groups (Table 3.12). Actually we excluded all subsets that contain the fifth group from being the subset of best groups at $T_0 = 4.15$, which is even before we would decide to exclude the fifth group from being the best group (one-group comparison, Table 3.11) at $T_0 = 6.5$.

At $T_0 = 6.5$, we see from Table 3.9 that we can select $\{1, 2, 3\}$ as the subset of best groups containing 3 groups. However, at this time, we see from Table 3.12 that we do not have a strong indication to select one subset for being the subset of best groups among $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ and $\{2, 3\}$, although $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are subsets of $\{1, 2, 3\}$, but the subset $\{1, 4\}$ contains only one group from $\{1, 2, 3\}$. Later, at $T_0 = 8.27$ we exclude the subset $\{2, 3\}$ ($\subset \{1, 2, 3\}$) from being the subset of best groups.

At $T_0 = 25.5$, see Table 3.10, we do not have a strong indication to select one of $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$ and $\{1, 3, 5\}$ as the subset including the best group. Here all possible 3-group subsets which include the best group (i.e. group 1) are still under comparison except for $\{1, 4, 5\}$, which contains the two worst groups (i.e. groups 4 and 5) according to Table 3.11. However, we exclude the subset $\{1, 4, 5\}$, at $T_0 = 25.5$ from being the subset including the best group. However at this time (Table 3.13) we do not have a strong indication in favour of selecting one of $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$ and $\{2, 3\}$ as the subset including the best group. At $T_0 = 25.5$, the subset $\{2, 3\}$ is also still under comparison for being the subset including the best group while group 1 (the best group) is not included in this subset. However, at least all subsets that include group 1 (the best group) are still under comparison for the subset including the best group when breakdown times have been observed for all units on test. $\triangle$

3.6 Concluding remarks

In this chapter we have presented NPI for comparison of several groups through experiments that may be terminated before the event of interest has been observed for all units. This work generalizes the NPI approach for selection presented by Coolen and van der Laan [25], with close links to methods for precedence testing which are explicitly developed to deal with such early termination.

We note that when the stopping time increases from T_0 to $T_0 + \epsilon$ such that only one extra failure occurs from one particular group, the lower probability remains either constant or increases and the upper probability remains constant or decreases. Hence, when T_0 increases to $T_0 + \epsilon$, the imprecision remains constant or decreases. It is shown that the lower probability for a certain event can be interpreted as quantifying the amount of information in favour of the event while the upper probability can be interpreted as quantifying the amount of information against the event.

The NPI method presented here is not considered to be a competitor for established classical methods for precedence testing and selection, but it provides an interesting alternative which may be suitable particularly in cases where interest is explicitly in a future observation from one or more selected groups. It may well be the case that these different methods lead to quite different conclusions, so care must be taken about the actual inferential conclusions. As always, applying a variety of suitable statistical methods to a practical problem might give valuable insights into the problem and the different methods, where differences typically occur due to different underlying assumptions and explicitly different inferential goals of the methods.

Chapter 4

Comparisons of lifetime data with early termination

4.1 Introduction

Coolen and Yan [26] introduced NPI for comparison of two groups of lifetime data that contain right-censored observations, using the suitable rc-$A_{(n)}$ assumption per group. However, they did not consider situations with more than two groups, nor the effect of early termination of the lifetime experiment. In Chapters 2 and 3 we presented NPI for comparison of groups of data with early termination of the experiment, say at time T_0, but all observations prior to T_0 were required to be actual failure times, so no right-censoring was possible apart from the right-censoring at T_0 of all units that had not yet failed. In this chapter we generalise that by developing NPI for comparison of multiple groups of lifetime data including right-censored observations, and with possible early termination of the lifetime experiment.

A short overview about the classical precedence tests is given in Section 4.2. The NPI lower and upper probabilities for comparing k groups in order to select the best group, with possible early termination of the lifetime experiment, are presented in Section 4.3, and this approach is illustrated and discussed via examples in Section 4.4. Some concluding remarks are given in Section 4.5.

4.2 Classical precedence tests

Suppose we have $k \geq 2$ independent groups, for group j $(j = 1, \ldots, k)$ n_j units are placed on a lifetime experiment, and let their random times to failure be denoted by $X_{j,1}, \ldots, X_{j,n_j}$. In classical statistics, it is typically assumed that these random quantities are independent and identically distributed, with continuous distribution function F_j. Several nonparametric tests have been proposed in the literature for comparing k groups of units placed simultaneously on a lifetime experiment [15, 16].

Classical precedence testing methods consider the null hypothesis $H_0 : F_1(x) = \ldots = F_k(x)$ for all x, which is tested against several alternative hypotheses, e.g. the most general alternative $H_1 : F_i(x) \neq F_j(x)$ for at least one pair of i and j and some value of x. Another alternative hypothesis that has been used is the one-sided alternative $H_2 : F_i(x) \leq F_1(x)$, with strict inequality for at least one $i = 2, 3, \ldots, k$ and some x [16]. This is of particular use in applications where one wants to compare a control population, with distribution function F_1, to other populations, with distribution functions $F_2, \ldots, F_k$, to test if any of the other populations are better than the control population. Tests for several alternative hypotheses are presented in [15, 16]. For given $p \in (0, 1)$, these tests typically depend on the statistics $U_{jp} = n_j \hat{F}_j \hat{F}_1^{-1}(p)$, $j = 2, 3, \ldots, k$, where $\hat{F}_j$ denotes the Kaplan-Meier estimator of $F_j(x)$ [44] and $\hat{F}_1^{-1}(u)$ is the Kaplan-Meier quantile function corresponding to $\hat{F}_1$. The asymptotic distribution of some functions of these statistics U_{jp} are given in [16], which also presents more details of such nonparametric precedence tests.

Let us consider one of these precedence tests proposed by Chakraborti and Desu [15] where, for a test of size α, one may reject H_0 in favour of H_2 if

$$V = (N\, \hat{\sigma}_1^2)^{-1/2} \sum_{j=2}^{k} (U_{jp} - n_j p) < -z_\alpha \tag{4.1}$$

where $N = \sum_{j=1}^{k} n_j$, and z_α is the upper 100α-percentile of the standard normal distribution. Let $x_{j,(i)}$ be the (distinct) ith largest failure time from group j, also let h_{ji} and $\tilde{n}_{x_{j,(i)}}$ be the number of failures and the number of units at risk, respectively, at $x_{j,(i)}$. Then, under H_0, $\hat{\sigma}_1^2 = (N/n_1)\,(1 - (n_1/N))^2\, \hat{J}_1^0 + \sum_{j=2}^{k} (n_j/N)\, \hat{J}_j^0$, where

$$\hat{J}_j^0 = (1-p)^2 \sum_{i: x_{j,(i)} \leq \hat{F}^{-1}(p)} \frac{n_j\, h_{ji}}{\tilde{n}_{x_{j,(i)}} (\tilde{n}_{x_{j,(i)}} - h_{ji})}\ ,\ j = 1, \ldots, k.$$

In the NPI approach presented in this chapter, no null hypothesis is tested. Instead, different groups are compared by considering one further unit from each group, the lifetime of which is assumed to be exchangeable with those of units that were actually tested for the corresponding group. The NPI approach uses lower and upper probabilities to quantify the uncertainties involved with the comparisons of such random quantities, this enables meaningful inferences without the need for further assumptions.

4.3 NPI for lifetime data with early termination

In this section, we consider a life-testing experiment to compare units of $k \geq 2$ groups, which are assumed to be completely independent, with the experiment starting on all units at time 0. The experiment can be terminated before all units have failed, say at time T_0, which is assumed not to hold any information on residual time-to-failure for units that have not yet failed. We also allow right-censoring to occur before the experiment is stopped, due to a censoring process that is assumed to be independent of the failure process. So we consider both right-censored observations in the original data and right-censoring due to stopping the experiment at T_0. For group j ($j = 1, \ldots, k$) n_j units are in the experiment, of which u_j units fail before (or at) T_0, with ordered failure times $x_{j,1} < \ldots < x_{j,u_j} \leq T_0$, and $c_{j,1} < \ldots < c_{j,v_j} < T_0$ are right-censoring times (we assume no tied observations for ease of notation, generalization is straightforward as discussed in Subsection 1.3.5). Let $x_{j,0} = 0$ and $x_{j,u_j+1} = \infty$. Let s_{j,i_j} be the number of right-censored observations in the interval (x_{j,i_j}, x_{j,i_j+1}), $i_j = 0, \ldots, u_j - 1$, with $x_{j,i_j} < c^{i_j}_{j,1} < \ldots < c^{i_j}_{j,s_{j,i_j}} < x_{j,i_j+1}$. Similarly, let s_{j,u_j} be the number of right-censored observations in the interval (x_{j,u_j}, T_0), with $x_{j,u_j} < c^{u_j}_{j,1} < \ldots < c^{u_j}_{j,s_{j,u_j}} < T_0$ and $\sum_{i_j=0}^{u_j} s_{j,i_j} = v_j$, so $n_j - (u_j + v_j)$ units from group j are right-censored at T_0.

To compare the k groups, we consider a hypothetical further unit from each group which would also have been involved in this experiment, with X_{j,n_j+1} the random failure time for the further unit from group j which is assumed to be exchangeable with the failure times of the n_j units of the same group included in the experiment. The assumption rc-$A_{(n_j)}$ implies the M-function values for X_{j,n_j+1} presented in Theorem 4.1.

Theorem 4.1. For NPI with lifetime data containing right-censored observations, and with early termination of the experiment at time T_0, the assumption rc-$A_{(n_j)}$ implies that the following M-function values apply for X_{j,n_j+1}, on the basis of data consisting of u_j failure times and $(n_j - u_j)$ right-censored observations:

$$M^j_{i_j} = M_{X_{j,n_j+1}}(x_{j,i_j}, x_{j,i_j+1}) = \frac{1}{n_j+1} \prod_{\{r: c_{j,r} < x_{j,i_j}\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}}$$

$$M^j_{i_j,t_j} = M_{X_{j,n_j+1}}(c^{i_j}_{j,t_j}, x_{j,i_j+1}) = \frac{1}{(n_j+1)} \left(\tilde{n}_{j,c^{i_j}_{j,t_j}}\right)^{-1} \prod_{\{r: c_{j,r} < c^{i_j}_{j,t_j}\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}}$$

$$M^j_{T_0} = M_{X_{j,n_j+1}}(T_0, \infty) = \frac{n_j - (u_j + \upsilon_j)}{n_j+1} \prod_{\{r: c_{j,r} < T_0\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}}$$

where $i_j = 0, \ldots, u_j$, $t_j = 1, \ldots, s_{j,i_j}$, and $\tilde{n}_{j,c_{j,r}}$ and $\tilde{n}_{j,c^{i_j}_{j,t_j}}$ are the number of units from group j in the risk set just prior to time $c_{j,r}$ and $c^{i_j}_{j,t_j}$, respectively. Also

$$\begin{aligned} P^j_{i_j} &= P(X_{j,n_j+1} \in (x_{j,i_j}, x_{j,i_j+1})) = \frac{1}{n_j+1} \prod_{\{r: c_{j,r} < x_{j,i_j+1}\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}} \\ P^j_{T_0} &= P(X_{j,n_j+1} \in (T_0, \infty)) = M_{X_{j,n_j+1}}(T_0, \infty) = M^j_{T_0}. \end{aligned}$$

Proof. This is similar to the proof of Theorem 2.1. For group j, suppose there are $w_j (= n_j - u_j - \upsilon_j)$ right-censored times beyond T_0, denoted by $c^{T_0}_{j,1} < \ldots < c^{T_0}_{j,w_j}$. Then,

$$\begin{aligned} M_{X_{j,n_j+1}}(T_0, \infty) &= \sum_{i^*_j=1}^{w_j} M_{X_{j,n_j+1}}(c^{T_0}_{j,i^*_j}, \infty) = \frac{1}{n_j+1} \sum_{i^*_j=1}^{w_j} \frac{1}{\tilde{n}_{j,c^{T_0}_{j,i^*_j}}} \prod_{\{r: c_{j,r} < c^{T_0}_{j,i^*_j}\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}} \\ &= \frac{1}{n_j+1} \prod_{\{r: c_{j,r} < T_0\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}} \left\{ \sum_{i^*_j=1}^{w_j} \frac{1}{\tilde{n}_{j,c^{T_0}_{j,i^*_j}}} \prod_{\{r: T_0 < c_{j,r} < c^{T_0}_{j,i^*_j}\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}} \right\} \\ &= \frac{1}{n_j+1} \prod_{\{r: c_{j,r} < T_0\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}} \left\{ \prod_{\{r: T_0 < c_{j,r} < \infty\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}} - 1 \right\} \\ &= \frac{1}{n_j+1} \prod_{\{r: c_{j,r} < T_0\}} \frac{\tilde{n}_{j,c_{j,r}}+1}{\tilde{n}_{j,c_{j,r}}} \left\{ \tilde{n}_{j,c^{T_0}_{j,1}} + 1 - 1 \right\} \end{aligned}$$

The fourth equality follows from Lemma 1.1, and $\tilde{n}_{j,c^{T_0}_{j,1}} = n_j - u_j - \upsilon_j$. □

In this chapter we restrict attention to the events $X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}$, for $l = 1,\ldots,k$, where the NPI lower and upper probabilities, presented in Theorem 4.2, are specified with the use of Theorem 4.1.

Theorem 4.2. The NPI lower and upper probabilities for the event that the next observed lifetime from group l is the maximum of all next observed lifetimes for the k groups in the experiment, with one future lifetime per group considered, i.e. $\underline{P}^{(l)} = \underline{P}\left(X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}\right)$ and $\overline{P}^{(l)} = \overline{P}\left(X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}\right)$, are

$$\underline{P}^{(l)} = \sum_{i_l=0}^{u_l}\left\{\prod_{\substack{j=1\\j\neq l}}^{k}\left[\sum_{i_j=0}^{u_j}\mathbf{1}\{x_{j,i_j+1}<x_{l,i_l}\}P^j_{i_j}\right]M^l_{i_l} + \sum_{t_l=1}^{s_{l,i_l}}\prod_{\substack{j=1\\j\neq l}}^{k}\left[\sum_{i_j=0}^{u_j}\mathbf{1}\{x_{j,i_j+1}<c^{i_l}_{l,t_l}\}P^j_{i_j}\right]M^l_{i_l,t_l}\right\} +$$
$$M^l_{T_0}\prod_{\substack{j=1\\j\neq l}}^{k}\sum_{i_j=0}^{u_j}\mathbf{1}\{x_{j,i_j+1}< T_0\}P^j_{i_j} \tag{4.2}$$

$$\overline{P}^{(l)} = \sum_{i_l=0}^{u_l} P^l_{i_l}\prod_{\substack{j=1\\j\neq l}}^{k}\left\{\sum_{i_j=0}^{u_j}\mathbf{1}\{x_{j,i_j} < x_{l,i_l+1}\}M^j_{i_j} + \sum_{i_j=0}^{u_j}\sum_{t_j=1}^{s_{j,i_j}}\mathbf{1}\{c^{i_j}_{j,t_j} < x_{l,i_l+1}\}M^j_{i_j,t_j} +\right.$$
$$\left.\mathbf{1}\{T_0 < x_{l,i_l+1}\}M^j_{T_0}\right\} + P^l_{T_0} \tag{4.3}$$

Proof. First, we write the probability for the event of interest as

$$P^{(l)} = P\left(X_{l,n_l+1} = \max_{1\leq j\leq k} X_{j,n_j+1}\right) = P\left(\bigcap_{\substack{j=1\\j\neq l}}^{k}\left\{X_{j,n_j+1} < X_{l,n_l+1}\right\}\right)$$
$$= \sum_{i_l=0}^{u_l} P\left(\bigcap_{\substack{j=1\\j\neq l}}^{k}\left\{X_{j,n_j+1} < X_{l,n_l+1}, X_{l,n_l+1}\in(x_{l,i_l},x_{l,i_l+1})\right\}\right) +$$
$$P\left(\bigcap_{\substack{j=1\\j\neq l}}^{k}\left\{X_{j,n_j+1} < X_{l,n_l+1}, X_{l,n_l+1}\in(T_0,\infty)\right\}\right)$$

The NPI lower probability is derived as follows

$$
\begin{aligned}
P^{(l)} \geq & \sum_{i_l=0}^{u_l} \left\{ P\left(\bigcap_{\substack{j=1\\ j\neq l}}^{k} \left\{ X_{j,n_j+1} < x_{l,i_l} \right\} \right) M_{i_l}^{l} + \sum_{t_l=1}^{s_{l,i_l}} P\left(\bigcap_{\substack{j=1\\ j\neq l}}^{k} \left\{ X_{j,n_j+1} < c_{l,t_l}^{i_l} \right\} \right) M_{i_l,t_l}^{l} \right\} + \\
& P\left(\bigcap_{\substack{j=1\\ j\neq l}}^{k} \left\{ X_{j,n_j+1} < T_0 \right\} \right) M_{T_0}^{l} \\
= & \sum_{i_l=0}^{u_l} \left\{ \prod_{\substack{j=1\\ j\neq l}}^{k} P\left(X_{j,n_j+1} < x_{l,i_l} \right) M_{i_l}^{l} + \sum_{t_l=1}^{s_{l,i_l}} \prod_{\substack{j=1\\ j\neq l}}^{k} P\left(X_{j,n_j+1} < c_{l,t_l}^{i_l} \right) M_{i_l,t_l}^{l} \right\} + \\
& \prod_{\substack{j=1\\ j\neq l}}^{k} P\left(X_{j,n_j+1} < T_0 \right) M_{T_0}^{l} \\
\geq & \sum_{i_l=0}^{u_l} \left\{ \prod_{\substack{j=1\\ j\neq l}}^{k} \left[\sum_{i_j=0}^{u_j} \mathbf{1}\{x_{j,i_j+1} < x_{l,i_l}\} P_{i_j}^{j} \right] M_{i_l}^{l} + \sum_{t_l=1}^{s_{l,i_l}} \prod_{\substack{j=1\\ j\neq l}}^{k} \left[\sum_{i_j=0}^{u_j} \mathbf{1}\{x_{j,i_j+1} < c_{l,t_l}^{i_l}\} P_{i_j}^{j} \right] M_{i_l,t_l}^{l} \right\} + \\
& \prod_{\substack{j=1\\ j\neq l}}^{k} \left[\sum_{i_j=0}^{u_j} \mathbf{1}\{x_{j,i_j+1} < T_0\} P_{i_j}^{j} \right] M_{T_0}^{l}
\end{aligned}
$$

The first inequality follows by putting all probability masses for X_{l,n_l+1} corresponding to the intervals (x_{l,i_l}, x_{l,i_l+1}), for $i_l = 0, 1, \ldots, u_l$, and (T_0, ∞), in the left end points of these intervals, and by using Lemma 1.4 for the nested intervals. The second inequality follows by putting all probability masses for X_{j,n_j+1}, for $j = 1, \ldots, k, j \neq l$, corresponding to the intervals (x_{j,i_j}, x_{j,i_j+1}), with $(i_j = 0, 1, \ldots, u_j)$, and (T_0, ∞), in the right end points of these intervals. The NPI upper probability is obtained in a similar way, but now all probability masses for the random quantities involved are put at the opposite end points of the respective intervals, then

$$
\begin{aligned}
P^{(l)} \leq & \sum_{i_l=0}^{u_l} P\left(\bigcap_{\substack{j=1\\ j\neq l}}^{k} \left\{ X_{j,n_j+1} < x_{l,i_l+1} \right\} \right) P_{i_l}^{l} + P\left(\bigcap_{\substack{j=1\\ j\neq l}}^{k} \left\{ X_{j,n_j+1} < \infty \right\} \right) P_{T_0}^{l} \\
= & \sum_{i_l=0}^{u_l} P_{i_l}^{l} \prod_{\substack{j=1\\ j\neq l}}^{k} P\left(X_{j,n_j+1} < x_{l,i_l+1} \right) + P_{T_0}^{l}
\end{aligned}
$$

$$\leq \sum_{i_l=0}^{u_l} P^l_{i_l} \prod_{\substack{j=1\\ j\neq l}}^{k} \left\{ \sum_{i_j=0}^{u_j} \mathbf{1}\{x_{j,i_j} < x_{l,i_l+1}\} M^j_{i_j} + \sum_{i_j=0}^{u_j} \sum_{t_j=1}^{s_{j,i_j}} \mathbf{1}\{c^{i_j}_{j,t_j} < x_{l,i_l+1}\} M^j_{i_j,t_j} + \mathbf{1}\{T_0 < x_{l,i_l+1}\} M^j_{T_0} \right\} + P^l_{T_0}$$

□

It is easily seen that the value of T_0 only influences these lower and upper probabilities through the values of u_j. If $u_l = 0$ then $\overline{P}^{(l)} = 1$, while if $u_j = 0$ for at least one $j \neq l$ then $\underline{P}^{(l)} = 0$. If the experiment is terminated before a single unit has failed, then $\underline{P}^{(l)} = 0$ and $\overline{P}^{(l)} = 1$ for all groups. These extreme cases illustrate an attractive feature of these NPI lower and upper probabilities in quantifying the strength of statistical information, in an intuitive manner that is not possible with precise probabilities. If T_0 increases, $\underline{P}^{(l)}$ never decreases and $\overline{P}^{(l)}$ never increases, and they can only change if further events are observed as we will see later in the examples.

If the experiment is not ended before event times for all units have been observed (whether the units have failed or were right-censored), then the terms including T_0 in (4.2) and (4.3) disappear, and we get an extension of the results by Coolen and Yan [26], who only considered NPI for comparison of two groups of lifetime data. In this case, the NPI lower and upper probabilities are

$$\underline{P}^{(l)} = \sum_{i_l=0}^{u_l} \left\{ \prod_{\substack{j=1\\ j\neq l}}^{k} \left[\sum_{i_j=0}^{u_j} \mathbf{1}\{x_{j,i_j+1} < x_{l,i_l}\} P^j_{i_j} \right] M^l_{i_l} + \sum_{t_l=1}^{s_{l,i_l}} \prod_{\substack{j=1\\ j\neq l}}^{k} \left[\sum_{i_j=0}^{u_j} \mathbf{1}\{x_{j,i_j+1} < c^{i_l}_{l,t_l}\} P^j_{i_j} \right] M^l_{i_l,t_l} \right\}$$

$$\overline{P}^{(l)} = \sum_{i_l=0}^{u_l} P^l_{i_l} \prod_{\substack{j=1\\ j\neq l}}^{k} \left\{ \sum_{i_j=0}^{u_j} \mathbf{1}\{x_{j,i_j} < x_{l,i_l+1}\} M^j_{i_j} + \sum_{i_j=0}^{u_j} \sum_{t_j=1}^{s_{j,i_j}} \mathbf{1}\{c^{i_j}_{j,t_j} < x_{l,i_l+1}\} M^j_{i_j,t_j} \right\}$$

Another special case occurs if there are no right-censored observations before T_0, then these results are identical to those presented in Chapter 3 (Section 3.3). One can study these lower and upper probabilities, given in Theorem 4.2, in detail following the same argument as in Chapters 2 and 3.

4.4 Examples

In this section, two examples with data from the literature are presented to illustrate our method, and in the second example we also briefly discuss a classical precedence testing alternative for the same data.

Example 4.1. We use a data set from Desu and Raghavarao [33, p. 263], representing the recorded times (months) until promotion at a large company, for 19 employees in $k = 3$ departments, which we refer to as 'groups' in line with terminology used throughout this chapter. The data are as follows: For group 1: $15, 20^+, 36, 45, 58, 60$ ($n_1 = 6$), for group 2: $12, 25^+, 28, 30^+, 30^+, 36, 40, 45, 48$ ($n_2 = 9$), and for group 3: $30^+, 40, 48, 50$ ($n_3 = 4$), where '+' indicates that the employee left the company at that length of service before getting promotion, hence this can be considered to be a right-censored observation. One could argue about whether or not this right-censoring process is independent of the promotion process, but as we only use this data set to illustrate our new method, and have no further circumstantial information, we do not address this in more detail. We consider at which department the data suggest that one needs to work the longest to get a promotion. In this example, as we are looking at maximum time till promotion, the 'best group' in terminology from Section 4.3, of course actually represents the department where one has to work the longest to achieve a promotion. This data set contains tied observations, we deal with them as discussed in Subsection 1.3.5.

