Imprecise Reliability*

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Abstract

We present a concise overview of imprecise reliability, particularly focussing on reliability theory with uncertainty quantified via lower and upper probabilities. We discuss the main approaches and opportunities of the theory, we include references to guide further study, and we briefly discuss some research challenges.

Keywords: expert judgements; Imprecise Dirichlet Model; lower and upper probability; natural extension; Nonparametric Predictive Inference; robustness.

1 Introduction

Most methods in reliability and quantitative risk assessment assume that uncertainty is quantified via precise probabilities, all perfectly known or determinable. For example, for system reliability complete probabilistic information about the system structure is usually assumed, including dependence of components and subsystems. Such detailed information is often not available, due to limited time or money for analyses or limited knowledge about a system. In recent decades, several alternative methods for uncertainty quantification have been proposed, some also for reliability. For example, fuzzy

*Contribution to *Wiley Encyclopedia of Quantitative Risk Assessment* reliability theory [1] and possibility theory [2] provided solutions to problems that could not be solved satisfactorily with precise probabilities. We do not discuss such methods, but restrict attention to generalized uncertainty quantification via upper and lower probabilities, also known as 'imprecise probability' [3] or 'interval probability' [4, 5]. During the last decade, upper and lower probability theory has received increasing attention, and interesting applications have been reported. See [6] for a detailed overview of imprecise reliability and many references. It is widely accepted that, by generalizing precise probability theory in a mathematically sound manner, with clear axioms and interpretations, this theory provides a better approach to generalized uncertainty quantification then its current alternatives.

In classical theory, a single probability $P(A) \in [0, 1]$ is used to quantify uncertainty about event A. For statistics, probability requires an interpretation, the most common ones are in terms of 'relative frequencies' or 'subjective fair prices for bets'. Theory of lower and upper probabilities [3, 4, 5] generalizes probability by using with lower probability $\underline{P}(A)$ and upper probability $\overline{P}(A)$ such that $0 \leq \underline{P}(A) \leq \overline{P}(A) \leq 1$. The classical situation, so-called 'precise probability', occurs if $\underline{P}(A) = \overline{P}(A)$, whereas $\underline{P}(A) = 0$ and $\overline{P}(A) = 1$ represents complete lack of knowledge about A. This generalization allows indeterminacy about A to be taken into account, and lower and upper probabilities can also be interpreted in several ways. One can consider them as bounds for a precise probability, related to relative frequency of the event A, reflecting the limited information one has about A. Alternatively, from subjective perspective the lower (upper) probability can be interpreted as the maximum (minimum) price for which one would actually wish to buy (sell) the bet which pays 1 if A occurs and 0 if not. Generally, $\underline{P}(A)$ reflects the information and beliefs in favour of event A, while $\overline{P}(A)$ reflects such information and beliefs against A. Walley [3], from the subjective point of view, uses coherence arguments to develop theory for lower and upper probabilities and related statistical methods. His theory generalizes the Bayesian statistical theory for precise probabilities in a manner similar to 'robust Bayesian methods' [7], where sets of prior distributions are used and data is taken into account via a generalized version of Bayes' theorem. Important properties are that lower (upper) probability is superadditive (subadditive), i.e. $\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B)$ ($\overline{P}(A \cup B) \leq \overline{P}(A) + \overline{P}(B)$) for disjoint A and B, and $\underline{P}(A) = 1 - \overline{P}(\bar{A})$, with $\bar{A}$ the complementary event to A [3, 4, 5]. An advantage of lower and upper probability is that one only requires limited input with regard to the uncertainties about the events of interest, and one can (in principle) always derive corresponding lower and upper probabilities for all events, via 'natural extension' [3], if the inputs are not contradictory. This is attractive as one normally can only meaningfully assess a limited number of characteristics of random quantities. Computation of the lower and upper probabilities according to the natural extension might be complicated, as constrained optimisation problems must be solved. We discuss this topic in Section 2.

Walley [3] discussed many reasons why precise probability is too restrictive for practical uncertainty quantification. In reliability, the most important ones include limited knowledge and information about random quantities of interest, and possibly information from several sources which might appear to be conflicting if restricted to precise probabilities. Common aspects such as grouped data and censoring can be dealt with naturally via imprecision, and imprecise reliability offers attractive methods for inference in case data contain zero failures.

