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Nonparametric predictive inference for system reliability using the survival signature

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Abstract

The survival signature has recently been presented as an attractive concept to aid quantification of system reliability. It has similar characteristics as the system signature, which is well established, but contrary to the latter it is easily applicable to systems with multiple types of components. We present an introductory overview of the survival signature together with new results to aid computation. We develop nonparametric predictive inference for system reliability using the survival signature. The focus is on the failure time of a system, given failure times of tested components of the same types as used in the system.

Keywords: coherent systems, exchangeable components, lower and upper survival functions, nonparametric predictive inference, signature, survival signature, system reliability.

1. Introduction

In recent decades, the signature has become a popular tool for quantifying reliability of coherent systems consisting of components with exchangeable random failure times [36], where in the literature the assumption of exchangeability [25] is often replaced by the stronger assumption of independent and identically distributed (*iid*) component failure times. The signature can be

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used to quantify aspects of reliability of a system such as its failure time distribution. An attractive feature of describing system structures through signatures is the possibility to compare the reliability of different systems based on stochastic ordering of their signatures, as long as the failure times of all components in these systems are exchangeable [33]. A detailed introduction and overview to system signatures is presented by Samaniego [36], some recent advances are reviewed by Eryilmaz [26].

Consider a system consisting of m components with exchangeable failure times. Let the random failure time of the system be T_S, and let $T_{j:m}$ be the j-th order statistic of the m random component failure times for $j = 1, \ldots, m$, with $T_{1:m} \leq T_{2:m} \leq \ldots \leq T_{m:m}$. The system's signature is the m-vector q with j-th component

$$q_j = P(T_S = T_{j:m}) \tag{1}$$

so q_j is the probability that the system failure occurs at the moment of the j-th component failure. It is natural to assume that $\sum_{j=1}^{m} q_j = 1$, so the system functions if all components function, has failed if all components have failed, and system failure can only occur at times of component failures. The survival function of the system failure time can be derived by

$$P(T_S > t) = \sum_{j=1}^{m} q_j P(T_{j:m} > t) \tag{2}$$

If the components' failure times are *iid* with known cumulative distribution function $F(t)$, then

$$P(T_{j:m} > t) = \sum_{r=m-j+1}^{m} \binom{m}{r} [1 - F(t)]^r [F(t)]^{m-r} \tag{3}$$

The essential property of the system signature is that it enables information of the system structure to be fully taken into account through the signature, and this is separated from information about the random failure times of the components. The main disadvantage of system signatures, however, is that it becomes extremely complicated, and is indeed effectively impossible, to keep this separation when generalizing the concept to systems with multiple types of components, which is crucial for a practically applicable theory as most real-world systems consist of more than a single type of components [16, 34]. Such a generalization requires probabilities for orderings of order statistics from different probability distributions, corresponding

2

to the different types of components, which is difficult to implement. As an alternative to the signature, Coolen and Coolen-Maturi [16] introduced the survival signature. For systems with just one type of components, the survival signature is closely related to the signature, but the survival signature can be defined for, and easily applied to, systems with multiple types of components.

Section 2 of this paper provides a brief introductory overview of the survival signature. In Section 3 we present some new results that are of use for computation of the survival signature. In Section 4 we present nonparametric predictive inference for the system failure time, using the survival signature and data existing of failure times of components of the same types as those in the system. Section 5 illustrates these results via examples, where also the use of partially known survival signatures is discussed. Section 6 concludes the paper with a discussion of some related research challenges.

2. The survival signature

For a system with m components, we define the state vector $\underline{x} \in \{0,1\}^m$ with entry $x_i = 1$ if the ith component functions and $x_i = 0$ if not. The labelling of the components is arbitrary but must be fixed to define $\underline{x}$. The structure function $\phi : \{0,1\}^m \to \{0,1\}$, defined for all possible $\underline{x}$, takes the value 1 if the system functions and 0 if the system does not function for state vector $\underline{x}$. In this paper, we restrict attention to coherent systems, which means that $\phi(\underline{x})$ is not decreasing in any of the components of $\underline{x}$, so system functioning cannot be improved by worse performance of one or more of its components. We further assume that $\phi(\underline{0}) = 0$ and $\phi(\underline{1}) = 1$, so the system fails if all its components fail and it functions if all its components function. These assumptions could be relaxed but are reasonable for most practical systems, and they simplify the presentation in this paper.

Consider a system with $K \geq 2$ types of components, with m_k components of type $k \in \{1, 2, \ldots, K\}$ and $\sum_{k=1}^{K} m_k = m$. Assume that the random failure times of components of the same type are exchangeable [25], while full independence is assumed for the random failure times of components of different types. Due to the arbitrary ordering of the components in the state vector, components of the same type can be grouped together, leading to a state vector that can be written as $\underline{x} = (\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^K)$, with $\underline{x}^k =$

$(x_1^k, x_2^k, \ldots, x_{m_k}^k)$ the sub-vector representing the states of the components of type k.

Coolen and Coolen-Maturi [16] introduced the *survival signature* for such a system, denoted by $\Phi(l_1, l_2, \ldots, l_K)$, with $l_k = 0, 1, \ldots, m_k$ for $k = 1, \ldots, K$, which is defined to be the probability that the system functions given that *precisely* l_k of its m_k components of type k function, for each $k \in \{1, 2, \ldots, K\}$.