T_0	u_1	u_2	u_3	$\underline{P}^{(1)}$	$\overline{P}^{(1)}$	$\underline{P}^{(2)}$	$\overline{P}^{(2)}$	$\underline{P}^{(3)}$	$\overline{P}^{(3)}$
$[0, 12)$	0	0	0	0	1	0	1	0	1
$[12, 15)$	0	1	0	0	1	0	0.9029	0	1
$[15, 28)$	1	1	0	0	0.8629	0	0.9029	0.0114	1
$[28, 36)$	1	2	0	0	0.8629	0	0.7974	0.0243	1
$[36, 40)$	2	3	0	0	0.7140	0	0.6591	0.0887	1
$[40, 45)$	2	4	1	0.0678	0.7140	0.0248	0.5398	0.1135	0.8332
$[45, 48)$	3	5	1	0.0813	0.6148	0.0315	0.4341	0.1969	0.8332
$[48, 50)$	3	6	2	0.1670	0.6148	0.0315	0.3542	0.2161	0.7475
$[50, 58)$	3	6	3	0.2392	0.6148	0.0315	0.3542	0.2161	0.6617
$[58, 60)$	4	6	3	0.2392	0.6148	0.0315	0.3542	0.2161	0.6617
$[60, \infty)$	5	6	3	0.2392	0.6148	0.0315	0.3542	0.2161	0.6617

Table 4.1: The best group: NPI lower and upper probabilities

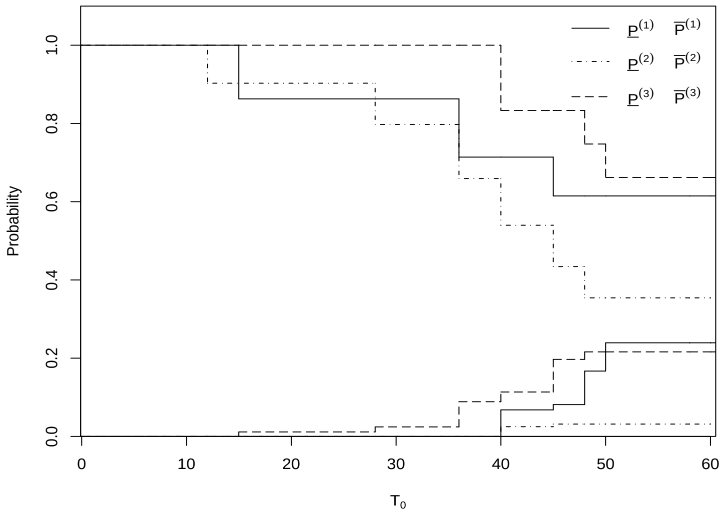


Figure 4.1: The best group: NPI lower and upper probabilities

We use these data to illustrate the NPI method proposed in this chapter, for which we also wish to illustrate the effect of possible early termination of a lifetime experiment. To enable this, we now assume that the recorded times until promotion are all measured from the same moment in time, and we consider the effect on our inferences if, instead of having the complete data as given above, the differences in time to promotion were studied after T_0 months. In this case, all observations that are larger than T_0 are replaced by right-censored observations at T_0. For several values of T_0, the NPI lower probabilities $\underline{P}^{(l)}$ and NPI upper probabilities $\overline{P}^{(l)}$, for $l = 1, 2, 3$, are presented in Table 4.1. For all values of T_0 until it is greater than the largest observation in the data set (60), these NPI lower and upper probabilities are also displayed in Figure 4.1. At no value for T_0 the data indicate strongly that one of the groups leads to longest time to promotion.

As mentioned before, T_0 only influences the NPI lower and upper probabilities considered here via the u_j, so the actually observed failure times, in the sense that, for increasing T_0, these lower and upper probabilities for each group are constant except when T_0 increases past an observed u_j. For example, for $15 \leq T_0 < 28$, the NPI lower and upper probabilities for all three groups remain constant since no observed failure times are in this interval, even though there are two right-censored observations in this interval. These right-censored observations affect the NPI lower and upper probabilities with larger values of T_0, at later failure times, as the jump sizes in these functions will increase. At $T_0 = 28$, when the experiment would include the failure time 28 for a unit of group 2, the upper probability for group 2 decreases and the lower probability for group 3 increases. However, the lower probabilities for groups 1 and 2 still remain 0 for $T_0 = 28$, as there has not yet been an observed failure for group 3 at that moment in time, so the data do not exclude the possibility that units in group 3 would never fail. If the experiment is ended before the first failure of a particular group occurs, as is the case for group 3 at T_0 less than 40 in this example, then the extreme case corresponding to these lower probabilities for groups 1 and 2, according to the NPI M-functions, allows the probability mass related to failure for units of group 3 to go to infinity, which explains why the lower probabilities for groups 1 and 2 remain equal to 0 until T_0 increases past 40, the smallest time at which a unit of group 3 fails.

If the experiment is stopped at $T_0 \in [15, 50)$, both the lower and upper probabilities for group 3 are greater than the lower and upper probabilities, respectively, for groups 1 and 2, as discussed before one could argue that this provides a weak indication that group 3 leads to the longest times until promotion. However, the large imprecision in these lower and upper probabilities indicates that the evidence for such a claim is weak, so care must be taken when formulating any conclusion along these lines. For larger values of T_0, such that event times for most units have been observed in the experiment, group 3 has most imprecision remaining, which reflects that there are only few observations for group 3. △

Example 4.2. In this example we use a data set considered by Lee and Desu [49], which gives leukemia remission times (in days) for patients undergoing three different treatments, so $k = 3$, and the numbers of patients per treatment are $n_1 = 25$, $n_2 = 19$ and $n_3 = 22$. The data are given in Table 4.2, where '+' again denotes that an observation is right-censored. In this example, 'better' means that a treatment leads to larger remission times. This data set was also used by Chakraborti and van der Laan [16] to illustrate precedence testing, with Treatment

Treatment 1					Treatment 2				Treatment 3				
4	5	9	10	10	8	10	10	12	8	10	11	12^+	23
12	13	20^+	23	28	14	20	48	70	25	25	28	28	31
28	28	29	31	32	75	99	103	161^+	31	40	48	89	124
37	41	41	57	62	162	169	195	199^+	143	159^+	190^+	196^+	197^+
74	100	139	258^+	269^+	217^+	220	245^+		205^+	219^+			

Table 4.2: The remission times (in days) of leukemia patients

1 considered as a control treatment, and with focus on the median of the remission times for the control treatment, i.e. $\hat{F}_1^{-1}(0.5) = 29.39$, $\hat{F}_2(29.39) = 0.3334$ and $\hat{F}_3(29.39) = 0.4207$, consequently the value of V, from equation (4.1), is -0.975. They tested the null hypothesis that all treatments have the same effect, against the alternative that at least one of Treatments 2 or 3 is better than Treatment 1. They concluded that, at 5% significance level, there is no evidence that any of the Treatments 2 or 3 is better than Treatment 1.

This data set also contains tied observations and we deal with them in the same manner as discussed in Subsection 1.3.5. Table 4.3 presents the NPI lower and upper probabilities for the events that Treatment l ($l = 1, 2, 3$) is the best, i.e. Treatment l leads to larger remission times than the other two treatments, for a number of times T_0 at which the experiment could have been stopped, where as before all units for which no event had yet been observed at T_0 would be considered to be right-censored at T_0. If we stop the experiment any time before 162 (i.e. $T_0 < 162$) then we have a weak indication that treatments 2 and 3 are better than treatment 1, since $\underline{P}^{(1)} < \underline{P}^{(j)}$ and $\overline{P}^{(1)} < \overline{P}^{(j)}$ for $j = 2, 3$. However, if we consider, for example, the case where the experiment would have been stopped at $T_0 = 162$, then the data would provide a strong indication that Treatments 2 and 3 are both better than Treatment 1, since $\overline{P}^{(1)} < \underline{P}^{(j)}$ for $j = 2, 3$. Of course, as these NPI lower (upper) probabilities never decrease (increase), the same indication holds if the experiment would have continued beyond time 162, no matter if or when it would have stopped. This is an interestingly different conclusion than that reached by Chakraborti and van der Laan [16], and is a good indication of the importance of using several statistical methods simultaneously. It should be noted that, if in this example the experiment is not terminated before an event for each unit has been

T_0	u_1	u_2	u_3	$\underline{P}^{(1)}$	$\overline{P}^{(1)}$	$\underline{P}^{(2)}$	$\overline{P}^{(2)}$	$\underline{P}^{(3)}$	$\overline{P}^{(3)}$
$[5,8)$	2	0	0	0	0.9232	0	1	0	1
$[8,10)$	3	1	1	0.0019	0.8851	0.0045	0.9505	0.0053	0.9570
$[10,11)$	5	3	2	0.0103	0.8102	0.0139	0.8535	0.0253	0.9156
$[11,13)$	6	4	3	0.0201	0.7734	0.0243	0.8059	0.0396	0.8741
$[13,20)$	7	5	3	0.0246	0.7366	0.0281	0.7586	0.0571	0.8741
$[20,23)$	7	6	3	0.0291	0.7366	0.0281	0.7113	0.0681	0.8741
$[23,25)$	8	6	4	0.0380	0.6992	0.0408	0.7113	0.0776	0.8339
$[25,28)$	8	6	6	0.0559	0.6992	0.0592	0.7113	0.0776	0.7536
$[28,31)$	12	6	8	0.0703	0.5598	0.1155	0.7113	0.1066	0.6772
$[31,37)$	14	6	10	0.0826	0.4927	0.1674	0.7113	0.1189	0.6034
$[37,40)$	15	6	10	0.0826	0.4592	0.1793	0.7113	0.1250	0.6034
$[40,48)$	17	6	11	0.0876	0.3934	0.2232	0.7113	0.1362	0.5678
$[48,57)$	17	7	12	0.0992	0.3934	0.2417	0.6823	0.1501	0.5351
$[57,62)$	18	7	12	0.0992	0.3624	0.2550	0.6823	0.1559	0.5351
$[62,70)$	19	7	12	0.0992	0.3313	0.2682	0.6823	0.1618	0.5351
$[70,74)$	19	8	12	0.1047	0.3313	0.2682	0.6558	0.1773	0.5351
$[74,89)$	20	9	12	0.1091	0.3015	0.2803	0.6304	0.2004	0.5351
$[89,99)$	20	9	13	0.1125	0.3015	0.2985	0.6304	0.2004	0.5085
$[99,103)$	21	10	13	0.1173	0.2750	0.3093	0.6069	0.2225	0.5085
$[103,162)$	22	11	15	0.1260	0.2510	0.3510	0.5848	0.2450	0.4664
$[162,169)$	22	12	15	0.1291	0.2510	0.3510	0.5664	0.2581	0.4664
$[169,195)$	22	13	15	0.1322	0.2510	0.3510	0.5480	0.2713	0.4664
$[195,220)$	22	15	15	0.1353	0.2510	0.3510	0.5302	0.2840	0.4664
$[220,\infty)$	22	14	15	0.1404	0.2510	0.3510	0.5226	0.2840	0.4664

Table 4.3: The best group: NPI lower and upper probabilities

recorded (so $T_0 > 269$), then the NPI lower and upper probabilities corresponding to Treatment 3 have the largest imprecision, which is caused by the fact that for this treatment more observations are right-censored, particularly the larger observations, than for the other treatments. $\triangle$

4.5 Concluding remarks

This chapter has introduced NPI for comparing $k \geq 2$ independent groups of units placed simultaneously on a lifetime experiment, with the possibility that the experiment is ended before all event times have been observed. For each unit, the event time recorded, if it happens before the experiment is ended, is either the time of an observed failure or a right-censoring time. Where classical frequentist methods in statistics tend to base such comparison on hypotheses tests, the NPI approach directly compares random failure times of further units from these groups, which are assumed to be related to the observations per group through the assumption $A_{(n)}$.

We consider it an advantage that, as clearly shown in Examples 4.1 and 4.2 for smaller values of T_0, corresponding NPI lower and upper probabilities may differ so much that they do not point towards clear decisions. This makes clear that, in order to derive stronger guidance, more information is needed, which in this application area would imply to either continue the experiment or to repeat it with more units involved. Of course, if there are no possibilities to gain further information, the wide bounds do not lead to indecision, but they just make clear that the data and method used do not strongly indicate a preference for any of the groups. In this case, the data and NPI method may still provide some weak indications to support a specific choice, whereas alternative statistical methods, if they lead to a null hypothesis of 'equal probability distributions' not being rejected, would provide very little guidance on what group to choose if one must do so.

We only considered comparison of different groups by focusing on a single group being best, defined in terms of maximum value of the random lifetime for a future observation. Generalization to consider subsets of groups, either such that they contain the best one or that all selected groups are better than all not-selected groups, is achievable along the lines of Chapter 3 and Coolen and van der Laan [25].

Chapter 5

Progressive Censoring

5.1 Introduction

One topic that has led to a substantial literature in frequentist statistics involving right-censored data is progressive censoring [3, 47], where, during a lifetime experiment, non-failing units are withdrawn from the experiments. This could be done to save cost or time, but it may also be useful, at the moment a unit fails, to study the unit in detail in comparison with units in the same experiment that have not failed, to get better knowledge about the underlying cause of failure [61, 67]. There may also be specific circumstances which cause some units to fail due to reasons unrelated to the experimentation [18], and it may occur that an individual or unit drops out of the study before the end of the experiment [3], which also makes progressive censoring schemes useful. Several progressive censoring schemes have been considered in the literature, including progressive Type-I censoring, progressive Type-II censoring [3] and Type-II progressively hybrid censoring [46].

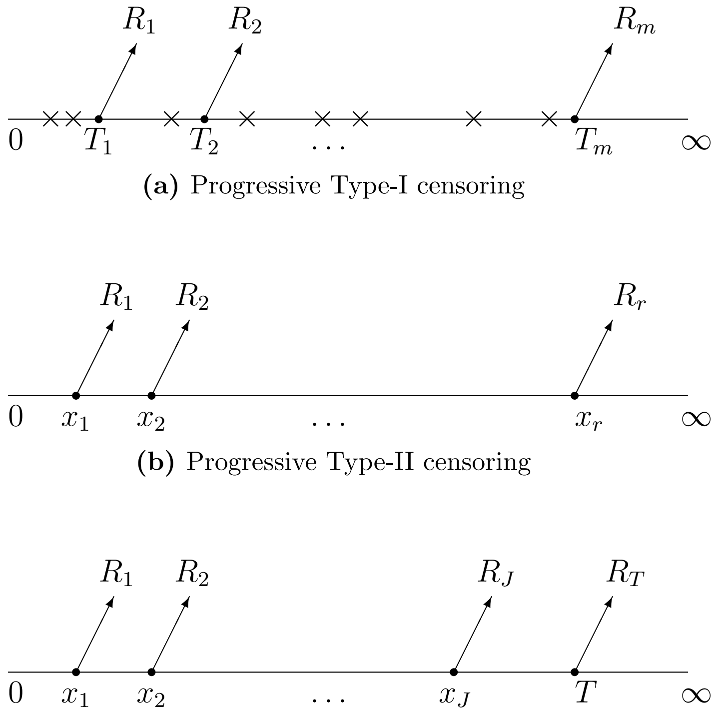
In this chapter we present Nonparametric Predictive Inference (NPI) for comparison of two groups under different progressive censoring schemes. Section 5.2 provides a short overview of progressive censoring schemes for as far as required in this chapter. In Sections 5.3, 5.4 and 5.5 we present the main results of this chapter, namely the NPI lower and upper probabilities for the event that the lifetime of a future unit from group Y is greater than the lifetime of a future unit from group X, under the three different progressive censoring schemes discussed in Section 5.2.

The main focus is on the progressive Type-II censoring scheme, as it has received most attention in the literature, for which the results stated are proved in detail. However, we also briefly present NPI for the two other schemes, the proofs of these follow the same lines of reasoning as for the progressive Type-II censoring results and are not included in full detail. NPI for these scenarios, and for a few related situations, is illustrated and discussed via an example, and the chapter ends with some concluding remarks in Section 5.7.

5.2 Progressive censoring schemes

This section provides a brief overview of three progressive censoring schemes, called progressive Type-I censoring, progressive Type-II censoring and Type-II progressively hybrid censoring. In a progressive Type-I censoring scheme, see Figure 5.1(a), n units are placed on a lifetime experiment. Of these n units, r fail during the experiment, we assume (in order to simplify presentation of the approach presented in this chapter) that they fail at r different failure times $x_1 < x_2 < \ldots < x_r$. At m times $T_1 < T_2 < \ldots < T_m$, some further units may be randomly withdrawn from the experiment, leading to right-censored observations for their corresponding lifetimes. At such a time T_j $(j = 1, \ldots, m)$ where progressive censoring is taking place, let R_j denote the number of units that are removed from the experiment without having failed. We assume that the experiment finishes at time T_m, hence $T_m > x_r$ and $R_m = n - r - \sum_{j=1}^{m-1} R_j$. For use later in this chapter, we define s_j to be the number of failures between the consecutive right-censoring times T_{j-1} and T_j, so $s_j = \#\{T_{j-1} < x_i \leq T_j : i = 1, \ldots, r\}$ $(j = 2, \ldots, m)$, and s_1 is the number of failures before T_1. Then the data from this experiment, under a progressive Type-I censoring scheme, consist of $r = \sum_{j=1}^{m} s_j$ observed failure times and $n - r$ right-censoring times.

In a progressive Type-II censoring scheme, see Figure 5.1(b), the number of units to be observed to fail is fixed, let this number be r. At each observed failure time, which we again assume to be r different times $x_1 < x_2 < \ldots < x_r$, some further units which have not failed are randomly removed from the experiment, and at the



(a) Progressive Type-I censoring

(b) Progressive Type-II censoring

(c) Type-II progressively hybrid censoring

Figure 5.1: Three different progressive censoring schemes

last failure time x_r all the remaining units are removed from the experiment. Let R_i denote the number of units that have not failed but are removed from the experiment at failure time x_i, for $i = 1, \dots, r$, then $R_r = n - r - \sum_{i=1}^{r-1} R_i$. The data consist of the r observed failure times $x_1 < x_2 < \dots < x_r$, together with the numbers of units with right-censored lifetimes at each of these failure times, which information we denote by $\breve{R} = (R_1, R_2, \dots, R_r)$. Some special cases of this censoring scheme occur if $r = n$, then $R_i = 0$ for all $i = 1, \dots, r$, which means that there is no censoring actually occurring, and if we have $R_i = 0$ for all $i = 1, \dots, r-1$ and $R_r = n-r$ then we obtain a conventional Type-II censored sample with censoring only due to the experiment being stopped before all units have failed. A special case of progressive Type-II censoring is the 'throw away scheme', presented by Cohen and Clifford [17], in which a fixed number of units is withdrawn from the experiment at each observed breakdown time. Such special cases are not highlighted further in this chapter, but are briefly considered in Example 5.1.

In a progressive Type-II censored experiment, it might take a very long time to reach the prefixed number r. Therefore, it may be attractive to consider an experiment which is ended as soon as either r failures have been observed, or at a specific time, say T, whichever occurs first. In the latter case, the lifetimes of all the remaining units in the experiment at time T are right-censored at this time, in addition to the right-censored lifetimes of units that were progressively censored during the experiment at different failure times before T. This scenario is called Type-II progressive hybrid censoring, and is a mixture of progressive Type-II censoring and conventional Type-I censoring, see Figure 5.1(c). Let x_J denote the largest observed failure time prior to T, and again we assume that no failures coincide, so the observed failure times are $x_1 < x_2 < \ldots < x_J < T$. At x_i, for $i = 1, \ldots, J$, R_i units are randomly withdrawn from the experiment. Finally, all the remaining R_T units are withdrawn from the experiment at time T, so $R_T = n - J - \sum_{i=1}^{J} R_i$.

As mentioned above, one may be interested in comparing two independent populations or treatments, say X and Y. For example, X may refer to a control group and Y to a new treatment group [7], where statistical inference would be aimed at investigating whether or not Y can be considered to provide an improvement compared to X. Most classical statistics methods presented in the literature [7], including several nonparametric methods, approach such comparison problems by hypothesis testing. In particular, they tend to assume continuous cumulative distribution functions for the random quantities of interest, say $F(\cdot)$ corresponding to X and $G(\cdot)$ corresponding to Y, and test the null hypothesis that the two groups X and Y are the same with regard to the random quantity of interest, so they test $H_0 : F(x) = G(x)$, for all x, against the hypothesis that group Y tends to have greater lifetimes than group X, expressed via the stochastic dominance hypothesis $H_1 : F(x) \geq G(x)$ with strict inequality holding for at least one x. The approach presented in this chapter is fundamentally different, with comparisons formulated directly in terms of a future observation for each of the two groups considered, a method which does not involve testing of hypotheses.

We focus on the progressive Type-II censoring scheme, for which in the literature two cases are considered, depending on whether progressive censoring has been applied to only one group or to both groups. In the first case, the progressive Type-II censoring scheme applies to only one group, say group Y, and it is assumed that the data from group X result from an experiment without progressive censoring, but which is also ended when the experiment of group Y ends, at failure time y_r from group Y, so the group X data consist of failure times prior to y_r and right-censored lifetimes at y_r, resulting from standard Type-II censoring at y_r. For this case, Ng and Balakrishnan [5, 67] have proposed several tests, including the weighted precedence test, the weighted maximal precedence test and the maximal Wilcoxon rank-sum precedence test as extensions of classical precedence tests that are suitable for this scenario. Bairamov and Eryilmaz [2] considered exceedance statistics for the same setting.

As the second case, one considers the situation with progressive Type-II censoring applied independently to both groups X and Y. Recently, Balakrishnan *et al.* [7] introduced a precedence test based on placement statistics with progressive censoring for both groups. The proposed precedence test statistic, $P_{(s)}$, is basically the number of failures from group X that precede the sth ($1 \leq s \leq r_y$) failure from group Y, where r_y is the number of failures from group Y. Two further precedence tests are proposed by Balakrishnan *et al.* [8, 9] for progressive censoring in both samples. The first is a Wilcoxon type rank-sum precedence test, $T_{(r_y)}$, where all censored items are assumed to fail immediately after the censoring occurs. Then $T_{(r_y)}$ is defined as the sum of the ranks of the X observations in the combined sample (i.e. X and Y combined together) [8, 9]. The second test statistic, $\bar{Q}_{(r_y)}$, is based on the Kaplan-Meier estimator [44] of the cumulative distribution functions F and G. Let Q_j be the number of failures from group X between the $(j-1)$th and jth failure from group Y for which $\hat{F}(x_i) \geq \hat{G}(y_j)$, for $j = 1, 2, \ldots, r_y$ where $y_0 = 0$. Then,

$$\bar{Q}_{(r_y)} = \sum_{j=1}^{r_y} Q_j + \frac{1}{2} Q_{r_y+1}$$

where the Q_{r_y+1} is obtained by assuming that the remaining unobserved failures from group X (i.e. the failures from group X that greater than y_{r_y}) occur before the censored items from group Y at y_{r_y} for which $\hat{F}(x_i) \geq \hat{G}(y_{r_y+1})$, where y_{r_y+1} is taken as the (r_y+1)th progressive Type-II censored order statistic with progressive censoring scheme $(R_1^y, \ldots, R_{r_y-1}^y, 0, n_y - r_y - 1 - \sum_{j=1}^{r_y-1} R_j^y)$. For the null distributions of these test statistics and for more details we refer to [8, 9]. The NPI approach is compared to these methods in Example 5.1.

5.3 NPI for progressive Type-II censoring

For the progressive Type-II censoring scheme, n units are placed on a lifetime experiment, and for r of these units actual failure times are observed during the experiment, while at each observed failure time for one of these r units, some of the remaining units may be withdrawn from the experiment, until the r-th failure time when the experiment is ended, and hence all remaining units are removed from the experiment. We can consider the $n-r$ progressively censored units (according to the scheme $\breve{R} = (R_1, R_2, \dots, R_r))$ as being grouped in blocks, each consisting of units censored at a specific observed failure time. Hence, this leads to all censored units in one block to be censored at the same time, which is dealt with by rc-$A_{(n)}$ as described below. For ease of notation, we assume throughout that there are no ties between the observed failure times, the tied right-censoring times do not provide complications and actually simplify the approach as discussed below. In addition, we use $x_0 = 0$ and $x_{r+1} = \infty$. The following theorem provides the M-functions required for NPI applied to comparison of lifetime data under the progressive Type-II censoring scheme, together with the total probability mass assigned to the interval (x_i, x_{i+1}).