In recent years, there has been increasing research into statistical methods using lower and upper probabilities. Following Walley [3], the majority of this work has been on robust-Bayes-like inferences, where sets of prior distributions are used for a parametric model. In particular Walley's Imprecise Dirichlet Model (IDM) [8], for multinomial inferences, is proving popular, though it is not undisputed (see the discussion to [8]). Some results on the application of the IDM to reliability problems are discussed in Section 3. An alternative statistical approach has been developed by Coolen, together with several co-authors [9, 10, 11]. It is called 'Nonparametric Predictive Inference' (NPI), and is explicitly aimed at only making limited modelling assumptions in addition to available data. Some NPI results in reliability are also discussed in Section 3. In Section 4 we discuss some research challenges for imprecise reliability.

2 Imprecise reliability via natural extension

Most of traditional reliability theory concerns analysis of systems, with probability distributions assumed to be known precisely. An attractive generalization provided by imprecise reliability theory enables such analyses under partial knowledge of probability distributions. In this approach, only some characteristics of probability distributions are assessed, or possibly even only bounds for such characteristics, and bounds are calculated for the inference of interest, which are the sharpest bounds consistent with the information provided. This process is called 'natural extension' [3], and involves constrained optimisation problems, where the inference of interest is the function to be maximised and minimised, to provide upper and lower bounds for the inference, respectively, and where all information available is formulated via constraints on the set of possible probability distributions. From this perspective, many basic problems in reliability have already been studied, a detailed recent overview with many references is given in [6]. Utkin [12, 13] considered system reliability, and provided algorithms for the optimisation problems. Example 1 [6] is a simple example of the possibilities of imprecise reliability in this context.

Example 1. [6]
Consider a series system consisting of two components, where only the following information about the reliability of the components is available. The probability of the first component to fail before 10 hours is 0.01. The mean-time-to-failure of the second component is between 50 and 60 hours. We want to draw inferences on the probability of system failure after 100 hours. The information provided does not suffice to deduce unique precise probability distributions on the times to failure of the components, so standard reliability methods which use precise probabilities would require additional assumptions. However, this information does restrict the possible probability distributions for these components' times to failure, and imprecise reliability now enables the inference in terms of the sharpest bounds for the probability of system failure after 100 hours, given that the probability distributions of the components' times to failure satisfy the constraints following from the information available. The information provided is extremely limited, and the corresponding bounds on the inferences are that the system will experience its failure only after at least 100 hours with lower probability 0 and upper probability 0.99. Suppose now, however, that we add one more piece of information, namely that the failure times of the two components are statistically independent. The upper probability of system failure after at least 100 hours now becomes 0.59. The corresponding lower probability remains 0, which agrees with intuition, as the information on both components does not exclude that either one of them might fail before 100 hours with probability one.

Although very basic, such an approach is of great value. It can give answers to many questions of interest, on the basis of whatever information is available. If the answer is considered to be too imprecise, then additional assumptions or information are needed. It then provides insight on the effect of the extra information on the upper and lower bounds of the inference. This approach also indicates if information or assumptions are conflicting, as no probability distributions would exist that satisfy all constraints.

In such cases, one has to reconsider the available information and assumptions. Utkin, partly in collaboration with several co-authors, has presented theory and methodology for many such imprecise reliability inferences for systems, including topics on monotone systems, multi-state and continuum-state systems, repairable systems, and reliability growth for software systems [6]. Further topics in imprecise reliability where natural extension provides exciting opportunities for inference under limited information include the use of expert judgements, and a variety of topics in risk analysis and decision support [6]. There are still many research topics that need to be addressed before a major impact on large-scale applications is possible.

3 Statistical inference in imprecise reliability

Walley [3] developed statistical theory based on lower and upper probabilities, where mostly the models suggested are closely related to Bayesian statistical methods with sets of prior distributions instead of single priors. An increasingly popular example is Walley's Imprecise Dirichlet Model (IDM) [8] for multinomial observations. The multinomial distribution provides a standard model for statistical inference if a random quantity can belong to any one of $k \geq 2$ different categories. In the Bayesian statistical framework [26], Dirichlet distributions are natural conjugate priors for the multinomial model, so corresponding posteriors are also Dirichlet distributions. A convenient notation is as follows. Let exchangeable random quantities X_i have a multinomial distribution with $k \geq 2$ categories $C_1, \ldots, C_k$ and parameter $\theta = (\theta_1, \ldots, \theta_k)$, with $\theta_j \geq 0$ and $\sum_j \theta_j = 1$, then $P(X \in C_j | \theta) = \theta_j$. A Dirichlet distribution with parameters $s > 0$ and $t = (t_1, \ldots, t_k)$, where $0 < t_j < 1$ and $\sum_j t_j = 1$, for the k-dimensional parameter θ, has probability density function (pdf) $p(\theta) \propto \prod_{j=1}^k \theta_j^{st_j - 1}$. Suppose that data are available, consisting of the categories to which the random quantities $X_1, \ldots, X_n$ belong, and that n_j of these belong to category C_j, with $n_j \geq 0$ and $\sum_j n_j = n$. The likelihood function corresponding to these data is $L(\theta | n_1, \ldots, n_k) \propto \prod_{j=1}^k \theta_j^{n_j}$. Bayesian updating leads to the posterior pdf