There are $\binom{m_k}{l_k}$ state vectors $\underline{x}^k$ with $\sum_{i=1}^{m_k} x_i^k = l_k$; let S_l^k denote the set of these state vectors for components of type k and let $S_{l_1, \ldots, l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$, $k = 1, 2, \ldots, K$. Due to the exchangeability assumption for the failure times of the m_k components of type k, all the state vectors $\underline{x}^k \in S_l^k$ are equally likely to occur, hence

$$\Phi(l_1, \ldots, l_K) = \left[\prod_{k=1}^K \binom{m_k}{l_k}^{-1} \right] \times \sum_{\underline{x} \in S_{l_1, \ldots, l_K}} \phi(\underline{x}) \tag{4}$$

Let $C_k(t) \in \{0, 1, \ldots, m_k\}$ denote the number of components of type k in the system which function at time $t > 0$. The probability that the system functions at time $t > 0$ is

$$P(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \ldots, l_K) P(\bigcap_{k=1}^K \{C_k(t) = l_k\}) \tag{5}$$

Assumed independence of the failure times of components of different types leads to, for $l_k \in \{0, 1, \ldots, m_k\}$ for each $k \in \{1, \ldots, K\}$,

$$P(\bigcap_{k=1}^K \{C_k(t) = l_k\}) = \prod_{k=1}^K P(C_k(t) = l_k)$$

The additional assumption of *iid* failures times of components of the same type with known CDF $F_k(t)$ for type k leads to

$$P(\bigcap_{k=1}^K \{C_k(t) = l_k\}) = \prod_{k=1}^K \binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k}$$

For the special case of a system consisting of only a single type of components, so with all m components exchangeable and $K = 1$ in the notation

4

above, the survival signature is linked to the signature through the following equality [16]

$$\Phi(l_1) = \sum_{j=m-l_1+1}^{m} q_j \tag{6}$$

This is logical, as the right-hand side is simply the probability that the system has not yet failed when l_1 of its m components function. This equality implies that, for this rather restrictive situation, the survival signature and the signature are effectively the same with just the order of summation in Equation (2) together with Equation (3) exchanged. This implies that all attractive properties of the signature also hold for the method using the survival signature [16], yet the survival signature is also easy to apply for systems with multiple types of components, and one could argue that it is easier to interpret than the signature.

The survival signature $\Phi(l_1, \ldots, l_K)$ must be derived for all $\prod_{k=1}^{K}(m_k+1)$ different $(l_1, \ldots, l_K)$. This information is anyhow required if one wishes to assess a system's reliability. The survival signature only has to be calculated once for any system, similar to the signature for systems with a single type of components. The main advantage of Equation (5) is that, as for the signature in case of systems with only one type of components, the information about system structure is fully separated from the information about the components' failure times. In Section 5 several examples of survival signatures are presented.

Coolen and Coolen-Maturi [16] presented two ways for comparison of reliability of two systems using the survival signature, but only restricted to systems with one type of components. If the two systems have the same single type of components, it might be possible to base comparison directly on the entries of the survival signatures, because if one dominates the other it directly corresponds to stochastic dominance of the corresponding signatures. If these systems do not have the same number of components, it is possible to extend the survival signature for the system with the smallest number of components to create a survival signature with the same number of entries as for the larger system, while still representing the same system failure time distribution. This is in line with a well-known method for the signature [36]. Perhaps a more attractive comparison of the random failure times T_A and T_B of systems A and B is by considering the probability for events $T_A > T_B + \delta$, which as function of δ allow detailed insight into the difference between these failure times. This can be used whether or not the two systems

have the same single type of components. It is relatively straightforward to generalize these two methods to compare two systems with multiple types of components. It follows the same reasoning and main mathematical steps as the results derived by Coolen and Coolen-Maturi [16], just with more complicated notation due to multiple types of components. We do not present this in detail here.

3. New results for survival signatures

Computation of the survival signature is complicated for systems of realistic size. In this section we briefly present two results that can simplify computation in specific situations. It may not be needed to compute a system's survival signature exactly for a specific inference, as bounds resulting from partial computations may be sufficient, similar to the use of bounds for signatures as presented by Al-nefaiee and Coolen [5]. As the survival signature of a coherent system is non-decreasing in all its components, the use of such bounds is pretty straightforward; we illustrate this briefly in Example 5 in Section 5, for more details see Al-nefaiee [4].

Gaofeng et al. [29] showed how the signature of a system with one type of components can be derived from the signatures of two subsystems, if the system consists of these two subsystems in either series or parallel configuration. Repeated application of their method enables quite straightforward computation of the signature of any system consisting of any number of subsystems as long as the overall structure can be created through a sequence of series or parallel configurations. The same idea can be used to compute the survival signature of a system consisting of two subsystems in either series or parallel configuration; by repeated use this enables the survival signatures for quite a substantial range of systems to be computed relatively easily.