Theorem 5.1. To apply NPI to data from an experiment with a progressive Type-II censoring scheme with $\breve{R} = (R_1, R_2, \dots, R_r)$, the assumption rc-$A_{(n)}$ implies that the probability distribution for a nonnegative random quantity X_{n+1} on the basis of data including r real and $(n-r)$ progressively censored observations, is partially specified by the following M-function values, for $i = 0, 1, \dots, r$,

$$M^X(x_i, x_{i+1}) = M_{X_{n+1}}(x_i, x_{i+1}) = \frac{1}{n+1} \prod_{k=1}^{i-1} \frac{n-k-\sum_{l=1}^{k-1} R_l + 1}{n-k-\sum_{l=1}^{k} R_l + 1} \tag{5.1}$$

$$M^X(x_i^+, x_{i+1}) = M_{X_{n+1}}(x_i^+, x_{i+1}) = \left[\frac{R_i}{n-i-\sum_{l=1}^{i} R_l + 1}\right] M^X(x_i, x_{i+1}) \tag{5.2}$$

where x_i^+ is used to indicate a value infinitessimally greater than x_i, which can be interpreted as representing the lower bound for the interval that would contain the actual lifetimes for all units censored at x_i. Then the total probability mass assigned to the interval (x_i, x_{i+1}) is the sum of the two M-functions corresponding to (x_i, x_{i+1}) and (x_i^+, x_{i+1}) (for $i = 0, 1, \ldots, r$), and is given by

$$P^X(x_i, x_{i+1}) = P(X_{n+1} \in (x_i, x_{i+1})) = \frac{1}{n+1} \prod_{k=1}^{i} \frac{n - k - \sum_{l=1}^{k-1} R_l + 1}{n - k - \sum_{l=1}^{k} R_l + 1} \tag{5.3}$$

Proof. We can write the observations, both failure times and right-censoring times, of n units from group X as given below, in which we assume that all observations are different values for ease of presentation, but for tied right-censored observations one can derive the exact NPI results as limiting situation with the difference between such right-censoring times becoming infinitesimally small. We follow Coolen and Yan [27] in assuming, which is also standard in the wider literature, that coinciding failure and right-censoring times are actually such that the latter is slightly larger than the failure time. Let the data be

$$0 < x_1 < c_1^1 < \ldots < c_{R_1}^1 < x_2 < c_1^2 < \ldots < c_{R_2}^2 < x_3 < \ldots$$

$$< \ldots < x_i < c_1^i < \ldots < c_{R_i}^i < x_{i+1} < \ldots < x_r < c_1^r < \ldots < c_{R_r}^r < \infty$$

For the setting considered in this chapter, $c_{l_i}^i$ is actually the right-censoring time of the l_ith unit censored at x_i, for $i = 1, \ldots, r$ and $l_i = 1, \ldots, R_i$.

For any block k ($k = 1, \ldots, r$), $x_k < c_1^k < \ldots < c_{l_k}^k < \ldots < c_{R_k}^k < x_{k+1}$, $\tilde{n}_{c_{l_k}^k}$ is the number of units at risk at $c_{l_k}^k$, that is $\tilde{n}_{c_{l_k}^k} = n - k - (l_k - 1) - \sum_{l=1}^{k-1} R_l$. Then, and from (1.3),

$$\begin{aligned} M^X(x_i, x_{i+1}) &= \frac{1}{n+1} \prod_{\{k: c_k < x_i\}} \frac{\tilde{n}_{c_k} + 1}{\tilde{n}_{c_k}} = \frac{1}{n+1} \prod_{k=1}^{i-1} \prod_{l_k=1}^{R_k} \frac{\tilde{n}_{c_{l_k}^k} + 1}{\tilde{n}_{c_{l_k}^k}} \\ &= \frac{1}{n+1} \prod_{k=1}^{i-1} \frac{n - k - \sum_{l=1}^{k-1} R_l + 1}{n - k - \sum_{l=1}^{k} R_l + 1} \end{aligned}$$

similar, and from (1.6),

$$P^X(x_i, x_{i+1}) = \frac{1}{n+1} \prod_{\{k: c_k < x_{i+1}\}} \frac{\tilde{n}_{c_k} + 1}{\tilde{n}_{c_k}} = \frac{1}{n+1} \prod_{k=1}^{i} \frac{n - k - \sum_{l=1}^{k-1} R_l + 1}{n - k - \sum_{l=1}^{k} R_l + 1}$$

since $M^X(x_i^+, x_{i+1}) = P^X(x_i, x_{i+1}) - M^X(x_i, x_{i+1})$, then

$$M^X(x_i^+, x_{i+1}) = \frac{1}{n+1}\left[\prod_{k=1}^{i-1} \frac{n-k-\sum_{l=1}^{k-1} R_l + 1}{n-k-\sum_{l=1}^{k} R_l + 1}\right]\left[\frac{R_i}{n-i-\sum_{l=1}^{i} R_l + 1}\right]$$

□

In this section we present NPI to compare two groups, say X and Y, when one (or both) is (are) progressively censored. Throughout, we consider the two groups to be completely independent. In NPI, the comparison of groups X and Y is in terms of lower and upper probabilities for the event that a single future observation from group Y is greater than a single future observation from group X, where lower and upper probabilities are used in order to keep inferential assumptions, added to the data observed, restricted.

Suppose that we have two independent groups, X and Y, consisting of n_x and n_y units, all placed on a lifetime experiment. Units of both groups are progressively Type-II censored with the schemes $\breve{R}^x = (R_1^x, R_2^x, \ldots, R_{r_x}^x)$ and $\breve{R}^y = (R_1^y, R_2^y, \ldots, R_{r_y}^y)$. In practice, for example, group X could be a control group, with a new treatment applied to units in group Y, and the aim might be to draw conclusions on whether or not the new treatment group tends to provide improved lifetimes. Given the data, $\breve{R}^x$, $\breve{R}^y$, and with the appropriate assumptions rc-$A_{(n_x)}$ and rc-$A_{(n_y)}$ for the respective groups, Theorem 5.2 presents the NPI lower and upper probabilities for the event that the next future observation from group Y, Y_{n_y+1}, is greater than the next future observation from group X, X_{n_x+1}.

Theorem 5.2. The NPI lower and upper probabilities for the event that the next future observation from group Y is greater than the next future observation from group X, under the progressive Type-II censoring scheme for both groups, are

$$\underline{P}\left(Y_{n_y+1} > X_{n_x+1}\right) = \sum_{j=0}^{r_y}\left\{\sum_{i=0}^{r_x} \mathbf{1}\{x_{i+1} < y_j\} P^X(x_i, x_{i+1})\right\} P^Y(y_j, y_{j+1}) \qquad (5.4)$$

$$\overline{P}\left(Y_{n_y+1} > X_{n_x+1}\right) = \sum_{j=0}^{r_y}\left\{\sum_{i=0}^{r_x} \mathbf{1}\{x_i < y_{j+1}\} P^X(x_i, x_{i+1})\right\} P^Y(y_j, y_{j+1}) \qquad (5.5)$$

with P^X and P^Y according to equation (5.3).

Proof. The NPI lower probability for the event $X_{n_x+1} < Y_{n_y+1}$, given the data and progressive Type-II censoring schemes $\breve{R}^x$ and $\breve{R}^y$, is derived as follows:

$$\begin{aligned}
P &= P(X_{n_x+1} < Y_{n_y+1}) = \sum_{j=0}^{r_y} P(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (y_j, y_{j+1})) \\
&\geq \sum_{j=0}^{r_y} \left\{ P(X_{n_x+1} < y_j) M^Y(y_j, y_{j+1}) + P(X_{n_x+1} < y_j^+) M^Y(y_j^+, y_{j+1}) \right\} \\
&= \sum_{j=0}^{r_y} P(X_{n_x+1} < y_j) \left\{ M^Y(y_j, y_{j+1}) + M^Y(y_j^+, y_{j+1}) \right\} \\
&= \sum_{j=0}^{r_y} P(X_{n_x+1} < y_j) P^Y(y_j, y_{j+1}) \\
&\geq \sum_{j=0}^{r_y} \sum_{i=0}^{r_x} \mathbf{1}\{x_{i+1} < y_j\} P^X(x_i, x_{i+1}) P^Y(y_j, y_{j+1})
\end{aligned}$$

The first inequality follows by putting all probability masses for Y_{n_y+1} corresponding to the intervals (y_j, y_{j+1}) and (y_j^+, y_{j+1}) $(j = 1, \ldots, r_y)$ to the left end points of these intervals, and by using Lemma 1.4 for the nested intervals (y_j, y_{j+1}) and (y_j^+, y_{j+1}). The second inequality follows by putting all probability masses for X_{n_x+1} corresponding to the intervals (x_i, x_{i+1}) and (x_i^+, x_{i+1}) $(i = 1, \ldots, r_x)$ to the right end points of these intervals. We should notice that $P(X_{n_x+1} < y_j^+) = P(X_{n_x+1} < y_j)$ since the R_j^y units that are right-censored at y_j do not cause these probabilities to be different due to the assumption of an infinitesimal difference between y_j^+ and y_j, and due to the fact that the M-functions in NPI are generally assigned to open intervals between observations.

The NPI upper probability is obtained in a similar way, but now all probability masses for the random quantities involved are put at the opposite end points of the respective intervals. We should notice that $\mathbf{1}\{x_i^+ < y_{j+1}\} = \mathbf{1}\{x_i < y_{j+1}\}$ by arguments similar to those used in the derivation above for the lower probability.

$$\begin{aligned}
P &= P(X_{n_x+1} < Y_{n_y+1}) = \sum_{j=0}^{r_y} P(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (y_j, y_{j+1})) \\
&\leq \sum_{j=0}^{r_y} P(X_{n_x+1} < y_{j+1}) P^Y(y_j, y_{j+1}) \\
&\leq \sum_{j=0}^{r_y} P^Y(y_j, y_{j+1}) \sum_{i=0}^{r_x} \left\{ \mathbf{1}\{x_i < y_{j+1}\} M^X(x_i, x_{i+1}) + \mathbf{1}\{x_i^+ < y_{j+1}\} M^X(x_i^+, x_{i+1}) \right\} \\
&= \sum_{j=0}^{r_y} \sum_{i=0}^{r_x} P^Y(y_j, y_{j+1}) \mathbf{1}\{x_i < y_{j+1}\} \left\{ M^X(x_i, x_{i+1}) + M^X(x_i^+, x_{i+1}) \right\} \\
&= \sum_{j=0}^{r_y} \sum_{i=0}^{r_x} \mathbf{1}\{x_i < y_{j+1}\} P^X(x_i, x_{i+1}) P^Y(y_j, y_{j+1})
\end{aligned}$$

□

The use of these NPI lower and upper probabilities is illustrated in Example 5.1. Next we present the NPI lower and upper probabilities for the two other progressive censoring schemes discussed in Section 5.2, for these all ingredients required for the complete derivations are provided, but detailed proofs are not presented as these follow the general lines of the proof above.

5.4 NPI for progressive Type-I censoring

In a progressive Type-I censoring scheme for n units on a lifetime experiment, as discussed in Section 5.2, R_j units are withdrawn from the experiment at T_j ($j = 1,\ldots,m$), and for a total of $r = \sum_{j=1}^{m} s_j$ units the actual failure times will be observed, where s_j is the number of observed failure times between T_{j-1} and T_j. Again assuming no ties among the observed failure times, the data can be written as

$$\cdots < \overset{\nearrow R_{j-1}}{T_{j-1}} < x_1^j < \cdots < x_{s_j}^j < \overset{\nearrow R_j}{T_j} < x_1^{j+1} < \ldots < x_{s_j+1}^{j+1} < \overset{\nearrow R_{j+1}}{T_{j+1}} < \ldots$$

where $x_{i_j}^j$ is the i_jth observed failure time between T_{j-1} and T_j ($i_j = 1,\ldots,s_j$, $j = 1,\ldots,m$). For this situation, the NPI approach for comparison of two groups, X and Y, similarly as presented in the previous section, is as follows. Let

$$\mathbf{B_j} = \frac{1}{n+1}\prod_{k=1}^{j}\frac{n-\sum_{l=1}^{k} s_l - \sum_{l=1}^{k-1} R_l + 1}{n-\sum_{l=1}^{k} s_l - \sum_{l=1}^{k} R_l + 1}$$

then the M-functions corresponding to a progressive Type-I censoring scheme, are (for $j = 1,\ldots,m$ and $i_j = 1,\ldots,s_j$)

$$M^X(0, x_1^1) = \mathbf{B_1} \quad , \quad M^X(x_{i_j}^j, x_{i_j+1}^j) = \mathbf{B_{j-1}} \quad ,$$

$$M^X(T_j, x_1^{j+1}) = \left[\frac{R_j}{n-\sum_{l=1}^{j} s_l - \sum_{l=1}^{j} R_l + 1}\right]\mathbf{B_{j-1}} \quad , \quad P^X(x_{i_j}^j, x_{i_j+1}^j) = \mathbf{B_j}$$

where x_1^{j+1} ($x_{s_j}^j$) is the first (last) failure time observed after (before) we removed R_j units at time T_j, and where $x_{s_j+1}^j = x_1^{j+1}$ and $x_1^{m+1} = \infty$.

Now we consider two groups X and Y under such a progressive Type-I censoring scheme, with right-censoring times T_a^x ($a = 1,\ldots,p$) and T_b^y ($b = 1,\ldots,q$), such that

R^x_a (R^y_b) units of group X (Y) that have not failed are withdrawn from the experiment at T^x_a (T^y_b). Then the numbers of failures from both groups are $r_x = \sum_{a=1}^p s^x_a$ and $r_y = \sum_{b=1}^q s^y_b$, where s^x_a (s^y_b) is the number of failures between the consecutive right-censoring times T^x_{a-1} and T^x_a (T^y_{b-1} and T^y_b). The NPI lower probability for the event $X_{n_x+1} < Y_{n_y+1}$ in this situation is

$$\underline{P} = \sum_{b=1}^{q} \left\{ \sum_{i_b=1}^{s^y_b} P(X_{n_x+1} < y^b_{i_b}) M^Y(y^b_{i_b}, y^b_{i_b+1}) + P(X_{n_x+1} < T^y_b) M^Y(T^y_b, y^{b+1}_1) \right\}$$

where

$$P(X_{n_x+1} < \cdot) = \sum_{a=1}^{p} \sum_{i_a=0}^{s^x_a} \mathbf{1}\{x^a_{i_a+1} < \cdot\} P^X(x^a_{i_a}, x^a_{i_a+1})$$

and the corresponding NPI upper probability is

$$\overline{P} = \sum_{b=1}^{q} \sum_{i_b=0}^{s^y_b} P(X_{n_x+1} < y^b_{i_b+1}) P^Y(y^b_{i_b}, y^b_{i_b+1})$$

where

$$P(X_{n_x+1} < \cdot) = \sum_{a=1}^{p} \left\{ \sum_{i_a=0}^{s^x_a} \mathbf{1}\{x^a_{i_a} < \cdot\} M^X(x^a_{i_a}, x^a_{i_a+1}) + \mathbf{1}\{T^x_a < \cdot\} M^X(T^x_a, x^{a+1}_1\} \right\}$$

As mentioned before, detailed justification of these results follows the same lines as the proof in the previous section. The special case where such progressive censoring is only applied to one of the two groups also follows straightforwardly, and will be briefly illustrated in Example 5.1.

5.5 NPI for Type-II progressively hybrid censoring

Under this scheme of progressive censoring, that was also introduced in Section 5.2, one only observes the J failure times which occur prior to time T, and at failure time x_i ($i = 1, \ldots, J$) R_i units that have not failed are removed, and finally the experiment is ended at time T, when the R_T remaining units are removed from the experiment. For this progressive censoring scheme, we can use the same M-functions as given in (5.1) and (5.2) for the intervals (x_i, x_{i+1}) and (x_i^+, x_{i+1}), where

$i = 0, 1, \ldots, J$, $x_0 = 0$ and $x_{J+1} = \infty$. However, for the additional interval (T, ∞), the M-function value is

$$M^X(T,\infty) = \frac{1}{n+1}\left[\frac{R_T}{n - J - \sum_{i=1}^{J} R_i - R_T + 1}\right]\prod_{k=1}^{J}\frac{n-k-\sum_{l=1}^{k-1} R_l + 1}{n-k-\sum_{l=1}^{k} R_l + 1}$$

This also leads to the same formula (5.3) being appropriate for the probability $P^X(x_i, x_{i+1})$, for $i = 0, 1, \ldots, J-1$, while for the last interval we have $P^X(x_J, \infty) = M^X(x_J, \infty) + M^X(x_J^+, \infty) + M^X(T, \infty)$.

NPI for comparison of two groups, X and Y, under such Type-II progressively hybrid censoring with $(R_1^x, R_2^x, \ldots, R_{J_x}^x, R_{T^x}^x)$ and $(R_1^y, R_2^y, \ldots, R_{J_y}^y, R_{T^y}^y)$, respectively, is again based on the NPI lower and upper probabilities for the direct comparison of one future observation from each group, so for the event $X_{n_x+1} < Y_{n_y+1}$. For this censoring scheme, these NPI lower and upper probabilities are

$$\underline{P} = \sum_{j=0}^{J_y} P(X_{n_x+1} < y_j)P^Y(y_j, y_{j+1}) + \{P(X_{n_x+1} < T^y) - P(X_{n_x+1} < y_{J_y})\}M^Y(T^y, \infty)$$

where

$$P(X_{n_x+1} < \cdot) = \sum_{i=0}^{J_x} \mathbf{1}\{x_{i+1} < \cdot\}P^X(x_i, x_{i+1})$$

and

$$\overline{P} = \sum_{j=0}^{J_y} P(X_{n_x+1} < y_{j+1})P^Y(y_j, y_{j+1}) + P(X_{n_x+1} < \infty)M^Y(T^y, \infty)$$

where

$$P(X_{n_x+1} < \cdot) = \sum_{i=0}^{J_x} \mathbf{1}\{x_i < \cdot\}P^X(x_i, x_{i+1}\} + \{\mathbf{1}\{T^x < \cdot\} - \mathbf{1}\{x_{J_x} < \cdot\}\}M^X(T^x, \infty)$$

Detailed justification of these results is again similar to the proof given for the progressive Type-II censoring scheme, and this case is also illustrated in Example 5.1.

Finally, let us briefly comment on what could be considered a special case of the progressive censoring schemes described above, namely if we just decide to terminate the lifetime experiment at a certain time point, say T_0, which could be a specific failure time, and with no other censoring applied. In this case, we have $\breve{R}^x = (0, 0, \ldots, R_{r_x}^x)$ and $\breve{R}^y = (0, 0, \ldots, R_{r_y}^y)$, where $R_{r_x}^x = n_x - r_x$ and $R_{r_y}^y = n_y - r_y$. NPI for comparison of two groups under this setting, co-called precedence testing, was presented in Chapter 2, whereas the generalization of such results to several groups was presented in Chapter 3. This precedence testing scenario is also included in Example 5.1.

5.6 Example

In this section, an example is given to illustrate the NPI approach for comparison of two groups of lifetime data under several progressive censoring schemes.

Example 5.1. In this example, we use a subset of Nelson's dataset [64, p. 462] on breakdown times (in minutes) of an insulating fluid that is subject to high voltage stress. The data are given in Table 5.1, for both groups there are 10 units involved in the experiment, hence $n_x = n_y = 10$. This data set was also used in Chapter 2 to illustrate the NPI approach for precedence testing (Example 2.2).

Group	Lifetimes									
X	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75
Y	1.34	1.49	1.56	2.10	2.12	3.83	3.97	5.13	7.21	8.71

Table 5.1: Lifetimes of two samples of an insulating fluid

Ng and Balakrishnan [67] used this data set to illustrate the weighted precedence and weighted maximal precedence tests, when progressive Type-II censoring is assumed to be applied to group Y. In their example, under the scheme $\breve{R}^y = (3, 0, 0, 0, 2)$, only 5 breakdown times from group Y are observed, for the other 5 units the observations are right-censored. They assume that the three units with actual observed breakdown times 2.10, 3.83 and 3.97 are instead removed from the experiment at the first breakdown time ($y_1 = 1.34$), and that the two units with the largest actual breakdown times, 7.21 and 8.71, are removed from the experiment at the fifth breakdown time, which then is $y_5 = 5.13$. So for all units of group X the actual breakdown times are observed, and such times are observed for 5 units from group Y, at times 1.34, 1.49, 1.56, 2.12 and 5.13. Ng and Balakrishnan [67] derived the weighted precedence test statistic as equal to 67, with p-value 0.009 for the test of the null-hypothesis that both groups' breakdown times are equally distributed, and the weighted maximal precedence test statistic is equal to 50 with corresponding p-value 0.006. Therefore, they conclude that there is a strong indication to reject this null-hypothesis, even with this specific result of progressive censoring applied to the Y group. Their analysis concludes that there is substantial evidence in the data to support a claim that breakdown times for group Y tend to be significantly larger than for group X.

Below we present the NPI results for this example, applying different progressive censoring schemes. We consider several cases with (mostly) progressive censoring, some in which it is applied only to group Y as done by Ng and Balakrishnan [67], and some cases with such censoring applied to both groups. We present the NPI lower and upper probabilities that group Y is better than group X, as before in the direct predictive sense by comparing one future observation from each group, X_{11} and Y_{11} in this example. Of course, the appropriate assumptions rc-$A_{(10)}$ are again made per group, and it is assumed that the groups are completely independent.

Case A: Progressive Type-II censoring applied to group Y

Consider the same setting as used by Ng and Balakrishnan [67] and described above, with three units withdrawn from the experiment at the first observed breakdown time for group Y (at $y_1 = 1.34$), and two units for this group withdrawn at the last observed breakdown time, $y_5 = 5.13$, so with $\breve{R}^y = (3, 0, 0, 0, 2)$. It is also assumed that all breakdown times for the units from group X are observed. So, with y^c denoting a right-censored observation at time y, the data actually used in this case are

$$X : 0.49, 0.64, 0.82, 0.93, 1.08, 1.99, 2.06, 2.15, 2.57, 4.75$$

$$Y : 1.34, 1.34^c, 1.34^c, 1.34^c, 1.49, 1.56, 2.12, 5.13, 5.13^c, 5.13^c$$

For this specific situation, the corresponding NPI lower and upper probabilities, as presented in Section 5.3, are $\underline{P}\,(Y_{11} > X_{11}) = 0.6139$ and $\overline{P}\,(Y_{11} > X_{11}) = 0.8052$. These values could be interpreted as pretty strongly supporting the explicit event of interest here, namely that if we would get one future value for each of these two groups, under exchangeability assumed per group, then the lower probability that

Y_{11} would be greater than X_{11} will be substantially larger than 0.5, which might be interpreted as reflecting a strong indication in favour of this event. This conclusion is in line with the test results by Ng and Balakrishnan [67] for exactly the same case. As this conclusion actually turns out to follow in each of the cases below (this is not necessarily the case in general, of course), it is not repeated nor further discussed there, and the NPI results are just given for illustration without further detailed discussion.

Case B: Progressive Type-II censoring applied to groups X and Y

Suppose that the progressive Type-II censoring scheme is applied to both groups X and Y, with $\breve{R}^x = (3, 1, 1, 0, 0)$ and $\breve{R}^y = (3, 2, 0, 0, 0)$ and resulting in the following data,

$$X : 0.49, 0.49^c, 0.49^c, 0.49^c, 0.64, 0.64^c, 0.93, 0.93^c, 2.06, 2.15$$

$$Y : 1.34, 1.34^c, 1.34^c, 1.34^c, 1.56, 1.56^c, 1.56^c, 2.10, 3.83, 7.21$$

If we calculate the test statistics proposed by [7, 8, 9], using notation as introduced in Section 5.2, we have $P_{(3)} = 4$, $P_{(5)} = 5$ and $T_{(5)} = 70$. To calculate $\bar{Q}_{(5)}$, we need to calculate the Kaplan- Meier estimator of $F(x_i)$ and $G(y_j)$ as follows.

$$\hat{F}(0.49) = 0.10, \hat{F}(0.64) = 0.25, \hat{F}(0.93) = 0.44, \hat{F}(2.06) = 0.72 \text{ and } \hat{F}(2.15) = 1$$

$$\hat{G}(1.34) = 0.10, \hat{G}(1.56) = 0.25, \hat{G}(2.10) = 0.50, \hat{G}(3.83) = 0.75 \text{ and } \hat{G}(7.21) = 1$$

Since $Q_1 = 3$, $Q_2 = 0$, $Q_3 = 1$, $Q_4 = 1$, $Q_5 = 0$ and $Q_6 = 0$, then $\bar{Q}_{(5)} = 5$. Using the near 5% critical values and the exact level of significance summarized in [7, 8, 9], we do not reject the null hypothesis for $P_{(3)}$, $P_{(5)}$ and $\bar{Q}_{(5)}$ at significance level 5%, however we reject the null hypothesis for $T_{(5)}$ at significance level 5%. The NPI results for the comparison of these two groups of breakdown times are $\underline{P}(Y_{11} > X_{11}) = 0.5448$ and $\overline{P}(Y_{11} > X_{11}) = 0.8678$.

Case C: Type-II progressively hybrid censoring applied to groups X and Y

In this example, a progressive Type-II censoring scheme is applied to groups X and Y, with $\breve{R}^x = (2, 1, 0, 1, 0, 0)$ and $\breve{R}^y = (1, 2, 0, 3)$. However, the experiment will be ended at $T = 2.11$, making this a Type-II progressively hybrid censoring scheme as discussed in Section 5.2. Suppose that the resulting data from this experiment are as follows,

$$X : 0.49, 0.49^c, 0.49^c, 0.64, 0.64^c, 0.93, 1.99, 1.99^c, 2.06, 2.11^c$$

$$Y : 1.34, 1.34^c, 1.49, 1.49^c, 1.49^c, 2.10, 2.11^c, 2.11^c, 2.11^c, 2.11^c$$

Then the corresponding NPI lower and upper probabilities are $\underline{P}(Y_{11} > X_{11}) = 0.5148$ and $\overline{P}(Y_{11} > X_{11}) = 0.8744$.