$$p(\theta | n_1, \ldots, n_k) \propto \prod_{j=1}^{k} \theta_j^{n_j + st_j - 1} \tag{1}$$

This is again a Dirichlet distribution, with compared to the prior, s replaced by $n + s$ and t_j replaced by $\frac{n_j + st_j}{n+s}$.

Walley [8] introduced the Imprecise Dirichlet Model (IDM) as follows. For fixed

$s > 0$, define $D(s)$ as the set of all Dirichlet(s, t) distributions, so the k-vector t varies over all values with $t_j > 0$ and $\sum_j t_j = 1$. The set of posterior distributions corresponding to $D(s)$, in case of data $(n_1, \dots, n_k)$ with $\sum_j n_j = n$, is the set of all Dirichlet distributions with pdf (1), with $t_j > 0$ and $\sum_j t_j = 1$. For any event of interest, the lower and upper probabilities according to the IDM are derived as the infimum and supremum, respectively, of the probabilities for this event corresponding to Dirichlet distributions in this set of posteriors. For example, if one is interested in the event that the next observation, X_{n+1}, belongs to C_j, then the IDM gives lower probability $\underline{P}(X_{n+1} \in C_j | n_1, \dots, n_k) = \frac{n_j}{n+s}$ and upper probability $\overline{P}(X_{n+1} \in C_j | n_1, \dots, n_k) = \frac{n_j+s}{n+s}$. The parameter s must be chosen independently of the data, Walley [8] advocates the use of $s = 1$ or $s = 2$, based on agreement of inferences with frequentist statistical methods.

The most obvious relevance of the IDM for reliability is with $k = 2$, when it reduces to the Binomial distribution with a set of Beta priors [3]. Binomial data often occur in reliability when success-failure data are recorded, for example when the number of faulty units over a given period of time is of interest. In case of system reliability, one often records data at all relevant levels (components, sub-systems, system) in terms of success or failure to perform their task. Hamada, *et al.* [14], showed how, in a standard Bayesian approach with precise probabilities, Binomial distributions with Beta priors at component level can be used, for a given system structure (e.g. expressed via a fault tree), together with failure data gathered at different levels. For example, some observations would just be 'system failure', without detailed knowledge of which component(s) caused the failure. Such multi-level data incur dependencies between the parameters of the components' failure distributions. Hamada, *et al.* [14], showed how modern computational methods (MCMC) can be used for such inferences. Researchers have recently attempted to combine this approach with the IDM. Wilson, *et al.* [15], applied the IDM to the same setting with multi-level failure data, although they expanded the binary approach from [14] to data with three categories ('failure', 'degraded', 'no failure'). They solve the computational problem, which is far more complex under imprecision then when precise probabilities are used, by brute force, as they perform multiple MCMCs, by first creating a fine grid over the space of prior distributions in the IDM, then running MCMC for each to derive the corresponding posterior probabilities, and finally computing the bounds over these. This only derives an upper (lower) bound for the lower (upper) probability, but if the MCMCs have been run long enough to ensure good convergence, and very many have been run, one is confident that such a brute method provides good approximations to the lower and upper probabilities. This indicates a major research challenge for imprecise probabilistic statistical inference, namely fast algorithms for computation. It is likely that optimisation methods can be combined with simulation-based methods like MCMC in a way that requires less computational effort than the method presented in [15].

Troffaes and Coolen [16] generalized the approach from [14] differently. First, restricting to $k = 2$, they analytically derive the IDM-based upper and lower probabilities for system and component failure for a very small system (two components) and multi-level failure data. Then, they delete the usual assumption of independence of the two components, and show how the IDM can be used for inference without any assumption on such independence, and also if there may be unknown bias in data collecting and reporting. These are situations which cannot be solved with precise probabilities without additional assumptions. Other reliability methods and applications of IDM include inference for survival data including right-censored observations [17] and reliability analysis of multi-state and continuum-state systems [18].