Suppose that a system consists of 2 subsystems for which the survival signatures are known. Let the system consist of $K \geq 1$ types of components, with m_k components of type k, for $k = 1, \ldots, K$, of which $m_k^r \geq 0$ are in subsystem r, for $r = 1, 2$. Let subsystem r consist in total of m^r components, so $m^r = \sum_{k=1}^{K} m_k^r$. We denote the survival signature for subsystem r by $\Phi^r(l_1^r, l_2^r, \ldots, l_K^r)$, for $l_k^r = 0, 1, \ldots, m_k^r$. For ease of notation, we define $\Phi^r(l_1^r, l_2^r, \ldots, l_K^r) = 0$ if $l_k^r > m_k^r$ for any $k \in \{1, \ldots, K\}$. If the two subsystems are in series configuration, then the survival signature of the system

6

can be derived, for $l_k \in \{0, 1, \ldots, m_k\}$, $k = 1, \ldots, K$, by

$$\Phi(l_1, \ldots, l_K) = \sum_{l_1^1=0}^{l_1} \cdots \sum_{l_K^1=0}^{l_K} \left[\Phi^1(l_1^1, \ldots, l_K^1) \Phi^2(l_1 - l_1^1, \ldots, l_K - l_K^1) \times \right.$$
$$\left. \prod_{k=1}^{K} \binom{m_k^1}{l_k^1} \binom{m_k^2}{l_k - l_k^1} \binom{m_k}{l_k}^{-1} \right] \qquad (7)$$

Similarly, if the two subsystems are in parallel configuration, then the survival signature of the system can be derived, for $l_k \in \{0, 1, \ldots, m_k\}$, $k = 1, \ldots, K$, by

$$\Phi(l_1, \ldots, l_K) =$$
$$\sum_{l_1^1=0}^{l_1} \cdots \sum_{l_K^1=0}^{l_K} \left[\{ 1 - (1 - \Phi^1(l_1^1, \ldots, l_K^1))(1 - \Phi^2(l_1 - l_1^1, \ldots, l_K - l_K^1)) \} \times \right.$$
$$\left. \prod_{k=1}^{K} \binom{m_k^1}{l_k^1} \binom{m_k^2}{l_k - l_k^1} \binom{m_k}{l_k}^{-1} \right] \qquad (8)$$

These results follow from straightforward combinatorial arguments, using the hypergeometric distribution for the probability that precisely l_k^1 of the l_k functioning components of type k are among the m_k^1 components of this type in subsystem 1, and the remaining ones are among the m_k^2 components of this type in subsystem 2. This computational method will be illustrated in Example 2 in Section 5.

In many situations where reliability of systems is of interest, activities such as maintenance or replacement of components are important. If one component of type k is replaced, the failure time distribution of the new component will typically differ from those components in the system that were of the same type before this replacement (unless the component's failure time is assumed to be Exponentially distributed, in which case the replaced component might still be considered to be of the same type as before). So, such an activity will, effectively, imply that there is a new type of component in the system, say type $K+1$, and that the number of components of its earlier type k has been reduced from m_k to $m_k - 1$. We now consider the effect of such a component replacement on the system's survival signature. We restrict

attention to replacement of a single component; this can be generalized quite easily, following the same principles and using the hypergeometric distribution for the weighting probabilities, to replacement of multiple components of the same type, while in case of replacement of multiple components of different types the result below can be used sequentially, for one component at a time.

Let $\Phi(l_1, \ldots, l_{k-1}, l_k, l_{k+1}, \ldots, l_K)$ be the survival signature for a system with K types of components, with $l_k \in \{0, 1, \ldots, m_k\}$ for $k = 1, \ldots, K$. Now suppose that one component of type k is replaced by a component of a new type, say type $K + 1$. This may really be a new type of component, or just similar to the one that is being replaced but with a different age, hence its failure time distribution over its remaining time till failure is not identical to that of the other components in the system. We must calculate the survival signature of this system with $K + 1$ types of components, which we denote by $\tilde{\Phi}(l_1 \ldots, l_{k-1}, \tilde{l}_k, l_{k+1}, \ldots, l_K, \tilde{l}_{K+1})$, where the tilde is added to emphasize a change compared to the survival signature of this system before the component was replaced.

The numbers m_j of components of types $j \neq k$ remain the same as before the replacement, and now there are $\tilde{m}_k = m_k - 1$ components of type k and $\tilde{m}_{K+1} = 1$ component of the new type $K + 1$. So the new survival signature $\tilde{\Phi}(l_1 \ldots, l_{k-1}, \tilde{l}_k, l_{k+1}, \ldots, l_K, \tilde{l}_{K+1})$ must be specified for $l_j \in \{0, 1, \ldots, m_j\}$ for $j \in \{1, \ldots, k-1, k+1, \ldots, K\}$, $\tilde{l}_k \in \{0, 1, \ldots, \tilde{m}_k\}$ and $\tilde{l}_{K+1} \in \{0, 1\}$. This specification is simplified by the following relationship,

$$
\begin{aligned}
\Phi(l_1, \ldots, l_{k-1}, l_k, l_{k+1}, \ldots, l_K) = \\
\frac{l_k}{m_k} \times \tilde{\Phi}(l_1 \ldots, l_{k-1}, l_k - 1, l_{k+1}, \ldots, l_K, 1) + \\
\frac{m_k - l_k}{m_k} \times \tilde{\Phi}(l_1 \ldots, l_{k-1}, l_k, l_{k+1}, \ldots, l_K, 0)
\end{aligned}
\tag{9}
$$