Case D: Progressive Type-I censoring applied to group Y

In this case, some units of group Y are removed from the experiment before breakdown, at different times, say at $T = (T_1, T_2, T_3) = (1.5, 3.5, 5.5)$. Suppose that one unit is removed at $T_1 = 1.5$, three at $T_2 = 2.5$, and one at $T_3 = 5.5$, and let us assume that this leads to the following data for group Y: 1.34, 1.49, 1.5^c, 1.56, 2.10, 3.5^c, 3.5^c, 3.5^c, 3.83 and 5.5^c. We assume that no progressive censoring is applied to group X. The corresponding NPI lower and upper probabilities for the comparison of groups X and Y are $\underline{P}(Y_{11} > X_{11}) = 0.6364$ and $\overline{P}(Y_{11} > X_{11}) = 0.8244$.

Case E: Throw away censoring scheme applied to groups X and Y

Suppose that the 'throw away scheme', as briefly discussed in Section 5.2, is applied to both groups X and Y, with one unit withdrawn each time, hence $\breve{R}^x = (1, 1, 1, 1, 1)$ and $\breve{R}^y = (1, 1, 1, 1, 1)$. Suppose further that the actually observed breakdown times (and corresponding right-censoring times) under this scheme are as follows,

$$X : 0.49, 0.49^c, 0.64, 0.64^c, 0.93, 0.93^c, 1.08, 1.08^c, 2.06, 2.06^c$$

$$Y : 1.34, 1.34^c, 1.56, 1.56^c, 2.10, 2.10^c, 3.83, 3.83^c, 7.21, 7.21^c$$

Then the corresponding NPI lower and upper probabilities are $\underline{P}(Y_{11} > X_{11}) = 0.5333$ and $\overline{P}(Y_{11} > X_{11}) = 0.9291$.

Case F: Precedence testing

Precedence testing can be considered as a special case of progressive censoring, as briefly explained at the end of Section 5.5, the corresponding NPI results for this approach are presented in Chapter 2. Suppose that the breakdown of insulating fluids experiment is terminated as soon as the fifth breakdown from group Y is observed, i.e. at time $y_5 = 2.12$. Then the breakdown times of five units from group Y are right-censored at that time, together with three units from group X. Then $\underline{P}\,(Y_{11} > X_{11}) = 0.5289$ and $\overline{P}\,(Y_{11} > X_{11}) = 0.8264$.

Case G: Complete data

Let us end this example by considering NPI comparison of these two groups of breakdown data using the complete data as presented in Table 5.1, so without any (progressive) censoring scheme applied. NPI for such a comparison of complete data from two groups was already presented by Coolen [19], and is also easily derived from the results in this chapter by obvious choices for the censoring schemes, namely $R_i^x = R_j^y = 0$ for all i and j, and hence $r_x = n_x$ and $r_y = n_y$. For this situation, the NPI results are $\underline{P}\,(Y_{11} > X_{11}) = 0.6364$ and $\overline{P}\,(Y_{11} > X_{11}) = 0.8099$. $\triangle$

5.7 Concluding remarks

In this chapter, we introduced NPI for comparison of two groups of lifetime data under several progressive censoring schemes. The NPI method has the attractive feature that it is applicable whether progressive censoring is adopted for one group or for both groups, and also for different censoring schemes. We have restricted attention to two groups, but the methods presented here are quite easily generalized to multiple groups, along the lines of the NPI methods for selection presented in Chapters 3 and 4. Although the ideas for such a generalization are indeed straightforward, deriving analytical expressions of the corresponding NPI lower and upper probabilities becomes somewhat tedious, it is more attractive to develop software routines that perform such calculations for any specific M-functions specified per group, and for any number of groups. In fact one can use the R commands provided in the appendix of this thesis for such purposes since it can be used for comparison of several groups and with different selection events of interest.

Chapter 6

Competing Risks

6.1 Introduction

In reliability, failure data often correspond to competing risks [13, 71], where several failure modes can cause a unit to fail, and where failure occurs due to the first failure event caused by one of the failure modes. Coolen *et al.* [23] introduced Nonparametric Predictive Inference (NPI) to some reliability applications, including lower and upper survival functions for a future unit, illustrated with an application with competing risks data. They illustrated the lower and upper marginal survival functions, which are restricted to a single failure mode. In this chapter, the main question considered is which failure mode will cause the next unit to fail, or for example in survival analysis terminology, which disease causes the next individual considered to die. From now on, terminology from reliability will be used, so events considered are failures of units, but the methods proposed are of course more generally applicable.

In this chapter, NPI lower and upper probabilities are presented for the event that a future unit, say unit $n + 1$, will fail due to a specific failure mode, based on data consisting of times of failures resulting from competing risks for n units. It also illustrates the effect of grouping different failure modes together, and some special cases and features are discussed. This approach uses NPI for right-censored data as presented by Coolen and Yan [27], see also Subsection 1.3.2. The use of lower and upper probabilities to quantify uncertainty has gained increasing attention during the last decade, short and detailed overviews of theories and applications in reliability, together called 'imprecise reliability', are presented by Coolen and Utkin [24, 74].

Some aspects of competing risks are briefly reviewed in Section 6.2. Section 6.3 presents NPI for the competing risks problem. In Section 6.4 we consider the special case of two failure modes which leads to some interesting results. The NPI method is illustrated by some examples in Section 6.5. NPI can also be applied for different censoring mechanisms, which is illustrated in Section 6.6 for competing risks inferences under progressive censoring. Some concluding remarks are given in Section 6.7.

6.2 Competing risks

In competing risks, several failure modes can cause a unit to fail. Throughout this chapter, we assume that each unit cannot fail more than once and it is not used any further once it has failed, and that a failure is caused by a single failure mode which, upon observing a failure, is known with certainty. Tsiatis [73] showed that failure data resulting from such competing risks cannot be used to identify dependence between the failure modes. Effectively, this means that such data can only be used to learn about the marginal distributions, which are the distributions of failure times restricted to single failure modes, for which all failures caused by other failure modes lead to right-censored observations. Throughout this chapter we assume that the failure modes are independent, inclusion of assumed dependence would be an interesting topic for future research, but cannot be learned about from the data as considered here as shown by Tsiatis, and NPI has also not yet been developed to take dependence into account.

In this chapter, we consider competing risks, with k distinct failure modes that can cause a unit to fail. It is further assumed that such failure observations are obtained for n units. As is common in study of failure data under competing risks, for each unit k random quantities are considered, say T_j for $j = 1, \ldots, k$, where T_j represents the unit's time to failure under the condition that failure occurs due to failure mode j. These T_j are assumed to be independent continuous random quantities, which implies the assumption that the failure modes occur independently, and the failure time of the unit is $T = \min(T_1, \ldots, T_k)$. Therefore, each unit considered can have one failure time and it will be known with certainty which failure mode caused a failure. Hence, for the T_j corresponding to the other failure modes, which did not cause the failure of the unit, the unit's observed failure time is a right-censoring time.

In a sample of size n, suppose that there are q ($q \leq n$) distinct failure times $x_1 < x_2 < \ldots < x_q$. Let h_{ij} be the number of units that failed due to failure mode j at time x_i, and $\tilde{n}_{x_i}$ be the number of units at risk at x_i. Then, the marginal distribution function of T_j, also called the Cumulative Incidence Function (CIF)[70], $F_j(t)$, can be estimated as

$$\hat{F}_j(t) = \sum_{\text{all } i,\, x_i \leq t} \frac{h_{ij}}{\tilde{n}_{x_i}} \, \hat{S}(x_{i-1}) \tag{6.1}$$

where $\hat{S}(t)$ is the Kaplan-Meier (KM) estimator given by (1.7). For the case of comparing two competing risks, i.e. two failure modes only, Kochar *et al.* [45] discussed several tests from literature to test whether the difference between the two corresponding CIFs is different from zero.

In the competing risks literature (e.g. [30, 70]), one often consider a bivariate random quantity (T, C), where C is an indicator which equals 0 if the observation is censored and therefore T is the censoring time, or $C = j$ where j represents the failure mode that caused the failure, in which case T is the failure time due to failure mode j [30]. NPI has not yet been developed for such bivariate random quantities, which is an interesting challenge for future research.

6.3 NPI for Competing Risks

For the NPI approach, let the failure time of a future unit be denoted by X_{n+1}, and let the corresponding notation for the failure time including indication of the actual failure mode, say failure mode j, be $X_{j,n+1}$ (so X_{n+1} corresponds to an observation T for unit $n+1$, and $X_{j,n+1}$ to T_j, according to the notation in the previous section). As the different failure modes are assumed to occur independently, the competing risk data per failure mode consist of a number of observed failure times for failures caused by the specific failure mode considered, and right-censoring times for failures caused by other failure modes. Hence rc-$A_{(n)}$ can be applied per failure mode j, for inference on $X_{j,n+1}$. Let the number of failures caused by failure mode j be u_j, $x_{j,1} < x_{j,2} < \ldots < x_{j,u_j}$, and let $v_j (= n - u_j)$ be the number of the right-censored observations, $c_{j,1} < c_{j,2} < \ldots < c_{j,v_j}$, corresponding to failure mode j. Again we assume that no ties occur, however we deal with ties as discussed in Subsection 1.3.5. For notational convenience, let $x_{j,0} = 0$ and $x_{j,u_j+1} = \infty$. Suppose further that there are s_{j,i_j} right-censored observations in the interval (x_{j,i_j}, x_{j,i_j+1}), denoted by $c^{i_j}_{j,1} < c^{i_j}_{j,2} < \ldots < c^{i_j}_{j,s_{j,i_j}}$, so $\sum_{i_j=0}^{u_j} s_{j,i_j} = v_j$. It should be emphasized that it is not assumed that each of the n units in the data set actually has failed. If a unit has not failed then there will be a right-censored observation recorded for this unit for each failure mode, as it is assumed that the unit will then be withdrawn from the study, or the study ends, at some point. The random quantity representing the failure time of the next unit, with all k failure modes considered, is $X_{n+1} = \min\limits_{1 \le j \le k} X_{j,n+1}$. Before introducing the NPI lower and upper probabilities for the event of interest, the NPI M-function values for $X_{j,n+1}$ $(j = 1, \ldots, k)$ following from Definition 1.2, are given below.

Definition 6.1. The NPI M-functions for $X_{j,n+1}$ $(j = 1, \ldots, k)$ are

$$M^j(t^{i_j}_{j,i^*_j}, x_{j,i_j+1}) = M_{X_{j,n_j+1}}(t^{i_j}_{j,i^*_j}, x_{j,i_j+1}) = \frac{1}{n+1}(\tilde{n}_{t^{i_j}_{j,i^*_j}})^{\delta^{i_j}_{i^*_j}-1} \prod_{\{r: c_{j,r} < t^{i_j}_{j,i^*_j}\}} \frac{\tilde{n}_{c_{j,r}}+1}{\tilde{n}_{c_{j,r}}} \quad (6.2)$$

where $i_j = 0, 1, \ldots, u_j$, $i^*_j = 0, 1, \ldots, s_{j,i_j}$ and

$$\delta^{i_j}_{i^*_j} = \begin{cases} 1 & \text{if } i^*_j = 0 \quad \text{i.e. } t^{i_j}_{j,0} = x_{j,i_j} \quad \text{(failure time or time 0)} \\ 0 & \text{if } i^*_j = 1, \ldots, s_{j,i_j} \quad \text{i.e. } t^{i_j}_{j,i^*_j} = c^{i_j}_{j,i^*_j} \quad \text{(censoring time)} \end{cases}$$

Again $\tilde{n}_{c_r}$ and $\tilde{n}_{t^{i_j}_{j,i^*_j}}$ are the numbers of units in the risk set just prior to times c_r and $t^{i_j}_{j,i^*_j}$, respectively. The corresponding NPI probabilities are

$$P^j(x_{j,i_j}, x_{j,i_j+1}) = P(X_{j,n_j+1} \in (x_{j,i_j}, x_{j,i_j+1})) = \frac{1}{n+1} \prod_{\{r: c_{j,r} < x_{j,i_j+1}\}} \frac{\tilde{n}_{c_{j,r}} + 1}{\tilde{n}_{c_{j,r}}} \quad (6.3)$$

where x_{j,i_j} and x_{j,i_j+1} are two consecutive observed failure times caused by failure mode j (and $x_{j,0} = 0$, $x_{j,u_j+1} = \infty$).

In this chapter, the main event of interest is that a single future unit, called the 'next unit', undergoing the same test or process as the n units for which failure data are available, fails due to a specific failure mode, say mode l. NPI lower and upper probabilities for this event are derived, for each $l = 1, \ldots, k$. The following notation is used for these NPI lower and upper probabilities, respectively, for the event of interest.

$$\underline{P}^{(l)} = \underline{P}\left(X_{l,n+1} = \min_{1\le j\le k} X_{j,n+1}\right) = \underline{P}\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\neq l}} X_{j,n+1}\right)$$

$$\overline{P}^{(l)} = \overline{P}\left(X_{l,n+1} = \min_{1\le j\le k} X_{j,n+1}\right) = \overline{P}\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\neq l}} X_{j,n+1}\right)$$

These NPI lower and upper probabilities for the event of interest are presented in the following theorem.

Theorem 6.1. The NPI lower and upper probabilities for the event that the next unit will fail due to failure mode l are

$$\underline{P}^{(l)} = \sum_{C_l(j,\, i_j,\, i_j^*)} \left[\sum_{i_l=0}^{u_l} \mathbf{1}\{x_{l,i_l+1} < \min_{\substack{1\le j\le k\\ j\neq l}} \{t_{j,i_j^*}^{i_j}\}\} P^l(x_{l,i_l}, x_{l,i_l+1})\right] \prod_{\substack{j=1\\ j\neq l}}^{k} M^j(t_{j,i_j^*}^{i_j}, x_{j,i_j+1}) \quad (6.4)$$

$$\overline{P}^{(l)} = \sum_{C_l(j,\, i_j)} \left[\sum_{i_l=0}^{u_l} \sum_{i_l^*=0}^{s_{l,i_l}} \mathbf{1}\{t_{l,i_l^*}^{i_l} < \min_{\substack{1\le j\le k\\ j\neq l}} \{x_{j,i_j+1}\}\} M^l(t_{l,i_l^*}^{i_l}, x_{l,i_l+1})\right] \prod_{\substack{j=1\\ j\neq l}}^{k} P^j(x_{j,i_j}, x_{j,i_j+1}) \quad (6.5)$$

where $\sum_{C_l(j,\, i_j,\, i_j^*)}$ denotes the sums over all i_j^* from 0 to s_{j,i_j} and over all i_j from 0 to u_j for $j = 1, \ldots, k$ but not including $j = l$. Similarly, $\sum_{C_l(j,\, i_j)}$ denotes the sums over all i_j from 0 to u_j for $j = 1, \ldots, k$ but not including $j = l$.

Proof. The NPI lower and upper probabilities (6.4) and (6.5) are derived as the sharpest bounds, based on the relevant rc-$A_{(n)}$ assumptions, for the probability

$$P = P\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\neq l}} X_{j,n+1}\right) = \sum_{C_l(j,\, i_j)} P\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\neq l}} \{X_{j,n+1}\}, \bigcap_{\substack{j=1\\ j\neq l}}^{k} \{X_{j,n+1} \in (x_{j,i_j}, x_{j,i_j+1})\}\right)$$

First consider the lower probability (6.4), which is derived as the sharpest general lower bound for the above probability P,

$$\begin{aligned} P &\geq \sum_{C_l(j,\, i_j,\, i_j^*)} P\left(X_{l,n+1} < \min_{\substack{1\leq j\leq k\\ j\neq l}}\{t^{i_j}_{j,i_j^*}\}\right) \cdot \prod_{\substack{j=1\\ j\neq l}}^{k} M^j(t^{i_j}_{j,i_j^*}, x_{j,i_j+1}) \\ &\geq \sum_{C_l(j,\, i_j,\, i_j^*)} \left[\sum_{i_l=0}^{u_l} \mathbf{1}\{x_{l,i_l+1} < \min_{\substack{1\leq j\leq k\\ j\neq l}}\{t^{i_j}_{j,i_j^*}\}\} P^l(x_{l,i_l}, x_{l,i_l+1})\right] \prod_{\substack{j=1\\ j\neq l}}^{k} M^j(t^{i_j}_{j,i_j^*}, x_{j,i_j+1}) \end{aligned}$$

The first inequality follows by putting all probability masses for each $X_{j,n+1}$ ($j = 1, \ldots, k$ and $j \neq l$) assigned to the intervals $(t^{i_j}_{j,i_j^*}, x_{j,i_j+1})$ ($i_j = 0, \ldots, u_j$ and $i_j^* = 0, 1, \ldots, s_{j,i_j}$) at the left end points of these intervals, and by using Lemma 1.4 for the nested intervals. The second inequality follows by putting all probability masses for $X_{l,n+1}$ in each of the intervals $(t^{i_l}_{l,i_l^*}, x_{l,i_l+1})$ ($i_l = 0, \ldots, u_l$ and $i_l^* = 0, 1, \ldots, s_{l,i_l}$) at the right end points of these intervals. The upper probability is obtained in a similar way, but now all probability masses for the random quantities involved are put at the opposite end points of the respective intervals, when compared to the derivation of the lower probability which leads to

$$\begin{aligned} P &\leq \sum_{C_l(j,\, i_j)} P\left(X_{l,n+1} < \min_{\substack{1\leq j\leq k\\ j\neq l}}\{x_{j,i_j+1}\}\right) \prod_{\substack{j=1\\ j\neq l}}^{k} P^j(x_{j,i_j}, x_{j,i_j+1}) \\ &\leq \sum_{C_l(j,\, i_j)} \left[\sum_{i_l=0}^{u_l} \sum_{i_l^*=0}^{s_{l,i_l}} \mathbf{1}\{t^{i_l}_{l,i_l^*} < \min_{\substack{1\leq j\leq k\\ j\neq l}}\{x_{j,i_j+1}\}\} M^l(t^{i_l}_{l,i_l^*}, x_{l,i_l+1})\right] \prod_{\substack{j=1\\ j\neq l}}^{k} P^j(x_{j,i_j}, x_{j,i_j+1}) \end{aligned}$$

□

6.4 Two competing risks

Before illustrating and discussing this method in examples in Section 6.5, let us consider the special case of the competing risks problem in which there are only two failure modes (so $k = 2$), say modes l and j, and in which each of the n units considered actually fails due to one of these two failure modes. Therefore, any unit which fails due to failure mode l leads to a right-censored observation for failure mode j, and vice versa. In this case, the number of failures due to failure mode l

(j) is equal to the number of right-censored observations for failure mode j (l), so $\upsilon_l = u_j$ and $\upsilon_j = u_l$. Let R_l (R_j) be the set of ranks of all ordered failure times due to failure mode l (j), so $R_l \subset \{1, 2, \ldots, n\}$ and $R_j = \{1, 2, \ldots, n\} \setminus R_l$. The NPI lower and upper probabilities for this scenario are presented in Theorem 6.2.

Theorem 6.2. The NPI lower and upper probabilities (6.4) and (6.5) for the event that the next unit will fail due to failure mode l, in case of only two failure modes, l and j, are

$$\underline{P}^{(l)} = \frac{1}{n+1} \sum_{r_l \in R_l} \frac{\tilde{n}_{x_{l,(r_l)}}}{\tilde{n}_{x_{l,(r_l)}} + 1} = \frac{1}{n+1} \sum_{r_l \in R_l} \frac{n+1-r_l}{n+2-r_l} \tag{6.6}$$

$$\overline{P}^{(l)} = 1 - \frac{1}{n+1} \sum_{r_j \in R_j} \frac{\tilde{n}_{x_{j,(r_j)}}}{\tilde{n}_{x_{j,(r_j)}} + 1} = 1 - \frac{1}{n+1} \sum_{r_j \in R_j} \frac{n+1-r_j}{n+2-r_j} \tag{6.7}$$

Proof. In the case of two competing risks, the NPI lower probability (6.4), i.e. $\underline{P}^{(l)} = \underline{P}(X_{l,n_l+1} < X_{j,n_j+1})$, becomes

$$\underline{P}^{(l)} = \sum_{i_j=0}^{u_j} \sum_{i_j^*=0}^{s_{j,i_j}} \left\{ \sum_{i_l=0}^{u_l} \mathbf{1}\{x_{l,i_l+1} < t_{j,i_j^*}^{i_j}\} P^l(x_{l,i_l}, x_{l,i_l+1}) \right\} M^j(t_{j,i_j^*}^{i_j}, x_{j,i_j+1}) \tag{6.8}$$

Above we have assumed that all n units considered have actually failed due to one of these two failure modes. As any failure of a unit due to failure mode l leads to a right-censored observation for failure mode j for that unit, and vice versa, then $x_{l,(r_l)} = c_{j,(r_l)}$ ($x_{j,(r_j)} = c_{l,(r_j)}$) for $r_l \in R_l$ ($r_j \in R_j$). Let $\sum_{C\left(i_j, i_j^*, c_{j,(r_l)}\right)}$ denote the sums over all i_j from 0 to u_j and over all i_j^* from 0 to s_{j,i_j} such that $t_{j,i_j^*}^{i_j} \geq c_{j,(r_l)}$. Then the NPI lower probability (6.8) can be written as

$$\begin{aligned}
\underline{P}^{(l)} &= \sum_{r_l \in R_l} P^l(x_{l,(r_l-1)}, x_{l,(r_l)}) \sum_{C\left(i_j, i_j^*, c_{j,(r_l)}\right)} M^j(t_{j,i_j^*}^{i_j}, x_{j,i_j+1}) \\
&= \sum_{r_l \in R_l} P^l(x_{l,(r_l-1)}, x_{l,(r_l)})\, \underline{S}_{X_{j,n+1}}(c_{j,(r_l)}) \\
&= \sum_{r_l \in R_l} \left(\frac{1}{n+1} \prod_{\{r: c_{l,r} < x_{l,(r_l)}\}} \frac{\tilde{n}_{c_{l,r}} + 1}{\tilde{n}_{c_{l,r}}} \right) \left(\frac{1}{n+1} \tilde{n}_{c_{j,(r_l)}} \prod_{\{r: c_{j,r} < c_{j,(r_l)}\}} \frac{\tilde{n}_{c_{j,r}} + 1}{\tilde{n}_{c_{j,r}}} \right) \\
&= \sum_{r_l \in R_l} \left(\frac{1}{n+1} \right)^2 \left(\frac{n+1}{n+2-r_l} \right) (n+1-r_l) \\
&= \frac{1}{n+1} \sum_{r_l \in R_l} \frac{n+1-r_l}{n+2-r_l}.
\end{aligned}$$

The second equality and the second term in the third equality follow immediately from the definition of the lower survival function [23] and its simplest closed-form (1.11) derived in Chapter 1, respectively. The fourth equality in this derivation results from the fact that, with all units assumed to fail due to one of the two failure modes considered, and $x_{l,(r_l)} = c_{j,(r_l)}$ and $x_{j,(r_j)} = c_{l,(r_j)}$ for all $r_l \in R_l$ and $r_j \in R_j$, the two product terms combine into a single product over all first $r_l - 1$ observations. This product simplifies to $\frac{n+1}{n+2-r_l}$, and $\tilde{n}_{c_{j,(r_l)}} = n + 1 - r_l$ completes the justification of the fourth equality.

The corresponding NPI upper probability (6.7) can be derived similarly, but it is easier to do so by use of the conjugacy property, $\overline{P}^{(l)} = 1 - \underline{P}(X_{l,n_l+1} > X_{j,n_j+1}) = 1 - \underline{P}^{(j)}$ where

$$\underline{P}^{(j)} = \frac{1}{n+1} \sum_{r_j \in R_j} \frac{n+1-r_j}{n+2-r_j}$$

is of course obtained directly from the above expression for $\underline{P}^{(l)}$. □

Furthermore, the imprecision for the event considered here, in this special case of competing risks with only two failure modes and all n units actually having failed, does not depend on the number of failures caused by each failure mode nor on their ordering. This is implied by the following theorem.

Theorem 6.3. The imprecision for the above scenario is equal to

$$\text{Imprecision} = \overline{P}^{(l)} - \underline{P}^{(l)} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{i}$$

Proof. For this situation with two failure modes and all n units failing due to one of them, the imprecision is

$$\begin{aligned}
\text{Imprecision} &= 1 - \left\{ \underline{P}^{(l)} + \underline{P}^{(j)} \right\} \\
&= 1 - \frac{1}{n+1} \left\{ \sum_{r_l \in R_l} \frac{n+1-r_l}{n+2-r_l} + \sum_{r_j \in R_j} \frac{n+1-r_j}{n+2-r_j} \right\} \\
&= 1 - \frac{1}{n+1} \sum_{i=1}^{n} \frac{n+1-i}{n+2-i} \\
&= \frac{1}{n+1} \left[1 + \sum_{i=1}^{n} \frac{1}{n+2-i} \right] = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{i}.
\end{aligned}$$

The second equality follows directly from Theorem 6.2. □

It should be emphasized that these attractive properties of the NPI results in the case of two competing risks do not generalize to more than two competing risks, due to the fact that the product terms in the NPI lower and upper probabilities (6.4) and (6.5) only disappear for this case with two failure modes and all n units actually having failed.