Imprecise probability enables inferential methods based on few mathematical assumptions if data are available. Coolen, with a number of co-authors, has developed Nonparametric Predictive Inference (NPI), where inferences are directly on future observable random quantities, e.g. the random time to failure of the next component to be used in a system. In this approach, imprecision depends in an intuitively logical way on the available data, as it decreases if information is added, yet aspects as censoring or grouping of data result in an increase of imprecision. Foundations of NPI, including proofs of its consistency in theory of interval probability, are presented in [9], an overview with detailed comparison to so-called 'objective Bayesian methods' is given in [10]. A first introductory overview of NPI in reliability is presented in [11], and theory for dealing with right-censored observations in NPI in [19, 20]. This framework is also suitable for guidance on high reliability demonstration, considering how many failure-free observations are required in order to accept a system in a critical operation [21]. The fact that, in such situations, imprecise reliability theory allows decisions to be based on the more pessimistic one of the lower and upper probabilities, e.g. lower probability of failure-free operation, is an intuitively attractive manner for dealing with indeterminacy. Recently, Coolen also considered probabilistic safety assessment from similar perspective [22]. NPI can also be used to support replacement decisions for technical equipment [23], giving decision support methods which are fully adaptive to failure data, and with imprecision reflected in bounds of cost functions. NPI can provide clear insights into the influence of a variety of assumptions which are often used for more established methods, and which may frequently be rather unrealistic if considered in detail. The fact that NPI can do without most of such assumptions and still be useful under reasonable data requirement is interesting.

Recent developments of NPI in reliability include the use of a new method for multinomial data for inferences on the possible occurrence of new failure modes [24] (see Example 3), and comparison of groups of success-failure data with specific attention to groups with zero failures [25]. To illustrate the use of NPI in reliability, we include two examples with references for further details.

Example 2. [11]
Studying 400 pumps in eight pressurized water reactors in commercial operation in the United States in 1972, 6 pumps failed to run normally for that whole year [26]. If we assume that all pumps are replaced preventively after one year, for example to avoid failure due to wear-out, and that the pumps and relevant process factors remain similar, then it may be relevant to consider NPI for the number of failing pumps out of 400 during the following year, giving the lower and upper probabilities presented in Table 1 (upper probabilities for not-reported r-values are less than 0.01).

r	$\underline{P}(r)$	$\overline{P}(r)$	r	$\underline{P}(r)$	$\overline{P}(r)$
0	0.0076	0.0153	9	0.0308	0.1533
1	0.0230	0.0541	10	0.0230	0.1222
2	0.0406	0.1088	11	0.0167	0.0939
3	0.0545	0.1641	12	0.0118	0.0700
4	0.0616	0.2059	13	0.0081	0.0508
5	0.0618	0.2270	14	0.0054	0.0359
6	0.0568	0.2273	15	0.0036	0.0249
7	0.0488	0.2111	16	0.0023	0.0169
8	0.0396	0.1844	17	0.0015	0.0113

Table 1: Lower and upper probabilities for r out of 400 pumps failing.

Such lower and upper probabilities provide insight into the evidence in favour and against these particular events. Lower and upper probabilities for a subset of possible values cannot be derived by adding the corresponding values in Table 1, due to the superadditivity (sub-) of lower (upper) probabilities, see [27] for more general results. This approach has the advantage over Bayesian methods [26] that no prior distributions are required, and their properties are such that they combine attractive features from both Bayesian and frequentist statistics [10].

Example 3. [24]
Suppose that a database contains detailed information on failures experienced during warranty periods of a particular product. Currently 200 failures have been recorded, with 5 different failure modes specified. The producer is interested in the event that the next failure of such a product during its warranty period is caused by another failure mode than these 5 already recorded. Let us first assume that there is no clear assumption or knowledge about the number of possible failure modes. Suppose that interest is in the event that the next reported failure is caused by any as yet unseen failure mode, then the NPI upper probability for this event is equal to 5/200 [24]. If, however, the producer has actually specified two further possible failure modes, which have not yet been recorded so far, and interest is in the event that the next failure mode will be one of these two, then the NPI method gives upper probability 2/200. This holds for any pair of such possible new failure modes that one wishes to specify, without restrictions on their number: this is possible due to the subadditivity of upper probability, which is an advantage over the use of precise probabilities. The NPI lower probabilities for both these events are 0, reflecting no strong evidence in the data that new failure modes can actually occur.