The proof of Equation (9) is based on the probability that the replaced component would be one of the l_k functioning ones out of the m_k components of type k in the original system, or one of the $m_k - l_k$ non-functioning ones. Hence, if one has the fully specified original survival signature Φ available, the computations required in order to fully specify the new survival signature $\tilde{\Phi}$ can, for example, be restricted to computing the values of $\tilde{\Phi}(l_1 \ldots, l_{k-1}, l_k - 1, l_{k+1}, \ldots, l_K, 1)$, from which the values of $\tilde{\Phi}(l_1 \ldots, l_{k-1}, l_k, l_{k+1}, \ldots, l_K, 0)$ follow by Equation (9), and together these fully specify $\tilde{\Phi}$. While this does

require new computations, the overall system structure remains the same, and attention can now be restricted to $l_k - 1$ components of type k, with the replaced component assumed to be functioning. So the computations to derive $\tilde{\Phi}$ are similar to those for Φ, which would already have been performed, but are actually easier as one component can be assumed to function. This is illustrated in Example 3 in Section 5.

4. Nonparametric predictive inference for system failure time

This section presents the use of the survival signature in nonparametric predictive inference (NPI) for the system failure time. NPI is a statistical method which gives a direct conditional probability for one or more future observable random quantities, conditional on observed values of related random quantities [8, 12, 13]. NPI can be considered suitable if there is hardly any knowledge about the random quantity of interest, other than the data which we assume consist of n observations, or if one does not want to use such further information, e.g. to study effects of additional assumptions underlying other statistical methods. NPI uses lower and upper probabilities, also known as imprecise probabilities, to quantify uncertainty [9, 19, 39, 40] and has strong consistency properties from frequentist statistics perspective [8, 13]. NPI provides a solution to some explicit goals formulated for objective (Bayesian) inference, which cannot be obtained when using precise probabilities [12], and it never leads to results that are in conflict with inferences based on empirical probabilities. Imprecise probabilities provide many exciting opportunities for reliability quantification [20, 37, 38]. The NPI method has already been used for system reliability [3, 14, 23, 31], but only for systems with quite restricted structures. NPI for system reliability using the signature has also been presented, for systems consisting of only one type of components [4, 5, 15]. NPI has also been presented for a variety of other problems in operational research and statistics, including predictive analysis for queueing problems [17], replacement problems [24], decision making under uncertain utilities [30] and classification with decision trees using maximum entropy [1, 2] (see also *www.npi-statistics.com*).

We now present NPI lower and upper survival functions for the failure time T_S of a system consisting of multiple types of components, using the system signature combined with NPI for Bernoulli data [11]. This enables the

9

NPI method to be applied to, in principle, all systems, so this methodology widely generalizes the earlier results on NPI for system reliability. The failure times of components of different types are assumed to be independent. NPI is used for learning about the components of a specific type in the system, from data consisting of failure times for components that are exchangeable with these. We assume therefore that such data are available, for example resulting from testing or previous use of such components. For $k \in \{1, \ldots, K\}$, let n_k denote the number of components of type k for which test failure data are available, and let $s_k(t)$ denote the number of these components which still function at time t. Throughout this paper we assume that the test data do not contain censored observations, a brief comment about this is included in Section 6.

The NPI lower survival function is derived as follows. Remember that $C_k(t)$ denotes the number of components of type k in the system which function at time t, where it is assumed that failure ends the functioning of a component and it is not repaired or replaced. Under the assumptions for the NPI approach [11], we derive the following lower bound for the survival function

$$P(T_S > t) \geq \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \overline{D}(C_k(t) = l_k)$$

where

$$
\begin{aligned}
\overline{D}(C_k(t) = l_k) &= \overline{P}(C_k(t) \leq l_k) - \overline{P}(C_k(t) \leq l_k - 1) \\
&= \binom{n_k + m_k}{n_k}^{-1} \binom{s_k(t) - 1 + l_k}{s_k(t) - 1} \binom{n_k - s_k(t) + m_k - l_k}{n_k - s_k(t)}
\end{aligned}
$$

In this expression, $\overline{P}$ denotes the NPI upper probability for Bernoulli data [11]. For each component type k, the function $\overline{D}$ ensures that maximum possible probability, corresponding to NPI for Bernoulli data [11], is assigned to the event $C_k(t) = 0$, so $\overline{D}(C_k(t) = 0) = \overline{P}(C_k(t) = 0)$. Then, $\overline{D}(C_k(t) = 1)$ is defined by putting the maximum possible remaining probability mass, from the total probability mass available for the event $C_k(t) \leq 1$, to the event $C_k(t) = 1$. This is achieved by $\overline{D}(C_k(t) = 1) = \overline{P}(C_k(t) \leq 1) - \overline{P}(C_k(t) = 0)$. This argument is continued, by assigning for increasing l_k the maximum possible remaining probability mass $\overline{D}(C_k(t) = l_k)$. As the survival signature is increasing in l_k for coherent systems, as assumed in this paper, and the

resulting $\overline{D}$ is a precise probability distribution, the right-hand side of the inequality above is indeed a lower bound and it is the maximum possible lower bound. As such, it is the NPI lower probability for the event $T_S > t$, giving the NPI lower survival function for the system failure time (for $t > 0$)

$$\underline{S}_{T_S}(t) = \underline{P}(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \overline{D}(C_k(t) = l_k) \quad (10)$$