The formulae (6.6) and (6.7) enable the derivation of some interesting results of the NPI approach in this specific setting, with only two failure modes and all n units actually having failed. Consider the following two specific scenarios in detail:

(A) all failures due to failure mode j come first, followed by all failures from failure mode l, meaning that the u_j failure times of failures due to mode j are all smaller than the u_l failure times of failures due to mode l. In this case, the NPI lower and upper probabilities for the event that the next unit will fail due to failure mode l are

$$\underline{P}^{(l),\,\mathrm{A}} = \frac{1}{n+1}\sum_{i=1}^{u_l}\frac{i}{i+1} \qquad \text{and} \qquad \overline{P}^{(l),\,\mathrm{A}} = 1 - \frac{1}{n+1}\sum_{i=u_l+1}^{n}\frac{i}{i+1}$$

(B) all failures due to failure mode l come first, followed by all failures from failure mode j, in which case the NPI lower and upper probabilities for the event of interest are

$$\underline{P}^{(l),\,\mathrm{B}} = \frac{1}{n+1}\sum_{i=u_j+1}^{n}\frac{i}{i+1} \qquad \text{and} \qquad \overline{P}^{(l),\,\mathrm{B}} = 1 - \frac{1}{n+1}\sum_{i=1}^{u_j}\frac{i}{i+1}$$

These NPI lower and upper probabilities follow straightforwardly from the general expressions (6.6) and (6.7) for these two special cases. Because $\frac{i}{i+1}$ is increasing in i, these results imply that case (A) leads to the minimal NPI lower and upper probabilities when all possible orderings of u_j failures due to mode j and u_l failures due to mode l are considered, while case (B) leads to the maximal NPI lower and upper probabilities in this setting. More generally, these results imply a nice monotonicity result, namely that the NPI lower and upper probabilities (6.6) and (6.7) increase whenever any failure caused by failure mode l would move to an earlier place in the ordering. This is illustrated in Example 6.1 in Section 6.5.

6.5 Examples

In this section three examples of NPI for competing risks are presented to illustrate the method and to discuss some of its properties. Example 6.1 is a small example which serves to illustrate the results presented in Section 6.4. Examples 6.2 and 6.3 involve a substantial competing risks data set from the literature, with different groupings of failure modes and also including some units which did not fail at all during the study, hence leading to right-censored observations for each of the failure modes considered. This will illustrate a further important aspect of NPI in this setting, and will also lead to a conjecture.

Example 6.1. Consider an experiment in which five units are subjected to two failure modes, FM1 and FM2, which are competing risks in the manner discussed in this chapter. Suppose that all five units are observed to fail, and that three units fail due to FM1 and two units due to FM2. So the failure times of the three units failing due to FM1 are right-censored observations for FM2, and the failure times of the two units which fail due to FM2 are right-censored observations for FM1. As all five units actually fail during the experiment, no further right-censored observations occur in this example. Suppose that there had actually been a sixth unit in the experiment, and this was randomly selected before the start of the experiment as the unit for which the failure information would not be revealed to us. The method presented in this chapter provides inferences for the event that this sixth unit fails due to FM1 or due to FM2 (instead 'will fail' could be used, if the inferences are interpreted as involving a future unit undergoing the same experiment, both are convenient ways to think about the setting and inferences).

In this NPI approach, the actual failure times of the five units are not important, only their ordering with regard to failure modes is important. Of course, NPI can also be used for inference on the actual failure time of the sixth unit, for example by considering the event that this unit will not fail before a specified time, in which case the failure times of the five units are explicitly used, not only their ordering with regard to the failure modes, this is briefly illustrated for Examples 6.2 and 6.3 at the end of this section. There are 10 possible orderings for the failure modes FM1 and FM2, with three units failing due to FM1 and two due to FM2. The NPI lower and upper probabilities that the sixth unit fails due to FM1, for the ten possible orderings of the two failure modes, are given in Table 6.1.

	FM Orderings					$\underline{P}(X_6^{FM1} < X_6^{FM2})$	$\overline{P}(X_6^{FM1} < X_6^{FM2})$
O_1	1	1	1	2	2	0.3972	0.8056
O_2	1	1	2	1	2	0.3833	0.7917
O_3	1	1	2	2	1	0.3556	0.7639
O_4	1	2	1	1	2	0.3750	0.7833
O_5	1	2	1	2	1	0.3472	0.7556
O_6	1	2	2	1	1	0.3333	0.7417
O_7	2	1	1	1	2	0.3694	0.7778
O_8	2	1	1	2	1	0.3417	0.7500
O_9	2	1	2	1	1	0.3278	0.7361
O_{10}	2	2	1	1	1	0.3194	0.7278

Table 6.1: NPI lower and upper probabilities for the sixth unit to fail due to FM1

Consider the ordering O_1, in which the three failures due to FM1 happen before the two failures caused by FM2, and which corresponds to case (B) discussed in Section 6.4. The NPI lower and upper probabilities for the event that the sixth unit fails due to FM1 are, for this ordering O_1, greater than the corresponding lower and upper probabilities for all other orderings of the failure modes. On the other hand, ordering O_{10}, in which the two failures due to FM2 happen before the three failures caused by FM1, and which corresponds to case (A) in Section 6.4, leads to the minimum lower and upper probabilities, over all orderings, for the event that the sixth unit will fail due to FM1. Table 6.1 also illustrates the monotonicity result mentioned in Section 6.4, namely that the NPI lower and upper probabilities for the next unit to fail due to FM1 increase if any failure caused by FM1 moves to an earlier place in the ordering.

The NPI lower and upper probabilities for the event that the sixth unit fails due to FM2, for the different orderings of the failure modes for the data, follow from those for FM1 reported in Table 6.1 by the conjugacy property [1, 76], i.e.

$$\underline{P}(X_6^{FM2}<X_6^{FM1})=1-\overline{P}(X_6^{FM1}<X_6^{FM2}),\ \overline{P}(X_6^{FM2}<X_6^{FM1})=1-\underline{P}(X_6^{FM1}<X_6^{FM2}).$$

Consider, for example, the ordering O_5, for which the corresponding NPI lower and upper probabilities that the sixth unit fails due to FM1 are 0.3472 and 0.7556, while for this unit to fail due to FM2 they are 0.2444 and 0.6528. On the basis of these NPI lower and upper probabilities alone, one could conclude that there is a weak indication that failure due to FM1 is more likely than due to FM2, as

$$\underline{P}(X_6^{FM1} < X_6^{FM2}) = 0.3472 > 0.2444 = \underline{P}(X_6^{FM2} < X_6^{FM1})$$

and

$$\overline{P}(X_6^{FM1} < X_6^{FM2}) = 0.7556 > 0.6528 = \overline{P}(X_6^{FM2} < X_6^{FM1})$$

One could speak about a strong indication for the event that failure of the sixth unit will be caused by FM1 if $\underline{P}(X_6^{FM1} < X_6^{FM2}) > \overline{P}(X_6^{FM2} < X_6^{FM1})$, which does not occur for any of the orderings in this example. Finally, the imprecision in this example, for all orderings of the two failure modes, is equal to 0.4084, illustrating the property presented in Theorem 6.3. △

Example 6.2. In this example and in Example 6.3, a well-known data set from the literature [48] is used to illustrate some aspects of the NPI method for dealing with competing risks. The data contain information about 36 units of a new model of a small electrical appliance which were tested, and where the lifetime observation per unit consists of the number of completed cycles of use until the unit failed. These data are presented in Table 6.2, which also includes the specific failure mode (FM) that caused the unit to fail. In the study, there were 18 different ways in which an appliance could fail, so 18 failure modes, but to illustrate the NPI method this number is reduce to two (groups of) failure modes in the current example, while grouping into three failure modes is considered in Example 6.3, after which the differences between these examples are discussed. Three units in the test did not fail before the end of the experiment, so for these units right-censored observations (2565, 6367 and 13403) are recorded for all failure modes considered, indicated by '-' for the failure mode in Table 6.2.

The two most frequently occurring failure modes in these data are FM9, which caused 17 units to fail, and FM6 which caused 7 failures. It is considered how likely

# cycles	FM	# cycles	FM	# cycles	FM
11	1	1990	9	3034	9
35	15	2223	9	3034	9
49	15	2327	6	3059	6
170	6	2400	9	3112	9
329	6	2451	5	3214	9
381	6	2471	9	3478	9
708	6	2551	9	3504	9
958	10	2565	-	4329	9
1062	5	2568	9	6367	-
1167	9	2702	10	6976	9
1594	2	2761	6	7846	9
1925	9	2831	2	13403	-

Table 6.2: Failure data for electrical appliance test

it is that the next unit, say unit 37, would fail due to FM9, assuming it would undergo the same test and its number of completed cycles would be exchangeable with these numbers for the 36 units reported. In this example, all failure modes other than FM9 are grouped together, and these are jointly considered as a single failure mode, which enables illustration of the NPI approach with 2 failure modes, FM9 and, say, ‘other failure mode’ (OFM). There are still three units that do not fail, and hence for which there are only right-censored observations (RC). For clarity, the data corresponding to this definition of failure modes are presented in Table 6.3.

FM9	1167	1925	1990	2223	2400	2471	2551	2568	3034	3034
	3112	3214	3478	3504	4329	6976	7846			
OFM	11	35	49	170	329	381	708	958	1062	1594
	2327	2451	2702	2761	2831	3059				
RC	2565	6367	13403							

Table 6.3: Failure data for electrical appliance test: FM9, OFM and RC

When the theory for NPI for competing risks data was presented in Section 6.3, it was assumed that there were no ties to avoid notational difficulties. In this example, however, there are tied observations, as two units have failed after 3034 completed cycles, both failed due to FM9. To deal with this, it is assumed that there is a small difference between these values, such that their ordering does not change with regard to observations of units in other groups. It is actually assumed that one of these two units failed after 3035 completed cycles. Implicit in the NPI method for competing risks data is that a failure time observation caused by one failure mode is simultaneously a right-censored observation for all other failure modes. This situation is dealt with in the NPI approach, as is common in many statistical approaches, by assuming that the right-censoring time is just beyond the failure time. The three right-censored observations, for units that were not observed to fail during the experiment, also lead to tied observations for the two failure modes (FM9 and OFM) considered, as for both the right-censoring times coincide. This is also dealt with by assuming that for one of the failure modes this event occurred fractionally later than for the other failure mode, and then the lower and upper probabilities for the event of interest are calculated by considering the minimum and maximum of the lower and upper probabilities, respectively, corresponding to the different possible orderings of these 'un-tied' right-censoring times.

The NPI lower and upper probabilities for the event that unit 37 will fail due to FM9 are

$$\underline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.4358 \ , \quad \overline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.5804$$

while the corresponding NPI lower and upper probabilities for unit 37 to fail due to OFM are

$$\underline{P}(X_{37}^{OFM} < X_{37}^{FM9}) = 0.4196 \ , \quad \overline{P}(X_{37}^{OFM} < X_{37}^{FM9}) = 0.5642$$

These lower and upper probabilities satisfy the conjugacy property [1, 76], which is due to the fact that, implicit in our method, it is assumed that the experiment on unit 37 would actually continue until it fails, and this is assumed to happen with certainty. On the basis of these NPI lower and upper probabilities, the data could be considered to contain a weak indication that the event that unit 37 will fail due to FM9 is a bit more likely than for it to fail due to another failure mode, with all the other failure modes grouped together as done in this example. △

Example 6.3. This example uses the same data as Example 6.2, but the failure modes are grouped differently. Both FM9 and FM6 are considered separately, with 17 and 7 units that failed due to them, respectively, and all other failure modes are grouped into one 'other failure mode' (OFM). For clarity, the data used here are given in Table 6.4.

FM9	1167	1925	1990	2223	2400	2471	2551	2568	3034	3034
	3112	3214	3478	3504	4329	6976	7846			
FM6	170	329	381	708	2327	2761	3059			
OFM	11	35	49	958	1062	1594	2451	2702	2831	
RC	2565	6367	13403							

Table 6.4: Failure data for electrical appliance test: FM9, FM6, OFM and RC

The NPI lower and upper probabilities for the event that unit 37 will fail due to FM9, due to FM6 or due to OFM, are

$$\underline{P}\left(X_{37}^{FM9}<\min\left\{X_{37}^{FM6},X_{37}^{OFM}\right\}\right)=0.3915\ ,\ \overline{P}\left(X_{37}^{FM9}<\min\left\{X_{37}^{FM6},X_{37}^{OFM}\right\}\right)=0.5804$$

$$\underline{P}\left(X_{37}^{FM6}<\min\left\{X_{37}^{FM9},X_{37}^{OFM}\right\}\right)=0.1749\ ,\ \overline{P}\left(X_{37}^{FM6}<\min\left\{X_{37}^{FM9},X_{37}^{OFM}\right\}\right)=0.3279$$

$$\underline{P}\left(X_{37}^{OFM}<\min\left\{X_{37}^{FM6},X_{37}^{FM9}\right\}\right)=0.2265\ ,\ \overline{P}\left(X_{37}^{OFM}<\min\left\{X_{37}^{FM6},X_{37}^{FM9}\right\}\right)=0.3808$$

Since

$$\underline{P}\left(X_{37}^{FM9}<\min\left\{X_{37}^{FM6},X_{37}^{OFM}\right\}\right)>\overline{P}\left(X_{37}^{FM6}<\min\left\{X_{37}^{FM9},X_{37}^{OFM}\right\}\right)$$

one could interpret the data as providing a strong indication that unit 37 is more likely to fail due to FM9 than due to FM6, in this setting with all other failure modes grouped into OFM. For example, if a person were to follow a subjective interpretation of lower and upper probabilities in terms of prices for desirable gambles, in line with Walley [76], then these lower and upper probabilities would imply that, for any price between 0.3279 and 0.3915, this person would be willing both to buy the gamble which pays 1 if unit 37 fails due to FM9 and to sell the gamble which pays 1 if unit 37 fails due to FM6. A quick look at the data may perhaps lead to some surprise that FM6 is not the more likely one to lead to failure, as it has caused relatively many early failures. However, one must not forget that it only caused failure of 7 out of the 36 units tested, the comparisons would be very different if the data were not competing risks data on the same units but completely independent failure times per group, see Chapter 4. Similarly, a strong indication that unit 37 is more likely to fail due to FM9 than due to OFM can be claimed because

$$\underline{P}\left(X_{37}^{FM9} < \min\left\{X_{37}^{FM6}, X_{37}^{OFM}\right\}\right) > \overline{P}\left(X_{37}^{OFM} < \min\left\{X_{37}^{FM6}, X_{37}^{FM9}\right\}\right)$$

It is interesting to compare the results presented in Examples 6.2 and 6.3, as they illustrate some features that are very different in statistics using lower and upper probabilities when compared to methods using precise probabilities. The NPI lower and upper probabilities for the event that unit 37 will fail due to FM9 are $[0.4358, 0.5804]$ in Example 6.2, where all other failure modes are grouped together, and $[0.3915, 0.5804]$ in Example 6.3, where FM6 is taken separately with all further failure modes grouped together. Hence, in the latter case, there is more imprecision in these upper and lower probabilities, while data are represented in more detail. This increase in imprecision, actually the fact that these upper and lower probabilities are nested with more imprecision if data are represented in more detail, is in line with a fundamental principle of NPI proposed and discussed by Coolen and Augustin [22] in the context of multinomial data. This leads to the conjecture that, for such competing risks data, if more failure modes are treated separately instead of grouped together, then lower and upper probabilities for an event that the next unit's failure is caused by a specific failure mode are nested, with imprecision increasing with the number of failure modes used. This conjecture has not been proven generally, due to the complexity of the expressions involved, but we strongly believe it to hold and all examples explored are in line with it.

One could also have considered the question whether or not unit 37 will fail due to FM9 from a basic Bernoulli variables perspective, taking only into account that of 33 observed failures so far (neglecting the 3 units with right-censored lifetimes), 17 failed due to FM9. NPI for Bernoulli random quantities [20] leads to lower probability $17/34 = 0.5$ and upper probability $18/34 = 0.5294$ (note also that these bound the empirical probability $17/33 = 0.5152$), which lie inside the intervals created by the lower and upper probabilities for this event in Examples 6.2 and 6.3. This is also in line with the observation that a more detailed data representation leads

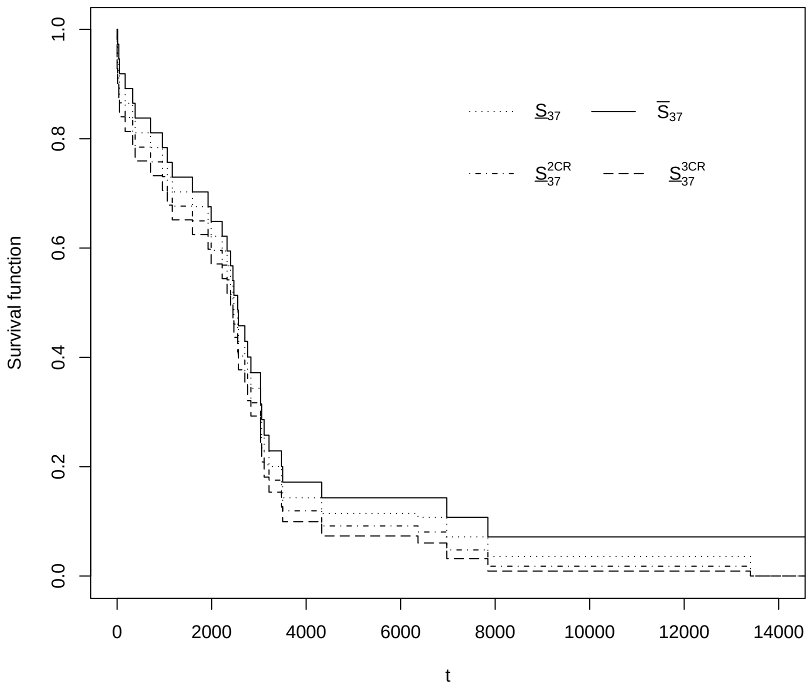


Figure 6.1: NPI lower and upper survival functions for unit 37

to increased imprecision in the NPI approach. This Bernoulli data representation would, of course, not enable any inferences with regard to actual failure time.

The two NPI upper probabilities for the event that unit 37 will fail due to FM9, for the cases with all other failure modes grouped together (Example 6.2) and with FM6 separated (Example 6.3), are both equal to 0.5804. This is a consequence of the fact that this upper probability is realized with the extreme assignments of probability masses in the intervals created by the data in accordance to the lower survival function for FM9 and the upper survival function for the other failure modes. With all failure modes assumed to be independent, the upper survival function for the other failure modes combined is actually the same, whether or not FM6 is considered separately, this was discussed by Coolen *et al.* [23], who presented individual NPI lower and upper survival functions and also considered the data used in Examples 6.2 and 6.3, but they did not develop the NPI method for multiple comparisons that underlies the NPI method for competing risks presented in this chapter.

To end discussion of Examples 6.2 and 6.3, it is useful to illustrate the NPI lower and upper survival functions that have been mentioned in these examples but which have not yet been presented. We can obtain these lower and upper survival functions using the simplest closed-form (1.11) and (1.12) derived in Chapter 1. Figure 6.1 shows the NPI lower and upper survival functions for unit 37 for three situations, for which the upper survival functions are identical hence only the lower survival functions differ. The lower survival function $\underline{S}_{X_{37}}$ results from total neglection of the information on different failure modes, hence just by applying rc-$A_{(36)}$ [27] with 33 observed failure times and 3 right-censoring times. The lower survival function $\underline{S}^{2CR}_{X_{37}}$ corresponds to the situation with two (groups of) failure modes in Example 6.2, and is derived by multiplying the lower survival functions which are conditional on the given failure modes. Similarly, the lower survival function $\underline{S}^{3CR}_{X_{37}}$ corresponds to the situation with three (groups of) failure modes in Example 6.3. These lower and upper survival functions show a similar nested structure, related to the level of detail of the data representation, as was discussed above for the event that FM9 causes the failure of unit 37.

Figure 6.2 shows the NPI lower and upper survival functions for unit 37 conditioned on the specific failure mode, for FM9 and for FM6, corresponding to Example 6.3. For example, $\underline{S}^{FM9}_{X_{37}}$ and $\overline{S}^{FM9}_{X_{37}}$ are based on rc-$A_{(36)}$ applied with the data set with the 17 failure times related to failures caused by FM9 treated as actual failure time observations, and the other 19 observations in the data set as right-censored data, and similar for FM6. This figure nicely illustrates the effect of the relatively many early failures due to FM6, and the fact that there are far fewer failures due to FM6 than due to FM9 is reflected in far greater imprecision (the difference between corresponding upper and lower survival functions) at larger times. Note that, in the NPI approach based on rc-$A_{(n)}$, the lower survival function is always equal to zero beyond the largest observation, no matter if this is an observed failure time or a right-censored observation, while the upper survival function remains positive, this is discussed in more detail by Coolen and Yan [27]. $\triangle$

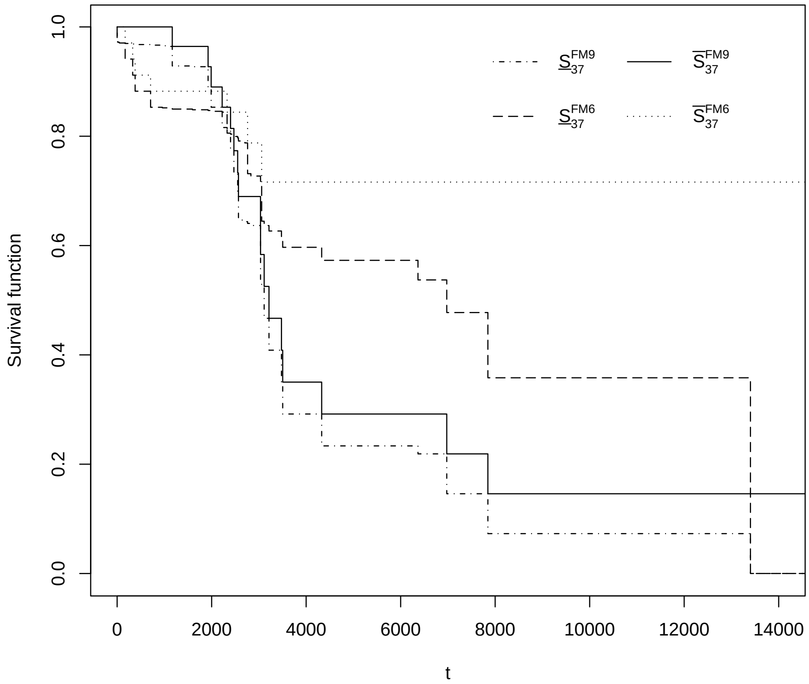


Figure 6.2: NPI lower and upper conditional survival functions for unit 37

6.6 Progressive Type-II censoring

In Chapter 5 we introduced NPI for comparing two groups of lifetime data under progressive censoring schemes, with careful discussion of different schemes and comparison to other frequentist approaches for such data. There we did not consider progressive censoring combined with competing risks data, which we briefly discuss in this section, and illustrate in an example which is based on Examples 6.2 and 6.3 in Section 6.5. The progressive censoring scheme considered here is known in the literature as 'progressive Type-II censoring' [3, 47], see also Chapter 5, for other progressive censoring schemes one can follow the same approach, a flexibility which is one of the advantages of NPI when compared to the more established frequentist statistical methods.

In progressive Type-II censoring, at each failure time regardless of the failure cause, some randomly chosen non-failing units may be removed from the experiment. Adding such possible censored data to the competing risks scenario presented in this chapter, the competing risks data per failure mode can consist of a number of observed failures caused by the specific failure mode considered, right-censored observations for failures caused by other failure modes, right-censored observations resulting from removing some non-failing units at failure times of other units (due to the progressive censoring scheme), and general right-censored observations due to unknown failure modes or other reasons, as was also allowed earlier in this chapter. The key thing here is that right-censored data of any kind are dealt with in the same manner, per failure mode, in NPI for competing risks, so effectively there is no difference in the way NPI for competing risks data deals with right-censored data of the last two types discussed, which are right-censored observations for all failure modes. In the case of tied observations, we deal with them in the same manner as discussed in Subsection 1.3.5.

Example 6.4. Suppose that, in the tests of the electrical appliances leading to the data in Examples 6.2 and 6.3, it had been decided that, in order to learn more about the physics underlying common failure modes, 3 non-failing units were to be removed from the experiment as soon as the third failure due to the same failure mode occurs, enabling detailed comparison of the condition of the failed units with units that did not yet fail. Assume that the non-failing units withdrawn from the experiment are selected randomly from those still in the study. At time 381, when the third failure caused by FM6 occurs, three non-failing units would be withdrawn, hence leading to three right-censored observations at that time. Assume that the unit which in the original data (Table 6.2) failed at time 1990 due to FM9 was one of the three withdrawn at time 381, and that the unit failing at time 2223 is the third one failing due to FM9. Then a further three units are withdrawn at that moment to enable detailed study of the processes underlying FM9 through comparison with non-failed units. Suppose that this process leads to the data presented in Table 6.5, where as before right-censoring times are indicated by '-' for failure mode.