Now suppose that these 200 failures were instead caused by 25 different failure modes. Then the upper probability for the next failure mode to be any as yet unseen failure mode changes to 25/200, but the upper probability for it to be either of any two described new failure modes remains 2/200. It is in line with intuition that the changed data affect this first upper probability, as the fact that more failure modes have been recorded suggests that there may be more failure modes. For the second event considered, the reasoning is somewhat different, as effectively interest is in two specific, as yet unseen, failure modes, and there is no actual difference in the data available that is naturally suggesting that either of these two failure modes has become more likely. If the producer has specific knowledge on the number of possible failure modes, that can also affect these upper probabilities [24].

4 Challenges for research and application

Imprecise probabilistic methods in reliability and risk assessment have clear practical advantages over more established and more restricted theory with precise probabilities. By not requiring or assuming perfect information, imprecise reliability acknowledges the limits on available knowledge and on time for assessment of uncertainties, and reduces the need for additional assumptions required for precise models, the effects of which are often hidden. Due to the possible interpretations of lower and upper probabilities, they are often convenient for practical risk assessment, as they reflect both 'pessimistic' and 'optimistic' inference. It is often clear on which of these to base decisions, for example one would focus on the upper probability of an accident to occur.

Imprecise reliability is still at an early stage of development [6, 30], as are general theory and applications of lower and upper probabilities. There are many research challenges, ranging from foundational aspects to development of implementation tools. Methods that use natural extension have been presented in general terms, with no specific restrictions to scale of problems that can be solved. However, upscaling from 'text-book style' examples to larger scale problems provides many challenges, not only for computational methods but also for effective elicitation of expert judgement. Combination of information from different sources must also be considered within the imprecise reliability framework. At the foundational level, further study of the relation between imprecision and (amount of) information is required. For example, the use of robust-Bayes type models in imprecise reliability seems logical, and was strongly advocated by Walley [3] from seemingly natural axioms of coherence. However, those axioms enable comparison of gambles over different moments in time, based on different information, and the consequential theory is similar in nature to precise Bayesian theory, implying that posterior (lower and upper) probabilities are equal to prior conditional (lower and upper) probabilities. It is not clear that this provides a universally acceptable framework for learning, with imprecision related to information available. An early approach that took a different view on this issue was presented in [28] within the context of reliability. There, the set of distributions used is reduced when more information becomes available, via an additional parameter which links imprecision to information. Further research on this topic could provide new insights and methods for inference.

There is huge scope for new models and methods in imprecise reliability, and further development of some methods is required. For example, Utkin and Gurov [29] presented attractive classes of lifetime distributions, with $H(r, s)$, for $0 \leq r \leq s$, the class of all lifetime distributions with cumulative hazard function $\Lambda(t)$ such that $\Lambda(t)/t^r$ increases and $\Lambda(t)/t^s$ decreases in t. For example, $H(1, \infty)$ are all distributions with increasing hazard rates. Some results for related inference have been presented [29], but interesting research problems are still open, including how to fit such classes to available data. For most methods imprecision quickly becomes very large once the size of the problem increases, if only limited information is available. A possible solution is development of classes of distributions which are restricted with regard to properties such as smoothness or tail behaviour. The IDM and NPI have shown promising opportunities for applications in reliability, but statistical theory must be developed further to enhance applicability, for example on the use of covariates and generalization to multivariate data.

References

[1] Cai, K.Y. 1996. *Introduction to Fuzzy Reliability.* Boston: Kluwer Academic Publishers.

[2] Dubois, D. & Prade, H. 1988. *Possibility Theory: An Approach to Computerized Processing of Uncertainty.* New York: Plenum Press.

[3] Walley, P. 1991. *Statistical Reasoning with Imprecise Probabilities.* London: Chapman and Hall.

[4] Weichselberger, K. 2000. The theory of interval-probability as a unifying concept for uncertainty. *International Journal of Approximate Reasoning* 24: 149-170.

[5] Weichselberger, K. 2001. *Elementare Grundbegriffe einer Allgemeineren Wahrscheinlichkeitsrechnung I. Intervallwahrscheinlichkeit als Umfassendes Konzept.* Heidelberg: Physika.

[6] Utkin, L.V. & Coolen, F.P.A. 2007. Imprecise reliability: an introductory overview. In: *Computational Intelligence in Reliability Engineering*, Volume 2: *New Metaheuristics, Neural and Fuzzy Techniques in Reliability*, Gregory Levitin (Ed.), Springer: Chapter 10, 261-306.