The corresponding NPI upper survival function for T_S is similarly derived, using the upper bound

$$P(T_S > t) \leq \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \underline{D}(C_k(t) = l_k)$$

where

$$\begin{aligned} \underline{D}(C_k(t) = l_k) &= \underline{P}(C_k(t) \leq l_k) - \underline{P}(C_k(t) \leq l_k - 1) \\ &= \binom{n_k + m_k}{n_k}^{-1} \binom{s_k(t) + l_k}{s_k(t)} \binom{n_k - s_k(t) + m_k - l_k - 1}{n_k - s_k(t)} \end{aligned}$$

In this expression, $\underline{P}$ denotes the NPI lower probability for Bernoulli data [11]. This construction ensures that minimum possible weight is given to small values of $C_k(t)$, resulting in the NPI upper survival function for the system failure time (for $t > 0$)

$$\overline{S}_{T_S}(t) = \overline{P}(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \underline{D}(C_k(t) = l_k) \quad (11)$$

Before illustrating this NPI approach for system reliability with the use of survival signatures in Examples 4 and 5 in Section 5, it is important to comment on the special case of systems consisting of only $K = 1$ type of components. For such systems, NPI theory for the system survival time using the signature was presented by Coolen and Al-nefaiee [15]. This used NPI for future order statistics of real-valued observations [18]. It is not trivial that this leads to the same inferences as the method using the survival signature and NPI for Bernoulli quantities [11] as presented in this section.
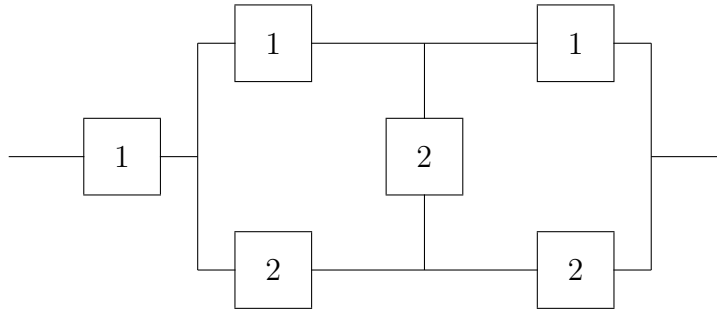
Figure 1: System with 2 types of components (Ex. 1)

However, the resulting inferences for such systems, from these two different NPI approaches, are identical, which is proven in detail by Al-nefaiee [4].

5. Examples

We present five examples involving small system structures, which are not related to actual physical systems but presented in order to illustrate the new methods that have been introduced in this paper. These examples also include some further discussions of aspects of these methods.

Example 1.
Consider the system with $K = 2$ types of components, types 1 and 2, as presented in Figure 1. The survival signature for this system is presented in Table 1, it is easily verified by checking all possible combinations of the specific components of each type which function or not. We will also use this system in Examples 3 and 4.

l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_2	$\Phi(l_1, l_2)$
0	0	0	2	0	0
0	1	0	2	1	0
0	2	0	2	2	4/9
0	3	0	2	3	6/9
1	0	0	3	0	1
1	1	0	3	1	1
1	2	1/9	3	2	1
1	3	3/9	3	3	1

Table 1: Survival signature of the system in Figure 1 (Ex. 1)

Example 2.
We illustrate the method presented in Section 3 to use the survival signatures of subsystems to compute a system's survival signature. Consider the system in Figure 2, which has three subsystems, labelled by A, B and C, and $K = 3$ types of components. The survival signatures of the three subsystems are easily derived and presented in Table 2. This table presents the survival signatures for the subsystems as functions of the numbers of components of each of these three types, even if not all types of components occur in the subsystem, which is in line with the notation introduced in Section 3.