If, in analogy to Example 6.2, all failure modes other than FM9 are grouped together and jointly considered as one failure mode OFM, then the NPI lower and

# cycles	FM	# cycles	FM	# cycles	FM
11	1	1167	9	2568	9
35	15	1594	2	2761	6
49	15	1925	9	2831	2
170	6	2223	9	3034	9
329	6	2223	-	3034	9
381	6	2223	-	3112	9
381	-	2223	-	3214	9
381	-	2327	6	3504	9
381	-	2400	9	4329	9
708	6	2471	9	6976	9
958	10	2551	9	7846	9
1062	5	2565	-	13403	-

Table 6.5: Failure data for electrical appliance test under progressive censoring

upper probabilities for the event that unit 37 will fail due to FM9 are

$$\underline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.4658\ , \quad \overline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.6258$$

Note that these NPI lower and upper probabilities are not nested when compared to those in Example 6.2, which is due to the fact that now the information per failure mode is really different. If, as in Example 6.3, failure modes FM9 and FM6 are considered separately, with all the other failure modes grouped as OFM, then the resulting NPI lower and upper probabilities for the events that unit 37 will fail due to FM9, due to FM6 or due to OFM, are

$$\underline{P}\big(X_{37}^{FM9}<\min\{X_{37}^{FM6}, X_{37}^{OFM}\}\big)=0.4109\ , \ \overline{P}\big(X_{37}^{FM9}<\min\{X_{37}^{FM6}, X_{37}^{OFM}\}\big)=0.6258$$

$$\underline{P}\big(X_{37}^{FM6}<\min\{X_{37}^{FM9}, X_{37}^{OFM}\}\big)=0.1668\ , \ \overline{P}\big(X_{37}^{FM6}<\min\{X_{37}^{FM9}, X_{37}^{OFM}\}\big)=0.3349$$

$$\underline{P}\big(X_{37}^{OFM}<\min\{X_{37}^{FM6}, X_{37}^{FM9}\}\big)=0.1906\ , \ \overline{P}\big(X_{37}^{OFM}<\min\{X_{37}^{FM6}, X_{37}^{FM9}\}\big)=0.3593$$

These NPI lower and upper probabilities are again not nested in a specific general way with the NPI lower and upper probabilities in Example 6.3. However, they show the same nested behaviour as discussed in Example 6.3 with regard to the NPI lower and upper probabilities for the event $X_{37}^{FM9} < X_{37}^{OFM}$ in this setting with OFM including FM6. △

6.7 Concluding remarks

In this chapter, NPI for competing risks has been presented, with focus on the event that the next unit will fail due to a specific failure mode. Some specific properties and special cases are discussed and illustrated via examples in Section 6.5. As such, NPI is widely applicable and it is usually straightforward to implement different censoring scenarios, as briefly discussed and illustrated in Section 6.6 for a specific progressive censoring scheme. Developing NPI to take into account the dependence between failure modes could be an interesting and challenging topic for future research.

Chapter 7

Comparison with terminated tails

7.1 Introduction

There are many situations in statistical practice where the information available consists of precise measurements of real-valued data only within a specific range, with in addition the numbers of observations to the left and to the right of this range available. This can be due to many reasons related to experimental design or some problems with regard to data collection. For example, a lifetime experiment may be ended before all units have failed in order to save costs and time, see Chapters 2 and 3, or very small measurements may not be available in risk analyses due to limits of detection of the measurement method. It may also be the case that complete data are available, but that the statistician choses to disregard the precise values of very small or very large observations, often called 'outliers', due to doubt about the collection or recording of the data. A further possibility is that only a part of the data range is considered relevant for the inference, as may occur for medical diagnostics tests.

Coolen and Yan [27] presented the assumption rc-$A_{(n)}$ which is suitable for right-censored data, see also Subsection 1.3.2. As part of the justification of rc-$A_{(n)}$, Coolen and Yan [27] introduced and justified what they called the assumption $\tilde{A}_{(n)}$, which follows from $A_{(n)}$ and was suitable for data with the upper tail terminated. In Section 7.3, we will use this assumption, together with similar arguments for lower tail termination, to derive the assumption related to $A_{(n)}$ that is suitable and appropriate for the kind of data considered in this chapter. The assumption rc-$A_{(n)}$ is suitable for data sets with multiple right-censored observations at different time points. We do not combine terminated tails with such further right-censorings within the non-terminated part of the data, doing so would not cause difficulties but it adds little to the presentation of the main ideas and results in this chapter. An obvious solution is to develop a software package (e.g. in R) to enable calculation of the NPI lower and upper probabilities for such a scenario. The R commands for comparing two groups, provided in the appendix of this thesis, can be used as a starting point.

For the problem considered in this chapter, namely Nonparametric Predictive Inference (NPI) for comparison of two groups of real-valued data with terminated tails, we consider the two groups to be completely independent and apply the suitable $A_{(n)}$ assumption per group, as the basis of our inference. We present NPI lower and upper probabilities for the event that the value of a future observation from one group is less than the value of a future observation from the other group.

In Section 7.2, two classical tests are briefly reviewed. The specific details of NPI for real-valued data with terminated tails are presented in Section 7.3, followed by the general results for pairwise comparison with such data in Section 7.4. Some special cases are discussed in Section 7.5, and an example is provided, in Section 7.6, to illustrate the theory presented in this chapter. Some concluding remarks are made in Section 7.7.

7.2 Classical methods

There are several robust techniques in the literature for comparing two independent groups. In this section we briefly review two methods for such comparison, following [37, 80] in definitions and notation. The so-called Yuen-Welch test [82] is based on comparing the corresponding sample trimmed means of the two groups, it tests the null hypothesis that the two groups have equal trimmed means. Suppose n_x and n_y are the numbers of observations from group X and Y, respectively. Let γ be the amount of trimming from both tails, then the remaining observations from both groups are $h_x = n_x - 2\lfloor \gamma n_x \rfloor$ and $h_y = n_y - 2\lfloor \gamma n_y \rfloor$, where $\lfloor a \rfloor$ is the largest integer not greater than a. The trimmed means, calculated from these remaining observations, are denoted by $\bar{x}_t$ and $\bar{y}_t$. The Yuen-Welch test statistic is

$$T_\gamma = \frac{\bar{x}_t - \bar{y}_t}{\sqrt{d_x + d_y}}$$

where $d_x = (n_x - 1)s^2_{w_x}/h_x(h_x - 1)$ and $d_y = (n_y - 1)s^2_{w_y}/h_y(h_y - 1)$. The quantities $s^2_{w_x}$ and $s^2_{w_y}$ are the Winsorized sample variances, calculated from the sample where the trimmed observations from the left (right) tail are given the same value as the smallest (largest) observation from the non-trimmed observations. Under the null hypothesis, this test statistic T_γ has approximately a t-distribution with the following degrees of freedom,

$$\hat{\upsilon}_{T_\gamma} = (d_x + d_y)^2 \left(\frac{d_x^2}{h_x - 1} + \frac{d_y^2}{h_y - 1} \right)^{-1}$$

One may want to compare the two groups by testing the null hypothesis that $P(X < Y) = 0.5$. The well known Wilcoxon-Mann-Whitney test [51] can be used for this setting. In the case of unequal variances and with ties occurring, one may use the modified version of the Wilcoxon-Mann-Whitney test proposed by Brunner and Munzel [14], in which for tied observations the midranks (the average of their ranks) are used. Let M_x^i $(i = 1, \ldots, n_x)$ and M_y^j $(j = 1, \ldots, n_y)$ be the midranks of X and Y within the pooled sample, let $\bar{M}_x$ and $\bar{M}_y$ be the corresponding means of these midranks, and let V_x^i and V_y^j be the midranks of X and Y within each sample. Then the Brunner-Munzel test statistic is

$$B = (\bar{M}_y - \bar{M}_x)/(n_x + n_y)\sqrt{s^2_{b_x}/n_x n_y^2 + s^2_{b_y}/n_x^2 n_y}$$

where

$$s^2_{b_x} = \frac{1}{n_x - 1} \sum_{i=1}^{n_x} \left(M_x^i - V_x^i - \bar{M}_x + \frac{n_x + 1}{2} \right)^2, \; s^2_{b_y} = \frac{1}{n_y - 1} \sum_{j=1}^{n_y} \left(M_y^j - V_y^j - \bar{M}_y + \frac{n_y + 1}{2} \right)^2$$

The distribution of B is approximately a t-distribution with the following degrees of freedom,

$$\hat{\upsilon}_B = \left(\frac{s^2_{b_x}}{n_y} + \frac{s^2_{b_y}}{n_x} \right)^2 \left(\frac{s^4_{b_x}}{n_y^2(n_x - 1)} + \frac{s^4_{b_y}}{n_x^2(n_y - 1)} \right)^{-1}$$

For more details and for R commands functions, which can be used in calculation, we refer to [80].

7.3 NPI with terminated tails

To present NPI for data with terminated tails we need to introduce some notation. Suppose we have cut points $L < U$ for group X. These cut points divide the n observations into three parts, where observations which are less than L are not observed but their number is known, say $l = \#\{x_i | x_i < L, i = 1, \ldots, n\}$, and similarly for observations greater than U, with $u = \#\{x_i | x_i > U, i = 1, \ldots, n\}$. The observations between L and U (inclusive) are fully available and their number is $r = \#\{x_i | L \leq x_i \leq U, i = 1, \ldots, n\}$, so $l + r + u = n$. Throughout this chapter, it is assumed that the values of L and U do not hold any further information about the observations in the tails. We should emphasize that when we terminate the data via the two cut points, we do not remove the observations totally from the comparison but we only delete any information about the actual position or location of the terminated observations. So all information that we use about the observations on the left (right) of L (U) is that their observed values are less (greater) than L (U). We denote the r observations between these cut points by

$$-\infty < L \leq x_{(1)} < x_{(2)} < \ldots < x_{(r)} \leq U < \infty$$

where $x_{(i)}$ is actually the $(l + i)$th ordered observation of the whole data set. The data structure is illustrated in Figure 7.1.

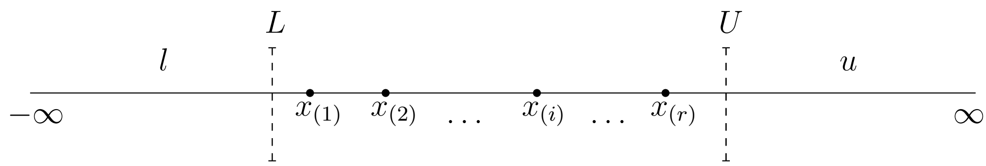


Figure 7.1: Data structure with terminated tails

For ease of notation, let $x_{(0)} = -\infty$ and $x_{(r+1)} = \infty$, of course these can be set at any other known bounds for the range of possible values for the observations, for example $x_{(0)}$ is set to zero when the inferences involve lifetimes. We should emphasize here that $x_{(r+1)}$ is not the first observation to the right of U. Again, we present the results assuming no ties in the data, but the method deals easily with ties as discussed in Subsection 1.3.5. To avoid a further complication, we assume throughout this chapter that there are observations in the interval $[L, U]$, so $r > 0$. The following theorem gives an assumption directly related to $A_{(n)}$, and indeed implied by $A_{(n)}$ taking the specific nature of the reported data into account.

Theorem 7.1 (The assumption $A^{tt}_{(n)}$)**.** The assumption $A^{tt}_{(n)}$ is that the probability distribution for a real-valued random quantity X_{n+1}, on the basis of the data terminated at two cut points L and U as described above, is partially specified by the following M-function values:

$$M_{X_{n+1}}(x_{(i)}, x_{(i+1)}) = \frac{1}{n+1} \quad , \quad i = 0, 1, \ldots, r$$

$$M_{X_{n+1}}(-\infty, L) = \frac{l}{n+1} \qquad \text{and} \qquad M_{X_{n+1}}(U, \infty) = \frac{u}{n+1}.$$

Proof. The justification of $A^{tt}_{(n)}$ is similar to the justification of $\tilde{A}_{(n)}$ given by Coolen and Yan [26], but that assumption is only for termination of the upper tail of data, which they then build upon to enable dealing with general right-censored data. Suppose that we actually had all n observations, and were interested in inference on X_{n+1}. Then $A_{(n)}$ would assign probability mass $1/(n+1)$ for X_{n+1} to each interval of the partition of the real-line created by the data. With l observations left of L, yet without any further assumptions on where these observations are, it is clear that a probability mass of $l/(n+1)$ has to be constrained to $(-\infty, L)$. In addition, there is a probability mass $1/(n+1)$ between the largest observation to the left of L and $x_{(1)}$, the smallest observation in the interval $[L, U]$. Again, without any further assumptions, this probability mass can only be assigned to $(-\infty, x_{(1)})$, or, of course, $(x_{(0)}, x_{(1)})$ if another lower limit, $x_{(0)}$, of the range of possible values for X_{n+1} is known. The arguments for the assignment of probability masses at the upper tail are identical. For the intervals $(x_{(i)}, x_{(i+1)})$, $i = 1, \ldots, r-1$, which are within $[L, U]$, this assignment is fully in line with the regular assumption $A_{(n)}$. □

The cut points L and U can arise from practical aspects of the experiments or data collection, or they can be chosen by the statistical analyst, for example to guard against influence of outliers which may be due to measurement or recording errors. It is crucial that they do not hold information on the observations in the tails, apart from this there are no restrictions on how they are chosen. For example, they could be chosen to terminate the data by a certain percentage from either a single tail or from both tails. One could argue that any combination of cut points together with explicitly observed values between the cut points may contain some information about data in the tails, for example related to extreme value theory in statistics [32], but this would always result from additional assumptions, as is always the case with such extrapolation. In NPI, we typically try to minimize additional assumptions, hence we make no assumptions about location of observations in the terminated tails at all. It should be emphasized that, although by terminating the tails of the data we are focusing on only a part of the real-line, this is only for as far as the data are concerned. The inferences for the future observation X_{n+1} are explicitly over the whole real-line (or known part of that, e.g. the non-negative values for lifetimes).

7.4 Comparing two groups with terminated tails

Suppose that $X_1, \ldots, X_{n_x}, X_{n_x+1}$ are exchangeable real-valued random quantities from group X and $Y_1, \ldots, Y_{n_y}, Y_{n_y+1}$ are exchangeable real-valued random quantities from group Y, with complete independence of the two groups. We use similar notation as in the previous section, adding an index to indicate the specific group. Let $L_x < U_x$ be the cut points for group X and $L_y < U_y$ for group Y. For each group, these cut points divide the data per group into three parts. For group X (Y), there are l_x (l_y) observations which are only known to be less than L_x (L_y), u_x (u_y) which are only known to be greater than U_x (U_y), while the r_x (r_y) ordered observations between the cut points are fully known and denoted by

$$-\infty < L_x \leq x_{(1)} < x_{(2)} < \ldots < x_{(r_x)} \leq U_x < \infty$$

$$-\infty < L_y \leq y_{(1)} < y_{(2)} < \ldots < y_{(r_y)} \leq U_y < \infty$$

Let $x_{(0)} = y_{(0)} = -\infty$ and $x_{(r_x+1)} = y_{(r_y+1)} = \infty$.

The NPI method for comparison of groups X and Y is explicitly in terms of future observations X_{n_x+1} and Y_{n_y+1}, for which we assume $A^{tt}_{(n_x)}$ and $A^{tt}_{(n_y)}$, respectively, so their M-function values follow from Theorem 7.1. The NPI comparison of these two groups is based on the sharpest bounds for the probability for the event $X_{n_x+1} < Y_{n_y+1}$ that are in agreement with these M-function values, without making any further assumptions. These bounds are lower and upper probabilities [1, 76, 79], denoted by $\underline{P} = \underline{P}(X_{n_x+1} < Y_{n_y+1})$ and $\overline{P} = \overline{P}(X_{n_x+1} < Y_{n_y+1})$, respectively. These NPI lower and upper probabilities are given in Theorem 7.2.

Theorem 7.2. Based on data with terminated tails as discussed above, the NPI lower and upper probabilities for the event $X_{n_x+1} < Y_{n_y+1}$ are

$$\underline{P} = A\Bigg[\sum_{j=1}^{r_y}\Big\{l_x\,\mathbf{1}\{L_x < y_{(j)}\} + \sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\}\Big\} + u_y\Big\{l_x\,\mathbf{1}\{L_x < U_y\} + \sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < U_y\}\Big\}\Bigg] \tag{7.1}$$

$$\overline{P} = A\Bigg[\sum_{j=1}^{r_y}\Big\{u_x\,\mathbf{1}\{U_x < y_{(j)}\} + \sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\}\Big\} + l_y\Big\{u_x\,\mathbf{1}\{U_x < L_y\} + \sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < L_y\}\Big\} + (l_x+1)(l_y+r_y) + (u_y+1)(n_x+1)\Bigg] \tag{7.2}$$

where $A = ((n_x+1)(n_y+1))^{-1}$.

Proof. The M-function values for X_{n_x+1} and Y_{n_y+1}, based on the assumptions $A^{tt}_{(n_x)}$ and $A^{tt}_{(n_y)}$, respectively, together with the n_x (n_y) observations for group X (Y), are, according to Theorem 7.1,

$$M_{X_{n_x+1}}(x_{(i)}, x_{(i+1)}) = \frac{1}{n_x+1}\quad,\quad i = 0, 1, \dots, r_x$$

$$M_{X_{n_x+1}}(-\infty, L_x) = \frac{l_x}{n_x+1} \qquad \text{and} \qquad M_{X_{n_x+1}}(U_x, \infty) = \frac{u_x}{n_x+1}$$

$$M_{Y_{n_y+1}}(y_{(j)}, y_{(j+1)}) = \frac{1}{n_y+1}\quad,\quad j = 0, 1, \dots, r_y$$

$$M_{Y_{n_y+1}}(-\infty, L_y) = \frac{l_y}{n_y+1} \qquad \text{and} \qquad M_{Y_{n_y+1}}(U_y, \infty) = \frac{u_y}{n_y+1}$$

The probability for the event $X_{n_x+1} < Y_{n_y+1}$, i.e. $P = P(X_{n_x+1} < Y_{n_y+1})$, can be written as

$$P = P\left(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (-\infty, L_y)\right) + P\left(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (U_y, \infty)\right) + \sum_{j=0}^{r_y} P\left(X_{n_x+1} < Y_{n_y+1}, Y_{n_y+1} \in (y_{(j)}, y_{(j+1)})\right)$$

The NPI lower probability for the event $X_{n_x+1} < Y_{n_y+1}$ is obtained as follows:

$$P \geq P(X_{n_x+1} < -\infty)\frac{l_y}{n_y+1} + \sum_{j=0}^{r_y} P(X_{n_x+1} < y_{(j)})\frac{1}{n_y+1} + P(X_{n_x+1} < U_y)\frac{u_y}{n_y+1}$$

$$\geq A\left[\sum_{j=0}^{r_y}\left\{l_x\,\mathbf{1}\{L_x < y_{(j)}\} + \sum_{i=0}^{r_x}\mathbf{1}\{x_{(i+1)} < y_{(j)}\} + u_x\,\mathbf{1}\{\infty < y_{(j)}\}\right\} + \right.$$

$$\left. u_y\left\{l_x\,\mathbf{1}\{L_x < U_y\} + \sum_{i=0}^{r_x}\mathbf{1}\{x_{(i+1)} < U_y\} + u_x\,\mathbf{1}\{\infty < U_y\}\right\}\right]$$

$$= A\left[\sum_{j=1}^{r_y}\left\{l_x\,\mathbf{1}\{L_x<y_{(j)}\}+\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)}<y_{(j)}\}\right\}+u_y\left\{l_x\,\mathbf{1}\{L_x<U_y\}+\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)}<U_y\}\right\}\right]$$

The first inequality follows by putting all probability masses for Y_{n_y+1} corresponding to the intervals $(-\infty, L_y)$, $(y_{(j)}, y_{(j+1)})$ $(j = 0, \ldots, r_y)$ and (U_y, ∞) to the left end points of these intervals, and by using Lemma 1.4 for the nested intervals. The second inequality follows by putting all probability masses for X_{n_x+1} corresponding to the intervals $(-\infty, L_x)$, $(x_{(i)}, x_{(i+1)})$ $(i = 0, \ldots, r_x)$ and (U_x, ∞) to the right end points of these intervals. The upper probability is obtained in a similar way, but now all M-function masses for the random quantities involved are put at the opposite end points of the respective intervals, which leads to

$$P \leq P(X_{n_x+1} < L_y)\frac{l_y}{n_y+1} + \sum_{j=0}^{r_y} P(X_{n_x+1} < y_{(j+1)})\frac{1}{n_y+1} + P(X_{n_x+1} < \infty)\frac{u_y}{n_y+1}$$

$$= P(X_{n_x+1} < L_y)\frac{l_y}{n_y+1} + \sum_{j=1}^{r_y} P(X_{n_x+1} < y_{(j)})\frac{1}{n_y+1} + \frac{u_y+1}{n_y+1}$$

$$\leq A\left[l_y\left\{l_x\,\mathbf{1}\{-\infty < L_y\} + \sum_{i=0}^{r_x}\mathbf{1}\{x_{(i)} < L_y\} + u_x\,\mathbf{1}\{U_x < L_y\}\right\} + \right.$$

$$\left.\sum_{j=1}^{r_y}\left\{l_x\,\mathbf{1}\{-\infty< y_{(j)}\}+\sum_{i=0}^{r_x}\mathbf{1}\{x_{(i)}< y_{(j)}\}+ u_x\,\mathbf{1}\{U_x< y_{(j)}\}\right\}+(u_y+1)(n_x+1)\right]$$

$$= A\left[l_y\left\{\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < L_y\} + u_x\,\mathbf{1}\{U_x < L_y\}\right\} + \sum_{j=1}^{r_y}\left\{\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\} + \right.\right.$$

$$\left.\left. u_x\,\mathbf{1}\{U_x < y_{(j)}\}\right\} + (l_x+1)(l_y+r_y) + (u_y+1)(n_x+1)\right]$$

□

It is straightforward to show that these NPI lower and upper probabilities satisfy the conjugacy property. These NPI lower and upper probabilities are the most conservative lower and upper bounds that correspond to all possible orderings of the data in the terminated tails. Hence, if L_x or L_y increases, or U_x or U_y decreases, the number of data in the terminated tails can increase (it cannot decrease), which could lead to decrease (but not increase) of the lower probability (7.1) and to increase (but not decrease) of the upper probability (7.2).

7.5 Special cases

An advantage of presenting the general result of this chapter, in Section 7.4, is that many important inferential problems are special cases of such comparisons with terminated tails, hence the NPI comparison methods for such special cases follow immediately from Theorem 7.2. In this section, we briefly discuss four special cases.

1. *Equal lower and upper tails termination.*

If $L_x = L_y = L$ and $U_x = U_y = U$, then the NPI lower probability (7.1) and upper probability (7.2) are

$$\underline{P} = A\left[\sum_{j=1}^{r_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\} + l_x(r_y+u_y) + r_x u_y\right]$$
$$\overline{P} = A\left[\sum_{j=1}^{r_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\} + (l_x+1)(l_y+r_y) + (u_y+1)(n_x+1)\right]$$

This situation enables a straightforward analysis of the numbers of observations in the two groups for which the exact information could be deleted by terminating the tails, whilst still achieving $\underline{P} > 0.5$, which might be interpreted as a strong indication that $X_{n_x+1} < Y_{n_y+1}$. Such a study can be relevant from the perspective of robust inference, this is briefly discussed in Section 7.7. Suppose that $n_x = n_y = n$, and that the data within the interval $[L, U]$ are maximally supportive for the event $X_{n_x+1} < Y_{n_y+1}$, meaning that the corresponding NPI lower and upper probabilities $\underline{P}$ and $\overline{P}$ are maximal over all possible configurations of the data for groups X and Y over this interval. It is easily seen and verified that this holds if all x_i's in $[L, U]$ are less than all y_j's in this interval. For this situation, $\underline{P} > 0.5$ if and only if $(n-u_x)(n-l_y) > 0.5(n+1)^2$. For example, this implies that for $n = 20$ observations from each group, one could have $\underline{P} > 0.5$ if $l_y = 5$ and $u_x = 5$, if the x_i's in the interval $[L, U]$ were all less than the y_j's in that interval, but if either l_y or u_x were greater than 5, this would not be possible anymore. A further special case of interest is if the tails were cut off in this manner, with also $u_x = l_y = c$. Then the above necessary and sufficient condition for $\underline{P} > 0.5$ to be possible (for the maximally supportive data) reduces to $c < (1 - \sqrt{0.5})n - \sqrt{0.5} = c(n)$. Although this is only a rather weak result, it does provide some insight into the amount of data that can be cut in the manner studied in this chapter, in order to still possibly get a strong result for the comparison of the two groups. Stated differently, if the tails termination leads to the exact information for more observations to be discarded than $c(n)$ from both tails of both groups, then a strong indication of preference for one group over the other ($\underline{P} > 0.5$) cannot follow anymore within the NPI framework. Of course, in most situations the data within the interval $[L, U]$ will not be maximally supportive for the event $X_{n_x+1} < Y_{n_y+1}$ in the way considered here, and generally the number of observations that can be deleted by terminating the tails without affecting the inference of interest must be separately studied for each specific data set.