[7] Berger, J.O. 1990. Robust Bayesian analysis: sensitivity to the prior. *Journal of Statistical Planning and Inference* 25: 303-328.

[8] Walley, P. 1996. Inferences from multinomial data: learning about a bag of marbles (with discussion). *Journal of the Royal Statistical Society B* 58: 3-57.

[9] Augustin, T. & Coolen, F.P.A. 2004. Nonparametric predictive inference and interval probability. *Journal of Statistical Planning and Inference* 124: 251-272.

[10] Coolen, F.P.A. 2006. On nonparametric predictive inference and objective Bayesianism. *Journal of Logic, Language and Information 15*, 21-47.

[11] Coolen, F.P.A., Coolen-Schrijner, P. & Yan, K.J. 2002. Nonparametric predictive inference in reliability. *Reliability Engineering and System Safety* 78: 185-193.

[12] Utkin, L.V. 2004. Interval reliability of typical systems with partially known probabilities. *European Journal of Operational Research* 153: 790-802.

[13] Utkin, L.V. 2004. A new efficient algorithm for computing the imprecise reliability of monotone systems. *Reliability Engineering and System Safety* 86: 179-190.

[14] Hamada, M.S., Martz, H., Reese, C.S., Graves, T., Johnson, V. & Wilson A.G., A fully Bayesian approach for combining multilevel failure information in fault tree quantification and optimal follow-on resource allocation. *Reliability Engineering and System Safety* 86: 297-305.

[15] Wilson, A.G., Huzurbazar, A.V. & Sentz, K. 2007. The Imprecise Dirichlet Model for multilevel system reliability. *In submission.*

[16] Troffaes, M. & Coolen, F.P.A. 2006. Applying the Imprecise Dirichlet Model in cases with partial observations and dependencies in failure data. *In submission.*

[17] Coolen, F.P.A. 1997. An imprecise Dirichlet model for Bayesian analysis of failure data including right-censored observations. *Reliability Engineering and System Safety* 56: 61-68.

[18] Utkin, L.V. 2006. Cautious reliability analysis of multi-state and continuum-state systems based on the imprecise Dirichlet model. *International Journal of Reliability, Quality and Safety Engineering* 13: 433-453.

[19] Coolen, F.P.A. & Yan, K.J. 2004. Nonparametric predictive inference with right-censored data. *Journal of Statistical Planning and Inference* 126: 25-54.

[20] Coolen, F.P.A. & Yan, K.J. 2003. Nonparametric predictive inference for grouped lifetime data. *Reliability Engineering and System Safety* 80: 243-252.

[21] Coolen, F.P.A. & Coolen-Schrijner, P. 2005. Nonparametric predictive reliability demonstration for failure-free periods. *IMA Journal of Management Mathematics* 16: 1-11.

[22] Coolen, F.P.A. 2006. On probabilistic safety assessment in case of zero failures. *Journal of Risk and Reliability* 220: 105-114.

[23] Coolen-Schrijner, P. & Coolen, F.P.A. 2004. Adaptive age replacement based on nonparametric predictive inference. *Journal of the Operational Research Society* 55: 1281-1297.

[24] Coolen, F.P.A. 2006. The occurrence of not yet observed failure modes. In: *Safety and Reliability for Managing Risk* (Proceedings of ESREL'06), eds: C. Guedes Soares, E. Zio. Taylor & Francis, London, pp. 881-888.

[25] Coolen-Schrijner, P. & Coolen, F.P.A. 2007. Nonparametric predictive comparison of success-failure data in reliability. *Proceedings 17th ARTS*, Loughborough (UK), April 2007, to appear.

[26] Martz, H.F. & Waller, R.A. 1982. *Bayesian Reliability Analysis.* New York: Wiley.

[27] Coolen, F.P.A. 1998. Low structure imprecise predictive inference for Bayes' problem. *Statistics & Probability Letters* 36: 349-357.

[28] Coolen, F.P.A. & Newby, M.J. 1994. Bayesian reliability analysis with imprecise prior probabilities. *Reliability Engineering and System Safety* 43: 75-85.

[29] Utkin, L.V. & Gurov, S.V. 2002. Imprecise reliability for the new lifetime distribution classes. *Journal of Statistical Planning and Inference* 105: 215-232.

[30] Coolen, F.P.A. 2004. On the use of imprecise probabilities in reliability. *Quality and Reliability Engineering International* 20: 193-202.