We first use Equation (7) to derive the survival signature for the subsystem which consists of subsystems A and B in series configuration, we refer to this as subsystem AB. This leads to the survival signature presented in Table 3, where apart from $\Phi^{ab}(0, 0, 0) = 0$ all not presented $\Phi^{ab}(l_1^{ab}, l_2^{ab}, l_3^{ab})$ with $l_1^{ab} \in \{0, 1\}$, $l_2^{ab} \in \{0, 1, 2, 3\}$ and $l_3^{ab} \in \{0, 1, 2\}$ are equal to 1. We calculate the survival signature of the entire system by combining the survival signature Φ^{ab} for subsystem AB with the survival signature Φ^c for subsystem C, using Equation (8). This leads to the system's survival signature presented in Table 4, where apart from $\Phi(0, 0, 0) = 0$ all not presented $\Phi(l_1, l_2, l_3)$ with $l_1 \in \{0, 1\}$, $l_2 \in \{0, 1, 2, 3, 4\}$ and $l_3 \in \{0, 1, 2, 3, 4\}$ are equal to 1. To provide more insight into the survival signature, the values in this table are given as fractions that correspond to the number of combinations considered when deriving these values directly. We will also use this system and its survival signature in Example 5.
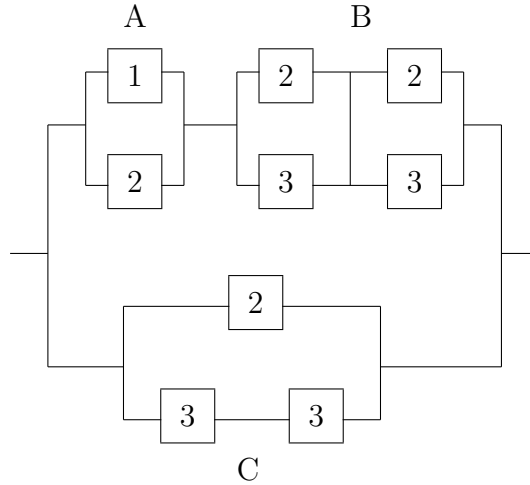
13

Figure 2: System with 3 types of components (Ex. 2)

Example 3.
To illustrate the effect of component replacement on a system's survival signature, as discussed in Section 3, consider again the system discussed in Example 1, as presented in Figure 1 with the survival signature given in Table 1. The numbers of components of types 1 and 2 in the original system are $m_1 = 3$ and $m_2 = 3$. Assume that one component of type 2 in the original system is replaced by a component of type 3, leading to the system in Figure 3. The numbers of components of types 1, 2 and 3 in the system after this component replacement are $m_1 = 3$, $\tilde{m}_2 = 2$ and $\tilde{m}_3 = 1$. The survival signature $\tilde{\Phi}$ for this system, after the component replacement, is given in Table 5. It is easy to verify that these values, together with the values of Φ in Table 1, satisfy Equation (9). For example, for $l_1 = 2$ and $l_2 = 2$, this equation becomes $\Phi(2,2) = \frac{2}{3}\tilde{\Phi}(2,1,1) + \frac{1}{3}\tilde{\Phi}(2,2,0) = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3} = \frac{4}{9}$. To calculate $\tilde{\Phi}$ in Table 5, we only have to calculate the values of $\tilde{\Phi}(l_1, \tilde{l}_2, 1)$, as given in the final column of Table 5. This is easier than the original calculation of Φ in Table 1, because it concerns the same system structure but with the new component of type 3 in Figure 3 certainly functioning. Given the fully specified survival signature Φ of the original system and the values of $\tilde{\Phi}(l_1, \tilde{l}_2, 1)$, the values of $\tilde{\Phi}(l_1, \tilde{l}_2, 0)$ in Table 5 are easily calculated using Equation (9).

14

	Subsystem A				Subsystem B				Subsystem C		
l_1^a	l_2^a	l_3^a	$\Phi^a(l_1^a,l_2^a,l_3^a)$	l_1^b	l_2^b	l_3^b	$\Phi^b(l_1^b,l_2^b,l_3^b)$	l_1^c	l_2^c	l_3^c	$\Phi^c(l_1^c,l_2^c,l_3^c)$
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	1	0	0	0	1	0
0	1	0	1	0	0	2	1	0	0	2	1
1	1	0	1	0	1	0	0	0	1	0	1
				0	1	1	1/2	0	1	1	1
				0	1	2	1	0	1	2	1
				0	2	0	1				
				0	2	1	1				
				0	2	2	1				

Table 2: Survival signatures of subsystems A, B and C in Figure 2 (Ex. 2)

l_1^{ab}	l_2^{ab}	l_3^{ab}	$\Phi^{ab}(l_1^{ab},l_2^{ab},l_3^{ab})$	l_1^{ab}	l_2^{ab}	l_3^{ab}	$\Phi(l_1^{ab},l_2^{ab},l_3^{ab})$
0	0	1	0	0	2	2	2/3
0	0	2	0	1	0	0	0
0	1	0	0	1	0	1	0
0	1	1	0	1	1	0	0
0	1	2	1/3	1	1	1	1/3
0	2	0	0	1	2	0	1/3
0	2	1	1/3	1	2	1	2/3

Table 3: Survival signature of subsystem AB in Figure 2 (Ex. 2)

Example 4.
To illustrate NPI for the system survival time, as presented in Section 4, consider again the system from Figure 1 with survival signature given in Table 1. Suppose that $n_1 = 2$ components exchangeable with those of type 1 and $n_2 = 2$ components exchangeable with those of type 2 were tested. First suppose that failure times $t_1^2 < t_1^1 < t_2^2 < t_2^1$ were observed, with t_j^k the j-th ordered failure time of a component of type k. The resulting NPI lower and upper survival functions for the system failure time T_S are specified in Table 6, together with the results for the case with the test failure times ordered as $t_1^1 < t_1^2 < t_2^1 < t_2^2$.