2. *No lower tails termination, equal upper tails termination.*

If there is no lower tail termination for both groups, so $L_x = L_y = -\infty$ and hence $l_x = l_y = 0$, while the upper cut points for both groups are equal, $U_x = U_y = U$, so with $u_x = n_x - r_x$ and $u_y = n_y - r_y$, then the NPI lower and upper probabilities (7.1) and (7.2) are coincide with those obtained in Chapter 2 (Theorem 2.2) for the application of NPI for the comparison of two groups based on precedence testing.

3. *No upper tails termination, equal lower tails termination.*

If there is no upper tail termination for both groups, so $U_x = U_y = \infty$ and $u_x = u_y = 0$, while the lower cut points for both groups are equal, $L_x = L_y = L$, so with $l_x = n_x - r_x$ and $l_y = n_y - r_y$, then the NPI lower and upper probabilities (7.1) and (7.2) are

$$\underline{P} = A\left[\sum_{j=1}^{r_y}\sum_{i=1}^{r_x} \mathbf{1}\{x_{(i)} < y_{(j)}\} + r_y\, l_x\right]$$

$$\overline{P} = A\left[\sum_{j=1}^{r_y}\sum_{i=1}^{r_x} \mathbf{1}\{x_{(i)} < y_{(j)}\} + (l_x + 1)n_y + (n_x + 1)\right]$$

This case is important in situations where exact values in the lower tails cannot be determined, which particularly occurs if measurement equipment has a lower limit of detection. For example, this frequently occurs in risk assessment with regard to, for example, food safety and environmental impact of chemicals, where small traces of chemicals may not be detectable but should still be considered, in particular in situations of exposure to multiple chemicals. A first study into the use of NPI in such risk assessments, focusing on a basic exposure model and also considering combination of NPI for some random quantities with Bayesian methods for others, has recently been presented by Montgomery [62].

4. *Tails termination for one group.*

Suppose that the lower and upper tails are terminated for one group, say X, whilst for the other group, Y, tails are not terminated so all observations from group Y are available and $L_y = -\infty$, $U_y = \infty$, $l_y = u_y = 0$ and $r_y = n_y$. Then the NPI lower and upper probabilities (7.1) and (7.2) are

$$\underline{P} = A\left[\sum_{j=1}^{n_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\} + l_x\sum_{j=1}^{n_y}\mathbf{1}\{L_x < y_{(j)}\}\right]$$

$$\overline{P} = A\left[\sum_{j=1}^{n_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\} + u_x\sum_{j=1}^{n_y}\mathbf{1}\{U_x < y_{(j)}\} + n_y(l_x+1) + (n_x+1)\right]$$

Moreover, if for group X only the lower tail is terminated, so $U_x = \infty$, $u_x = 0$ and $l_x = n_x - r_x$, then these NPI lower and upper probabilities become

$$\underline{P} = A\left[\sum_{j=1}^{n_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\} + (n_x - r_x)\sum_{j=1}^{n_y}\mathbf{1}\{L_x < y_{(j)}\}\right]$$

$$\overline{P} = A\left[\sum_{j=1}^{n_y}\sum_{i=1}^{r_x}\mathbf{1}\{x_{(i)} < y_{(j)}\} + n_y(n_x - r_x + 1) + (n_x+1)\right]$$

An important example from medical statistics where this case occurs is inference involving a partial area under the Receiver Operating Characteristic (ROC) curve, which is used to evaluate the accuracy of a diagnostic test which yields ordinal or continuous test results [34, 69]. The ROC curve can also be used to compare the accuracy of two or more continuous diagnostic tests. The use of ROC curves for diagnostic tests can also be considered within the NPI framework, where focus on a partial area under the ROC curve relates to the methods in this chapter with tails termination for one group. Work on this topic is ongoing and we aim to present the results soon elsewhere.

To end this section, it is worth mentioning the situation without tails termination, so with complete data for both groups, as presented by Coolen [19]. This is also a, rather trivial, special case of the general results presented in this chapter, with $L_x = L_y = -\infty$, $U_x = U_y = \infty$, $l_x = u_x = l_y = u_y = 0$, $r_x = n_x$ and $r_y = n_y$, for which the NPI lower and upper probabilities (7.1) and (7.2) are reduced to the formulae (1.1) and (1.2), in Chapter 1, respectively.

7.6 Example

The following example is used to illustrate the presented NPI approach for comparison of two groups with terminated tails and to discuss the special cases mentioned above.

Example 7.1. We consider a data set used by Nelson [64, p.462], which gives the breakdown times of units from 6 different groups. In this example, only the first two groups are used to illustrate the NPI method for pairwise comparison with tails termination. The data for these groups are presented in Table 7.1 and Figure 7.2. Both groups consist of 10 observations, so $n_x = n_y = 10$. The first unit of group X has a reported breakdown time of 0.00, we interpret this as a very small but positive breakdown time.

X	0.00	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	10.60
Y	0.31	0.66	1.54	1.70	1.82	1.89	2.17	2.24	4.03	9.99

Table 7.1: Breakdown times of units from groups X and Y

Figure 7.2 shows that there are 4 observations (1 in group X, 3 in group Y) which may be considered as outliers, using the established rule-of-thumb to highlight observations as possible outliers if they are more than 1.5 times the interquartile range below (above) the first (third) quartile of the data. The NPI approach presented in this chapter considers the lower and upper probabilities for the event that the

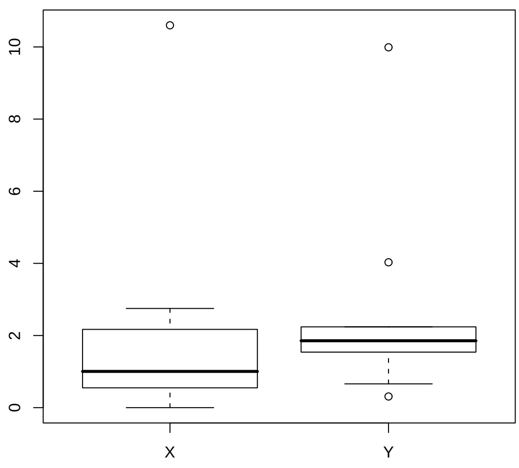


Figure 7.2: Breakdown times of units from groups X and Y

breakdown time of a future unit from group X, say X_{11}, is less than the breakdown time Y_{11} of a future unit from group Y. For both groups the inferences are based on the assumption $A^{tt}_{(10)}$ in combination with the respective data per group, and of course the breakdown times are non-negative.

If we consider the complete data without any tails termination, then the NPI lower and upper probabilities [19] are

$$\underline{P}(X_{11} < Y_{11}) = 0.5372, \quad \overline{P}(X_{11} < Y_{11}) = 0.7273$$

If one instead considers the event $Y_{11} < X_{11}$, then the NPI lower and upper probabilities are

$$\underline{P}(Y_{11} < X_{11}) = 0.2727, \quad \overline{P}(Y_{11} < X_{11}) = 0.4628$$

which is in line with the conjugacy property for lower and upper probabilities [76]. The fact that $\underline{P}(X_{11} < Y_{11}) > 0.5$ can be interpreted as a strong indication that group Y is better, in the sense of leading to longer breakdown times, than group X. This data set contains two pairs of tied observations, at 0.66 and 2.17. To deal with this, we follow the argument mentioned in Subsection 1.3.5.

Let us consider termination of these data by setting cut points $L_y = 0.5$, $U_y = 9$ and $U_x = 10$, so we terminate one observation from the upper tail from each group and one observation from the lower tail from group Y. This just means that for these observations the exact value is not taken into account, which might have happened if indeed the measurements in these tails were not available, or for example if one would have severe doubts about the accuracy of observations in these tails. The corresponding NPI lower and upper probabilities are

$$\underline{P}(X_{11} < Y_{11}) = 0.5207, \quad \overline{P}(X_{11} < Y_{11}) = 0.7355$$

Now suppose that we want to exclude the effect of the 4 possible outliers as discussed above, which can for example be achieved by cut points $L_y = 0.5$, $U_y = 4$ and $U_x = 10$, This leads to NPI lower and upper probabilities

$$\underline{P}(X_{11} < Y_{11}) = 0.5207, \quad \overline{P}(X_{11} < Y_{11}) = 0.7438$$

Compared to the situation discussed above with $U_y = 9$ and the other cut points the same, one more observation is now terminated from the upper tail of group Y. The effect of this is that the NPI lower probability for the event $X_{11} < Y_{11}$ remains the same, but the NPI upper probability increases, so the imprecision $(\overline{P} - \underline{P})$ increases due to more observations being terminated. Again, there is a strong indication that group Y is better than group X.

If all units were put simultaneously on the lifetime experiment and this is terminated at time 4, so $U_y = U_x = 4$ with no termination of the lower tails, then for all units with observations greater than 4 the actual observations would not have been available, instead we would only have right-censored observations at time 4 for these units. The corresponding NPI lower and upper probabilities are

$$\underline{P}(X_{11} < Y_{11}) = 0.5372, \quad \overline{P}(X_{11} < Y_{11}) = 0.7438$$

This lower probability exceeds 0.5, hence one may reach the same conclusion as discussed above for the case that the experiment had not been terminated. By terminating the experiment at time 4, 3 units have not broken down and could possibly be used for other purposes, and, possibly more importantly, reducing the time of the experiment may lead to cost savings. Actually, we could have ended the experiment earlier while still getting the lower probability greater than 0.5 (and this could not decrease by running the experiment longer), as e.g. ending the experiment at time 2.18, so with $U_y = U_x = 2.18$, would lead to 5 units not having broken down and NPI lower and upper probabilities

$$\underline{P}(X_{11} < Y_{11}) = 0.5207, \quad \overline{P}(X_{11} < Y_{11}) = 0.7603$$

If the experiment had been ended before time 2.17, the NPI lower probability for the event $X_{11} < Y_{11}$ would be less than 0.5. For example, with $U_y = U_x = 2.16$ the NPI lower and upper probabilities are

$$\underline{P}(X_{11} < Y_{11}) = 0.4959, \quad \overline{P}(X_{11} < Y_{11}) = 0.7769$$

If we terminate both tails of the data at the same cut points for both groups, for example with $L_y = L_x = 0.5$ and $U_y = U_x = 4$, then two units would have been terminated from the lower tail of group X and one unit from its upper tail, while for group Y one unit would have been terminated from its lower tail and two units from its upper tail. Then the corresponding NPI lower and upper probabilities are

$$\underline{P}(X_{11} < Y_{11}) = 0.5207, \quad \overline{P}(X_{11} < Y_{11}) = 0.7438$$

These discussed cases illustrate that, as discussed at the end of Section 7.4, the NPI lower (upper) probability is maximal (minimal) when all observations are exactly included in the comparison, while deleting some of the exact information leads to increased imprecision. This example also makes clear that varying the cut points may have no, or only a very small effect on the actual inference. Clearly, the lower and upper probabilities considered can only change if a change in cut point is such that it leads to more or fewer observations in the terminated tails. For example, for any specific cut point U_x between 2.75 and 10.60 in this example it does not matter that the actual largest observation of group X was 10.60, the inferences would have been the same if it were any larger value.

Before ending this example it is interesting to report some results of the classical methods presented in Section 7.2. For the Yuen-Welch test, $T_{0.20} = 1.67$ and $T_{0.10} = 1.24$ with corresponding p-values 0.141 and 0.235, respectively, so at 5% significance level we do not reject the null hypothesis that the trimmed means of the two groups are equal. The same conclusion is obtained from the Brunner-Munzel test where $B = 1.21$ and the p-value is 0.244. $\triangle$

7.7 Concluding remarks

In this chapter we have introduced NPI for comparison of two groups of real-valued data with tails termination, which brings together a number of important applications of statistics. The main contribution of this chapter is in simultaneously dealing with possible termination of the lower and upper tails, which was not considered from the NPI perspective before, and which enables several important special cases to be brought together as shown in Section 7.5. We have kept presentation relatively basic, there are several generalizations which are important for statistical practice, and which are relatively straightforward but which require far more complicated notation. For example, the NPI approach presented here can quite easily be generalized to comparison of more than two groups, and it is also conceptually easy to deal with further right-censored observations within the data by using the more general M-functions following from rc-$A_{(n)}$ [27].

The problem considered in this chapter could be considered as a special case of pairwise comparison based on interval-censored data, such that each observation is only known to belong to an interval (which may be a single point or open-ended). The NPI approach for this general problem is not straightforward, and provides an exciting challenge for future research. The main problem is that the assumption rc-$A_{(n)}$ does not have a straightforward generalization to deal with finite upper bounds for the interval-censored data nor for dealing with (partially) overlapping intervals corresponding to different observations. One could derive bounds for the probabilities of interest by assuming that the data for one group are as large as possible and for the other group as small as possible, and then apply the method of Coolen [19] for such specifically assumed data, but that would lead to wide bounds as it would neglect the exchangeability of censored data with other observations from the same group.

There are interesting links between NPI and methods from the robust statistics literature (see e.g. [10, 39, 43]). The NPI method for pairwise comparisons for real-valued data presented in this chapter only takes the ranks of the non-terminated observations into account, and as such it is insensitive to outliers even without tails being terminated. By terminating the tails, the focus shifts explicitly to the information in the non-terminated part of the data, which has some conceptual similarities to robust statistics procedures such as trimmed means. However, our method does not disregard the fact that there are observations in the terminated tails and it takes the numbers of such observations into account, making such tails termination different from truncation of the tails, which refers to situations where such numbers are not available. As illustrated in Example 7.1, one can study how many observations can be in the terminated tails in order to still get similar inferences, in particular we considered the NPI lower probability for the event of interest to still exceed 0.5.

Chapter 8

Conclusions

In this thesis we have presented Nonparametric Predictive Inference (NPI) for several comparisons problems. We introduced NPI for comparison of multiple groups of data including right-censored observations. Different right-censoring schemes discussed are early termination of an experiment, progressive censoring and competing risks. Several selection events of interest are considered including selecting the best group, the subset of best groups, and the subset including the best group. We also discussed the situation when only a part of the data range is considered relevant for the inference, with in addition the numbers of observations to the left and to the right of this range available. In the appendix, we have included the R commands that have been used for calculating NPI lower and upper probabilities for different multiple comparison problems.

NPI is a fully nonparametric statistical approach, which explicitly does not use any information or assumptions about the random quantities of interest other than the relevant $A_{(n)}$ or rc-$A_{(n)}$ assumptions per group. These inferences have a frequentist justification, but explicitly use the available data and do not require the use of counterfactual data (i.e. data that could have occurred, under a specific experimental set-up, but which did not occur), which for example happens in many frequentist methods for hypothesis testing.

Our method has the advantage that the comparison is not based on testing the hypothesis of equality of the distributions, which, although a well established approach in classical statistics, is a somewhat surprising starting point as the reasons for making a comparison of different groups may make it very unlikely that observations from all groups would actually have identical distributions. In addition, in both cases of rejection or not of such a hypothesis, it is not clear what such a conclusion implies for the next future observation. Application of our method leads to lower and upper probabilities for certain events of interest, which enables conservative decisions by basing these on the worst possible situation for the event of interest.

A further advantage is related to the similar general advantage of statistical methods that adhere to the likelihood principle, for which stopping rules tend not to affect the inferences, and is a direct consequence of the fact that no hypotheses are being tested, hence no counterfactual data play any role. Of course, one must be happy to accept the assumptions, related to exchangeability, underlying NPI. In addition, one does not have to restrict attention to specific censoring schemes as presented in this thesis, as censoring can take place at any time without causing problems for the NPI approach, as long as the censoring mechanism is independent of the lifetime random quantities, and as long as one can reasonably assume complete independence of the groups being compared.

As for any new statistical method, it is important to consider how it can be applied. Of course, the assumption $A_{(n)}$ per group is crucial, if for example the data or knowledge about underlying processes are such that one does not consider the exchangeability assumption, implicit to $A_{(n)}$, to be appropriate, then this method should not be applied. All classical nonparametric methods in such applications tend to agree with this exchangeability assumption, but require additional assumptions (e.g. similarities in the probability distribution functions corresponding to different groups). One may well think that, in most applications, there is knowledge about the process and groups considered that can or should be taken into account, which NPI does not take on board. However, in all cases NPI can be considered to be a 'baseline method', it provides inferences without further assumptions or information, and this enables, for example, useful study of the outcomes of other statistical methods. If other methods lead to conclusions which differ substantially from those following from the NPI approach, then this will be due to the assumptions underlying the other method, which may often not have been made under complete awareness but more for mathematical convenience. Also, it is important to be aware of the fact that problems which appear to be identical are often formulated in substantially different manners in different statistical methods, with each method affected by specific features of the data. As such, we would strongly recommend the use of several statistical methods for a problem of interest, followed by careful study of the resulting inferences. If these all point in the same direction, then one can have great confidence in the inferences, but if not the value of such an extensive study may well be even greater, as detailed understanding of the different outcomes is likely to provide more insight in the data and the actual inferential problem, as well as in the different methods used.

The results presented in this thesis show how NPI can be applied to a variety of problems which have been considered in the literature, mostly from classical frequentist perspective and which are of great relevance in many applications. Most of these problems involve multiple comparisons, and generally NPI provides exciting opportunities for such problems via explicit focus on the next future observation per group.

In addition to applications to a wider variety of problems, there are many research challenges for the further development of NPI. These include the option to base inferences on more than one future observation per group, which is conceptually easy although one must not forget to take account of the fact that these future observations are inter-dependent. An interesting challenge is the requirement to formulate appropriate predictive events of interest for a variety of inferential problems, which in this thesis was a rather straightforward comparison of the single next observations per group. Often, however, such predictive inferences may be more in line with intuition than established statistical methods such as hypothesis testing. More generally, development of NPI for multivariate situations, including data with covariates, is a key challenge that promises exciting research opportunities, the results of which are strongly needed to enhance wide applicability of NPI.

Appendix A

An illustrative example of rc-$A_{(n)}$

The following example is provided to illustrate the assumption rc-$A_{(n)}$, the full theory is presented by Coolen and Yan [27].

First, suppose we have $n = 5$ observations which create 6 intervals, and all 5 observations are failure times. Then the assumption $A_{(5)}$ implies that the next observation X_6 will fall in any one of these intervals with probability 1/6.

Now suppose that one of these observations is right-censored at time c, so we have 4 failure times, then $A_{(5)}$ cannot be used directly but we can use rc-$A_{(5)}$, which is explained in detail in Figure A.1. As first step, shown in Figure A.1(a), the probability masses for the intervals created by the 4 failure times are equal to 1/6. There is also a probability mass 1/6 spread over the interval (c, ∞) since all we know, without making any further assumptions, that the lifetime of this observation will be at any point beyond c (see $\tilde{A}_{(n)}$ assumption in [27, p. 32]). We assume non-informative censoring, which means that the residual lifetime of this censored observation is independent of the censoring process. Therefore, one can apply $A_{(2)}$ but with the starting point shifted from 0 to the censoring time c, see Figure A.1(b) (see shifted-$\tilde{A}_{(n)}$ assumption in [27, p. 33]). This gives 3 equally likely possibilities for the actual lifetime of the censored observation, say x_c, see Figures A.1(b.1-b.3). In the case of Figure A.1(b.1), for example, this censored observation falls somewhere between (c, x_3), so the probability mass for X_6 that was assigned to the interval (x_2, x_3) (in Figures A.1(a-b)) would now be reassigned to the interval (x_2, x_c) instead. Moreover, the probability mass that was carried forward by the censored observation, c, would now be assigned to the interval (x_c, x_3). Without further assumptions, x_c is in (c, x_3) with probability 1/3, so a probability mass equal to 1/18 will be assigned for X_6 to belong to the interval (c, x_3). For the case where the censored observation falls between (x_3, x_4), Figure A.1(b.2), or between (x_4, ∞), Figure A.1(b.3), the explanation is the same. So the rc-$A_{(5)}$ assumption results by combining these probabilities per interval as shown in Figure A.1(c).

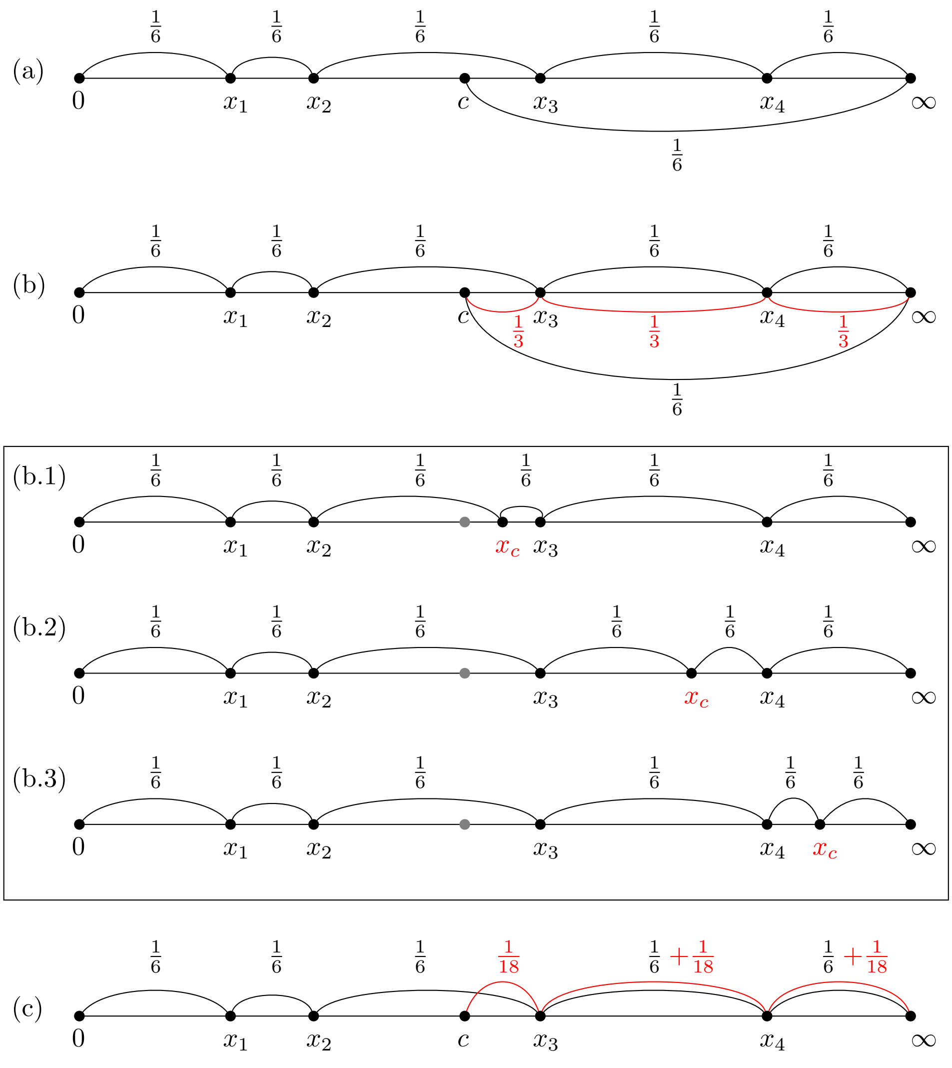


Figure A.1: rc-$A_{(5)}$ with one censored observation

Let us now consider the case with two censored observations and 3 failures. Using the same first argument as above (i.e. the $\tilde{A}_{(5)}$ assumption) leads to Figure A.2(a). That is each censored observation carries forward probability mass 1/6.