For the ordering $t_1^2 < t_1^1 < t_2^2 < t_2^1$, in the first interval in Table 6 we have not yet seen a failure in the test data, so the NPI upper probability that the system will function is equal to one. In the second interval, one failure of type 2 has occurred but we do not have any evidence from the data against the possibility that a component of type 1 will certainly function at times in this

l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$
0	0	1	0	0	2	3	22/24
0	0	2	1/6	1	0	0	0
0	0	3	1/2	1	0	1	0
0	1	0	1/4	1	0	2	2/6
0	1	1	1/4	1	1	0	1/4
0	1	2	10/24	1	1	1	6/16
0	1	3	12/16	1	1	2	16/24
0	2	0	3/6	1	2	0	4/6
0	2	1	14/24	1	2	1	18/24
0	2	2	27/36	1	2	2	32/36

Table 4: Survival signature of the system in Figure 2 (Ex. 2)

interval, so the NPI upper probability remains one. In the fourth interval, both type 2 components have failed but only one component of type 1 has failed. In this interval, to consider the lower survival function the system is effectively reduced to a series system consisting of three components of type 1, with one 'success' and one 'failure' as data, denoted by $(2,1)$. As such a series system only functions if all three components function, the NPI lower survival function within this fourth interval is equal to $\underline{S}_{T_S}(t) = \frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} = 0.100$, which follows by sequential reasoning, using that, based on n observations consisting of s successes and $n-s$ failures, denoted as data (n,s), the NPI lower probability for the next observation to be a success is equal to $s/(n+1)$ [11]. The NPI lower probability for the first component to function, given test data $(2,1)$, is equal to $1/3$. Then the second component is considered, conditional on the first component functioning, which combines with the test data to two out of three components observed (or assumed) to be functioning, so combined data $(3,2)$, hence this second component will also function with NPI lower probability $2/4$. Similarly, the NPI lower probability for the third component to function, conditional on functioning of the first two components in the system, so with combined data $(4,3)$, is equal to $3/5$. In the final interval, we are beyond the failure times of all the tested components, so we no longer have evidence in favour of the system to function, so $\underline{S}_{T_S}(t) = 0$, but the system might of course still function as reflected by $\overline{S}_{T_S}(t) = 0.148$.

For the second case in Table 6, with data ordering $t_1^1 < t_1^2 < t_2^1 < t_2^2$, we have $\overline{S}_{T_S}(t) = 0.667$ in the second interval, where one failure of type 1 has
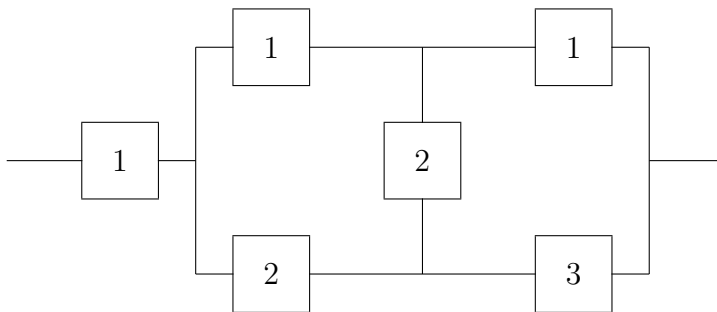
Figure 3: System with 3 types of components (Ex. 3)

occurred in the test data. In the fourth interval, both tested components of type 1 have failed, leading to $\underline{S}_{T_S}(t) = 0$. Both of these values are directly related to the required functioning of the left-most component in Figure 1.

Example 5.
For the system in Figure 2, the survival signature $\Phi(l_1, l_2, l_3)$ was specified, for all $l_1 \in \{0, 1\}$ and $l_2, l_3 \in \{0, 1, 2, 3, 4\}$, in Table 4. While this is still a small system, computation of the survival signature is feasible, either via the method illustrated in Example 2 or directly, although the number of combinations to be considered is already substantial. This raises the question whether or not it is necessary to calculate the survival signature entirely. Of course, as availability of the exact survival signature is convenient for reliability quantification, it is the ideal scenario. However, based on partial information, e.g. following from checking only a subset of all combinations of functioning and not functioning components, one can derive bounds for quantities of interest. If such bounds suffice for a specific inferential question, then of course one would not need to compute the survival signature further. This is a straightforward idea, which was also explored by Al-nefaiee and Coolen [5] for system signatures. Given the monotonicity of the survival signature in each of its components for coherent systems, working with bounds for it is straightforward as long as the inference of interest is monotone as function of the survival signature. If interest is in (NPI lower and upper)

17

l_1	$\tilde{l}_2$	$\tilde{l}_3$	$\tilde{\Phi}(l_1,\tilde{l}_2,\tilde{l}_3)$	l_1	$\tilde{l}_2$	$\tilde{l}_3$	$\tilde{\Phi}(l_1,\tilde{l}_2,\tilde{l}_3)$
0	0	0	0	0	0	1	0
0	1	0	0	0	1	1	0
0	2	0	0	0	2	1	0
1	0	0	0	1	0	1	0
1	1	0	0	1	1	1	1/6
1	2	0	0	1	2	1	1/3
2	0	0	0	2	0	1	0
2	1	0	0	2	1	1	1/2
2	2	0	1/3	2	2	1	2/3
3	0	0	1	3	0	1	1
3	1	0	1	3	1	1	1
3	2	0	1	3	2	1	1