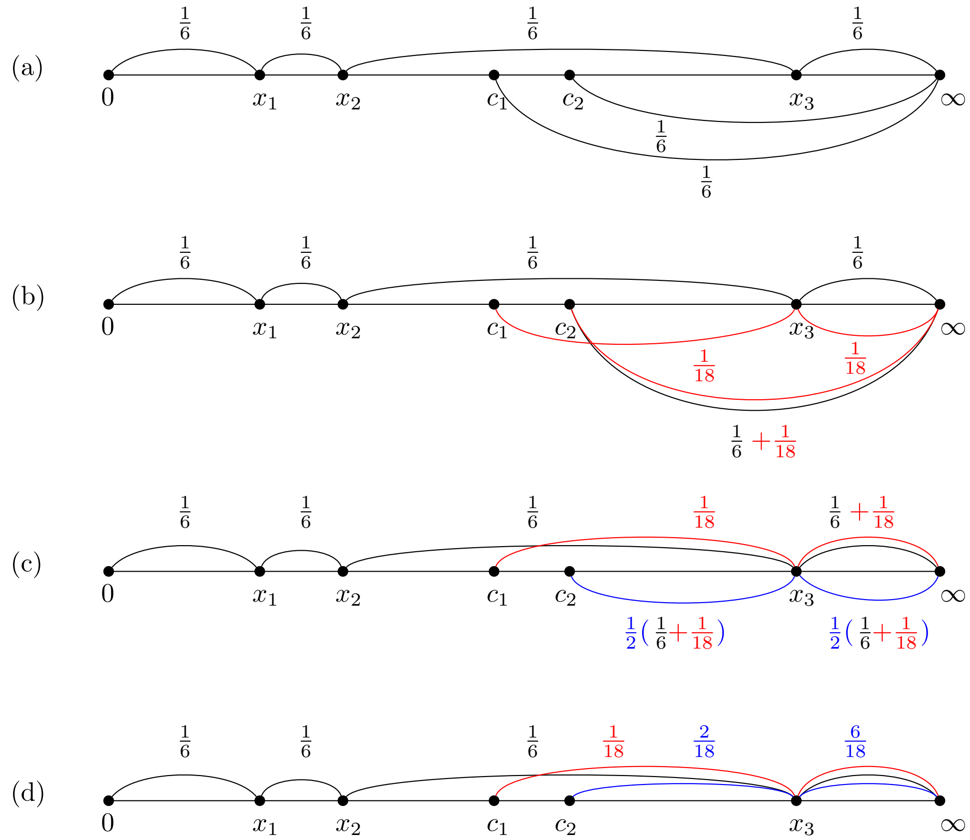


Figure A.2: rc-$A_{(5)}$ with two censored observations

Again by applying the shifted-$\tilde{A}_{(5)}$ assumption, the probability mass corresponding to the first censored observation, c_1, is divided equally to the intervals to the right of c_1. And since there is another censored observation, c_2, and all we know about the lifetime corresponding to this censored observation is that it will be at any time beyond c_2, then this probability mass resulting from c_1 will be assigned to the interval (c_2, ∞), see Figure A.2(b). Now there is a total probability mass $1/6+1/18$ assigned to the interval (c_2, ∞). Again by applying the shifted-$\tilde{A}_{(5)}$ assumption, this probability mass will be divided equally to the sub-intervals in (c_2, ∞), see Figure A.2(c). Then by using the same argument as above (i.e. Figure A.1), the rc-$A_{(5)}$ assumption results by combining these mass probabilities per interval as shown in Figure A.2(d). This can also be interpreted along the same lines as the probability redistribution algorithm for right-censored data as introduced by Efron [35] and also discussed by Coolen and Yan [27].

Appendix B

R programs

B.1 NPI for multiple comparison of lifetime data

```
# NPI for comparing several groups (complete data, right censoring, precedence,
# progressive, competing risks).
# Data consist of a list of groups, X11, X22,..., each of them is a matrix where
# the first column is the lifetime and the second column is the state of this
# observation;1 if failure & 0 if censored.
# S is the set of best group(s) or that includes the best group, e.g. S<-c(1,2,3).
# The length of S is three groups however one can easily extended, i.e. repeat
# the related commands.
# data<-list(X11,X22, ...)

best.select <-
function (data, S)
{
    k <- length(data)
    jk <- 1:k
    NS <- jk[-S]
    ls <- length(S)
    lns <- length(NS)
    X.c <- function(X) { # to get the censored data
        ifelse(length(X[X[, 2] == 0, ]) > 2, x1 <- X[X[, 2] ==
            0, ][, 1], x1 <- X[X[, 2] == 0, ][1])
        return(x1)
    }
```

```
X.u <- function(X) { # to get the failure data
    ifelse(sum(X[, 2] == 1) == 1, x1 <- X[X[, 2] == 1, ][1],
        x1 <- X[X[, 2] == 1, ][, 1])
    return(x1)
}
Xu1 <- function(X) { # all censored, no failure occurs
    ifelse(sum(X[, 2] == 1) == 0, Y <- Inf, Y <- c(X.u(X),
        Inf))
    return(Y)
}
Xt0 <- function(X) {
    Y <- c(0, X[, 1])
    return(Y)
}
data0 <- lapply(data, Xt0)  # add zero to the lifetime t0
data.u <- lapply(data, Xu1) # to get failure data
m2 <- function(data) { # create all possible values
    XX <- NULL
    for (i1 in 1:length(data[[1]])) {
        for (i2 in 1:length(data[[2]])) {
            XX <- rbind(XX, c(data[[1]][i1], data[[2]][i2]))
        }
    }
    return(XX)
}
m3 <- function(data) {
    XX <- NULL
    for (i1 in 1:length(data[[1]])) {
        for (i2 in 1:length(data[[2]])) {
            for (i3 in 1:length(data[[3]])) {
              XX <- rbind(XX, c(data[[1]][i1], data[[2]][i2],
                data[[3]][i3]))
            }
        }
    }
    return(XX)
}
```

```
# calculate the product terms to use later for Mfun and prob
cond <- function(X, y) {
    P1 <- NULL
    n <- nrow(X)
    Xc <- X.c(X)
    ncc <- function(X, cr) { # calculate the term in the product term
        (sum(X[, 1] >= cr) + 1)/sum(X[, 1] >= cr)
    }
    cr.obs <- Xc[Xc < y]
    n.cr.obs <- length(cr.obs) # calculate the condition under the product term
    ifelse(n.cr.obs == 0 | sum(X[, 2] == 0) == 0, P1 <- 1,
        for (j in 1:n.cr.obs) {
            P1[j] <- ncc(X, cr.obs[j])
        })
    P3 <- prod(P1)/(n + 1)
    return(P3)
}
# calculate Mfun and prob
Mfun <- function(X) {
    Y <- rbind(c(0, 1), X)
    ny <- nrow(Y)
    Mu <- NULL
    for (i in 1:ny) {
        Mu[i] <- (sum(X[, 1] >= Y[, 1][i]))^(Y[, 2][i] -
            1) * cond(X, Y[, 1][i])
    }
    return(Mu)
}
Prob <- function(X) {
    Y <- Xu1(X)
    ny <- length(Y)
    P4 <- NULL
    for (i in 1:ny) {
        P4[i] <- cond(X, Y[i])
    }
    return(P4)
}
```

```
# To calculate  X<Y times M-function or Prob
fun1 <- function(X, Y, MP) {
    d <- matrix(0, length(X), length(Y))
    for (i in 1:length(Y)) {
        for (j in 1:length(X)) {
            d[j, i] <- sum(X[j] < Y[i])
        }
    }
    d1 <- MP %*% d
    return(d1)
}
MM <- lapply(data, Mfun)
PP <- lapply(data, Prob)
# m=1
if (ls == 1)
    XL <- unlist(data0[S])
if (ls == 1)
    XU <- unlist(data.u[S])
if (ls == 1)
    MMS <- unlist(MM[S])
if (ls == 1)
    PPS <- unlist(PP[S])
# m=2
if (ls == 2)
    XL <- m2(data0[S])
if (ls == 2)
    XU <- m2(data.u[S])
if (ls == 2)
    MMS <- m2(MM[S])
if (ls == 2)
    PPS <- m2(PP[S])
# m=3
if (ls == 3)
    XL <- m3(data0[S])
if (ls == 3)
    XU <- m3(data.u[S])
if (ls == 3)
```

```
        MMS <- m3(MM[S])
    if (ls == 3)
        PPS <- m3(PP[S])
    ifelse(ls == 1, prod.M <- MMS, prod.M <- apply(MMS, 1, prod))
    ifelse(ls == 1, prod.P <- PPS, prod.P <- apply(PPS, 1, prod))
    # compute the minmum in subset best case
    ifelse(ls == 1, Min.XL <- XL, Min.XL <- apply(XL, 1, min))
    ifelse(ls == 1, Min.XU <- XU, Min.XU <- apply(XU, 1, min))
    # compute the maximum in subset include the best case
    ifelse(ls > 1, Max.XL <- apply(XL, 1, max), Max.XL <- XL)
    ifelse(ls > 1, Max.XU <- apply(XU, 1, max), Max.XU <- XU)
    # Select the best groups &  the subset include the best
    Lprob <- function(y) {
        s1 <- NULL
        ifelse(lns == 1, s1 <- fun1(unlist(data.u[NS]), y, unlist(PP[NS])),
            for (j in 1:lns) {
                s1 <- rbind(s1, fun1(data.u[NS][[j]], y, PP[NS][[j]]))
            })
        ifelse(lns > 1, Z <- sum(apply(s1, 2, prod) * prod.M),
            Z <- sum(s1 * prod.M))
        return(Z)
    }
    Uprob <- function(y) {
        s2 <- NULL
        ifelse(lns == 1, s2 <- fun1(unlist(data0[NS]), y, unlist(MM[NS])),
            for (j in 1:lns) {
                s2 <- rbind(s2, fun1(data0[NS][[j]], y, MM[NS][[j]]))
            })
        ifelse(lns > 1, Z <- sum(apply(s2, 2, prod) * prod.P),
            Z <- sum(s2 * prod.P))
        return(Z)
    }
    print(c("Lprob.best", "Uprob.best", "Lprob.include", "Uprob.include"))
    return(round(c(Lprob(Min.XL), Uprob(Min.XU), Lprob(Max.XL),
        Uprob(Max.XU)), 4))
}
### END ###
```

B.2 NPI for comparing two groups with terminated tails

```
# NPI for comparing two groups, Terminated tails (Y is the best group)

ttfun <-
function (X, Y, Lx, Ux, Ly, Uy)
{   # Data & choose the cut points
    nx <- length(X)
    ny <- length(Y)
    nly <- sum(Y < Ly)
    nuy <- sum(Y > Uy)
    nry <- sum(Y >= Ly & Y <= Uy)
    nlx <- sum(X < Lx)
    nux <- sum(X > Ux)
    nrx <- sum(X >= Lx & X <= Ux)
    Yr <- Y[Y >= Ly & Y <= Uy]
    Xr <- X[X >= Lx & X <= Ux]
    Mlx <- nlx/(nx + 1)
    Mrx <- rep(1, nrx)/(nx + 1)
    Mux <- nux/(nx + 1)
    Mly <- nly/(ny + 1)
    Mry <- rep(1, nry)/(ny + 1)
    Muy <- nuy/(ny + 1)
    fun1 <- function(X, Y, MP) { # To calculate  X<Y times M-function
        d <- matrix(0, length(X), length(Y))
        for (i in 1:length(Y)) {
            for (j in 1:length(X)) {
                d[j, i] <- sum(X[j] < Y[i])
            }
        }
        d1 <- MP %*% d
        return(d1)
    }
    # Lower and Upper prob. that Y is the best
    YU <- c(Yr, Uy)
    XL <- c(Lx, Xr)
```

```
    YL <- c(Ly, Yr)
    XU <- c(Xr, Ux)
    Lprob <- sum(fun1(XL, YU, c(Mlx, Mrx)) * c(Mry, Muy))
    Uprob <- sum(fun1(XU, YL, c(Mrx, Mux)) * c(Mly, Mry)) + ((nx +
        1) * (ny + 1))^(-1) * ((nlx + 1) * (nly + nry) + (nuy +
        1) * (nx + 1))
    return(round(c(Lprob, Uprob), 4))
}
### END ###
```

Bibliography

[1] Augustin T. and Coolen F.P.A. (2004). Nonparametric predictive inference and interval probability. *Journal of Statistical Planning and Inference*, 124(2), 251–272.

[2] Bairamov I. and Eryilmaz S. (2006). Spacings, exceedances and concomitants in progressive type II censoring scheme. *Journal of Statistical Planning and Inference*, 136(3), 527–536.

[3] Balakrishnan N. and Aggarwala R. (2000). *Progressive Censoring: Theory, Methods, and Applications*. Boston: Birkha.

[4] Balakrishnan N. and Frattina R. (2000). Precedence test and maximal precedence test. In *Recent Advances in Reliability Theory: Methodology, Practice and Inference*, (Editors) N. Limnios and M. Nikulin, pp. 355–378.

[5] Balakrishnan N. and Ng H.K.T. (2006). *Precedence-Type Tests and Applications*. Hoboken, N.J.: Wiley.

[6] Balakrishnan N., Ng H.K.T. and Panchapakesan S. (2006). A nonparametric procedure based on early failures for selecting the best population using a test for equality. *Journal of Statistical Planning and Inference*, 136(7), 2087–2111.

[7] Balakrishnan N., Tripathi R.C. and Kannan N. (2008). On the joint distribution of placement statistics under progressive censoring and applications to precedence test. *Journal of Statistical Planning and Inference*, 138(5), 1314–1324.

[8] Balakrishnan N., Tripathi R.C., Kannan N. and Ng H.K.T. (2008). Precedence-type Test based on Progressively Censored Samples. Working Papers 0054, College of Business, University of Texas at San Antonio.

[9] Balakrishnan N., Tripathi R.C., Kannan N. and Ng H.K.T. (2010). Some nonparametric precedence-type tests based on progressively censored samples and evaluation of power. *Journal of Statistical Planning and Inference*, 140(2), 559–573.

[10] Barnett V. and Lewis T. (1994). *Outliers in Statistical Data*. Chichester: Wiley, 3 edition.

[11] Bechhofer R.E. (1954). A single-sample multiple decision procedure for ranking means of normal populations with known variances. *Annals of Mathematical Statistics*, 25(1), 16–39.

[12] Bechhofer R.E., Santner T.J. and Goldsman D.M. (1995). *Design and Analysis of Experiments for Statistical Selection, Screening and Multiple Comparisons*. New York: Wiley.

[13] Bedford T., Alkali B. and Burnham R. (2008). Competing risks in reliability. In *Encyclopedia of Quantitative Risk Analysis and Assessment*, (Editors) E.L. Melnick and B.S. Everitt, pp. 307–312. Chichester: Wiley.

[14] Brunner E. and Munzel U. (2000). The nonparametric Behrens-Fisher problem: Asymptotic theory and a small-sample approximation. *Biometrical Journal*, 42(1), 17–25.

[15] Chakraborti S. and Desu M.M. (1990). Quantile tests for comparing several treatments with a control under unequal right-censoring. *Biometrical Journal*, 32(6), 697–706.

[16] Chakraborti S. and van der Laan P. (1997). An overview of precedence-type tests for censored data. *Biometrical Journal*, 39(1), 99–116.

[17] Cohen A. and Clifford J. (1963). Progressively censored samples in life testing. *Technometrics*, 5(3), 327–339.

[18] Cohen A.C. (1966). Query 18: Life testing and early failure. *Technometrics*, 8(3), 539–545.

[19] Coolen F.P.A. (1996). Comparing two populations based on low stochastic structure assumptions. *Statistics & Probability Letters*, 29(4), 297–305.

[20] Coolen F.P.A. (1998). Low structure imprecise predictive inference for Bayes' problem. *Statistics & Probability Letters*, 36(4), 349–357.

[21] Coolen F.P.A. (2006). On nonparametric predictive inference and objective Bayesianism. *Journal of Logic, Language and Information*, 15(1-2), 21–47.

[22] Coolen F.P.A. and Augustin T. (2009). A nonparametric predictive alternative to the Imprecise Dirichlet Model: the case of a known number of categories. *International Journal of Approximate Reasoning*, 50(2), 217–230.

[23] Coolen F.P.A., Coolen-Schrijner P. and Yan K.J. (2002). Nonparametric predictive inference in reliability. *Reliability Engineering & System Safety*, 78(2), 185–193.

[24] Coolen F.P.A. and Utkin L.V. (2008). Imprecise reliability. In *Encyclopedia of Quantitative Risk Analysis and Assessment*, (Editors) B.S. Everitt and E.L. Melnick, pp. 875–881. Chichester: Wiley.

[25] Coolen F.P.A. and van der Laan P. (2001). Imprecise predictive selection based on low structure assumptions. *Journal of Statistical Planning and Inference*, 98(1-2), 259–277.

[26] Coolen F.P.A. and Yan K.J. (2003). Comparing two groups of lifetime data. In *ISIPTA'03: Proceedings of the Third International Symposium on Imprecise Probabilities and their Applications*, (Editors) J.M. Bernard, T. Seidenfeld and M. Zaffalon, pp. 148–161.

[27] Coolen F.P.A. and Yan K.J. (2004). Nonparametric predictive inference with right-censored data. *Journal of Statistical Planning and Inference*, 126(1), 25–54.

[28] Coolen-Schrijner P., Maturi T.A. and Coolen F.P.A. (2007). Nonparametric predictive precedence testing for two groups. In *Proceedings of the 5th Mathematical Methods in Reliability*. Glasgow, United Kingdom.

[29] Coolen-Schrijner P., Maturi T.A. and Coolen F.P.A. (2009). Nonparametric predictive precedence testing for two groups. *Journal of Statistical Theory and Practice*, 3(1), 273–287. Also published in: P. Coolen-Schrijner, F.P.A. Coolen, M.C. Troffaes, T. Augustin and S. Gupta (Eds.) (2009), Imprecision in Statistical Theory and Practice, pp. 91-105. Greensboro (NC) USA: Grace Scientific Publishing LLC.

[30] Crowder M.J. (2001). *Classical Competing Risks*. London: Chapman & Hall.

[31] De Finetti B. (1974). *Theory of Probability: A Critical Introductory Treatment*. London: Wiley.

[32] De Haan L. and Ferreira A. (2006). *Extreme Value Theory: An Introduction*. New York: Springer.

[33] Desu M.M. and Raghavarao D. (2004). *Nonparametric Statistical Methods for Complete and Censored Data*. Boca Raton: Chapman & Hall.

[34] Dodd L.E. and Pepe M.S. (2003). Partial AUC estimation and regression. *Biometrics*, 59(3), 614–623.

[35] Efron B. (1967). The two sample problem with censored data. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, volume 4, pp. 831–853. New York: Prentice-Hall.

[36] Epstein B. (1955). Comparison of some non-parametric tests against normal alternatives with an application to life testing. *Journal of the American Statistical Association*, 50(271), 894–900.

[37] Fagerland M.W. and Sandvik L. (2009). Performance of five two-sample location tests for skewed distributions with unequal variances. *Contemporary Clinical Trials*, 30(5), 490–496.

[38] Gupta S.S. (1965). On some multiple decision (selection and ranking) rules. *Technometrics*, 7(2), 225–245.

[39] Hampel F.R., Ronchetti E.M., Rousseeuw P.J. and Stahel W.A. (1986). *Robust Statistics*. New York: Wiley.

[40] Hill B.M. (1968). Posterior distribution of percentiles: Bayes' theorem for sampling from a population. *Journal of the American Statistical Association*, 63(322), 677–691.

[41] Hill B.M. (1988). De Finetti's theorem, induction, and A_n, or Bayesian nonparametric predictive inference (with discussion). In *Bayesian Statistics 3*, (Editors) J.M. Bernando, M.H. DeGroot, D.V. Lindley and A. Smith, pp. 211–241. Oxford University Press.

[42] Hill B.M. (1993). Parametric models for A_n: Splitting processes and mixtures. *Journal of the Royal Statistical Society. Series B (Methodological)*, 55(2), 423–433.

[43] Huber P.J. (1981). *Robust Statistics.* New York: Wiley.

[44] Kaplan E.L. and Meier P. (1958). Nonparametric estimation from incomplete observations. *Journal of the American Statistical Association*, 53(282), 457–481.

[45] Kochar S.C., Lam K.F. and Yip P.S.F. (2002). Generalized supremum tests for the equality of cause specific hazard rates. *Lifetime Data Analysis*, 8(3), 277–288.

[46] Kundu D. and Joarder A. (2006). Analysis of Type-II progressively hybrid censored data. *Computational Statistics & Data Analysis*, 50(10), 2509 – 2528.

[47] Kundu D., Kannan N. and Balakrishnan N. (2004). Analysis of progressively censored competing risks data. In *Handbook of Statistics 23: Advances in Survival Analysis*, (Editors) N. Balakrishnan and C.R. Rao, pp. 331–348. Amsterdam: Elsevier.

[48] Lawless J.F. (2003). *Statistical Models and Methods for Lifetime Data.* Hoboken, N.J.: Wiley.

[49] Lee E.T. and Desu M.M. (1972). A computer program for comparing K samples with right-censored data. *Computer Programs in Biomedicine*, 2(4), 315–321.

[50] Lehmann E.L. (1975). *Nonparametrics: Statistical Methods Based on Ranks.* San Francisco: Holden-Day.

[51] Mann H.B. and Whitney D.R. (1947). On a test of whether one of two random variables is stochastically larger than the other. *The Annals of Mathematical Statistics*, 18(1), 50–60.

[52] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2008). Early termination of experiments in nonparametric predictive comparisons. In *Proceedings International Workshop on Applied Probability.* Compiegne, France.

[53] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2008). Lifetime comparisons with early termination of experiments. In *Proceedings International Seminar on Nonparametric Inference*, (Editors) J.C. Pardo-Fernandez and J. de Una-Alvarez, pp. 127–130. Vigo, Spain.

[54] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2009). Nonparametric predictive multiple comparisons with censored data and competing risks. In *ISIPTA'09: Proceedings 6th International Symposium on Imprecise Probability: Theories and Applications*, pp. 307–316. Durham, United Kingdom.

[55] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2009). Nonparametric predictive pairwise comparison for real-valued data with terminated tails. *International Journal of Approximate Reasoning*, 51(1), 141–150.

[56] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2009). On nonparametric predictive inference for competing risks. In *Proceedings of the 18th Advances in Risk and Reliability Technology Symposium*, (Editor) L. Bartlett, pp. 196–211. Loughborough, United Kingdom.

[57] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2010). Nonparametric predictive comparison of lifetime data under progressive censoring. *Journal of Statistical Planning and Inference*, 140(2), 515–525.

[58] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2010). Nonparametric predictive comparison of lifetime data with early termination of experiments. In submission.

[59] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2010). Nonparametric predictive inference for competing risks. *Journal of Risk and Reliability*, 224(1), 11–26.

[60] Maturi T.A., Coolen-Schrijner P. and Coolen F.P.A. (2010). Nonparametric predictive selection with early experiment termination. In submission.

[61] Montanari G.C. and Cacciari M. (1988). Progressively-censored aging tests on xlpe-insulated cable models. *IEEE Transactions on Electrical Insulation*, 23(3), 365–372.

[62] Montgomery V. (2009). *New Statistical Methods in Risk Assessment by Probability Bounds*. Ph.D. thesis, Durham University, Durham, UK. Available from www.npi-statistics.com.

[63] Nelson L.S. (1963). Tables for a precedence life test. *Technometrics*, 5(4), 491–499.

[64] Nelson W. (1982). *Applied Life Data Analysis*. New York: Wiley.

[65] Nelson W.B. (1972). Graphical analysis of accelerated life test data with the inverse power law model. *IEEE Transactions on Reliability*, 21, 2–11.

[66] Ng H.K.T. and Balakrishnan N. (2004). Wilcoxon-type rank-sum precedence tests. *Australia and New Zealand Journal of Statistics*, 46, 631–648.

[67] Ng H.K.T. and Balakrishnan N. (2005). Weighted precedence and maximal precedence tests and an extension to progressive censoring. *Journal of Statistical Planning and Inference*, 135(1), 197–221.

[68] Ng H.K.T., Balakrishnan N. and Panchapakesan S. (2007). Selecting the best population using a test for equality based on minimal Wilcoxon rank-sum precedence statistic. *Methodology and Computing in Applied Probability*, 9(2), 263–305.

[69] Pepe M.S. (2003). *The Statistical Evaluation of Medical Tests for Classification and Prediction*. Oxford: Oxford University Press.

[70] Pintilie M. (2006). *Competing Risks: A Practical Perspective*. Chichester: Wiley.

[71] Ray M.R. (2008). Competing risks. In *Encyclopedia of Quantitative Risk Analysis and Assessment*, (Editors) E.L. Melnick and B.S. Everitt, pp. 301–307. Chichester: Wiley.

[72] Shafer G. (1976). *A Mathematical Theory of Evidence*. Princeton (US): Princeton University Press.

[73] Tsiatis A. (1975). A nonidentifiability aspect of the problem of competing risks. In *Proceedings of the National Academy of Sciences of the United States of America*, 72, pp. 20–22.

[74] Utkin L.V. and Coolen F.P.A. (2007). Imprecise reliability: An introductory overview. In *Computational Intelligence in Reliability Engineering, Volume 2: New Metaheuristics, Neural and Fuzzy Techniques in Reliability*, (Editor) G. Levitin, pp. 261–306. New York: Springer.

[75] Verheijen J.H.M., Coolen F.P.A. and van der Laan P. (1997). Combining two classical approaches for statistical selection. *Communications in Statistics: Theory and Methods*, 26(6), 1291–1312.

[76] Walley P. (1991). *Statistical Reasoning with Imprecise Probabilities*. London: Chapman & Hall.

[77] Walley P. (1996). Measures of uncertainty in expert systems. *Artificial Intelligence*, 83(1), 1 – 58.

[78] Weichselberger K. (2000). The theory of interval-probability as a unifying concept for uncertainty. *International Journal of Approximate Reasoning*, 24(2-3), 149–170.

[79] Weichselberger K. (2001). *Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung I. Intervallwahrscheinlichkeit als umfassendes Konzept*. Heidelberg: Physika.

[80] Wilcox R.R. (2005). *Introduction to Robust Estimation and Hypothesis Testing*. San Diego, CA: Academic Press, 2 edition.

[81] Yan K.J. (2002). *Nonparametric Predictive Inference with Right-Censored Data*. Ph.D. thesis, Durham University, Durham, UK. Available from www.npi-statistics.com.

[82] Yuen K.K. (1974). The two-sample trimmed t for unequal population variances. *Biometrika*, 61(1), 165–170.