Table 5: Survival signature of the system in Figure 3 (Ex. 3)

$t_1^2 < t_1^1 < t_2^2 < t_2^1$			$t_1^1 < t_1^2 < t_2^1 < t_2^2$		
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$	$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0, t_1^2)$	0.553	1	$(0, t_1^1)$	0.553	1
(t_1^2, t_1^1)	0.458	1	(t_1^1, t_1^2)	0.230	0.667
(t_1^1, t_2^2)	0.148	0.553	(t_1^2, t_2^1)	0.148	0.553
(t_2^2, t_2^1)	0.100	0.458	(t_2^1, t_2^2)	0	0.230
(t_2^1, ∞)	0	0.148	(t_2^2, ∞)	0	0.148

Table 6: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system in Figure 1 and two data orderings (Ex. 4)

probabilities for the event that the system functions at time t, as considered in this paper, then indeed it is straigthforward to use the information about the survival signature, consisting of bounds for its values. Detailed theory of this use of bounds is presented by Al-nefaiee [4]. We restrict ourselves to an illustration using the system in Figure 2, for which the exact survival signature was derived in Example 2 and presented in Table 4. We combine illustration of the use of bounds for the survival signature with the NPI approach, as presented in Section 4, with more data observations than in Example 4.

Table 7 provides bounds for the survival signature, together with the exact survival signature (final column), for four cases, representing four possible subsequent stages of its direct computation (so not using the method pre-

sented in Section 3 and illustrated in Example 2). The bounds are denoted as intervals, entries which are either 0 or 1 for all stages and where this follows by monotonicity from other entries in the table have been deleted. Case 1 only involved an initial assessment for rather trivial values of (l_1, l_2, l_3) for which the system either functions or not with certainty. Without further calculations, the survival signature is only known to be in $[0, 1]$ at all other (l_1, l_2, l_3). Case 2 shows the effect of calculating $\Phi(0, 0, 3) = 1/2$, Case 3 of the additional calculations $\Phi(0, 1, 0) = \Phi(0, 1, 1) = 1/4$, these are all pretty trivial to derive. For Case 4 we calculated, by going through all relevant combinations, the precise values of the survival signature at 6 further points (l_1, l_2, l_3), as shown in Table 7. Most of these precise values affect some bounds at other points due to the monotinicity of the survival signature, but not all. However, all these calculations affect the related bounds for the inferences. It is also possible to calculate the survival signature only partially at a point (l_1, l_2, l_3), leading to bounds at that point which also affect bounds at other points (for an example we refer to Al-nefaiee [4]).

To illustrate the effect of such increased knowledge of the system's survival signature, we present its application in the NPI method presented in Section 4, using simulated failure times as given in Table 8, which for components of type $k \in \{1, 2, 3\}$ were simulated from the Weibull distribution with shape parameter k and scale parameter 1. Of course, using the lower (upper) bounds for the survival signature leads to a lower (upper) bound for the NPI lower (upper) survival signature. These bounds for all four cases are presented in Figure 4, where each plot also includes the NPI lower and upper survival functions based on the exact survival signature. Due to the monotonicities involved, additional calculations for the survival signatures lead to sharper bounds for the NPI lower and upper survival functions, with the effect of the rather straightforward calculations in Cases 2 and 3 already quite substantial. The additional calculations in Case 4 lead to bounds that are already mostly close to the actual NPI lower and upper survival functions. Depending on the inference of interest, these bounds may already be sufficient to derive the conclusion, in which case further calculation of the survival signature would not be required.

l_1, l_2, l_3	Case 1	Case 2	Case 3	Case 4	$\Phi(l_1, l_2, l_3)$
0, 0, 1	0	0	0	0	0
0, 0, 2	$[0,1]$	$[0,1/2]$	$[0,1/2]$	1/6	1/6
0, 0, 3	$[0,1]$	1/2	1/2	1/2	1/2
0, 0, 4	1	1	1	1	1
0, 1, 0	$[0,1]$	$[0,1]$	1/4	1/4	1/4
0, 1, 1	$[0,1]$	$[0,1]$	1/4	1/4	1/4
0, 1, 2	$[0,1]$	$[0,1]$	$[1/4,1]$	$[1/4,32/36]$	10/24
0, 1, 3	$[0,1]$	$[1/2,1]$	$[1/2,1]$	$[1/2,1]$	12/16
0, 1, 4	1	1	1	1	1
0, 2, 0	$[0,1]$	$[0,1]$	$[1/4,1]$	3/6	3/6
0, 2, 1	$[0,1]$	$[0,1]$	$[1/4,1]$	$[3/6,32/36]$	14/24
0, 2, 2	$[0,1]$	$[0,1]$	$[1/4,1]$	$[3/6,32/36]$	27/36
0, 2, 3	$[0,1]$	$[1/2,1]$	$[1/2,1]$	$[3/6,1]$	22/24
1, 0, 1	0	0	0	0	0
1, 0, 2	$[0,1]$	$[0,1]$	$[0,1]$	2/6	2/6
1, 0, 3	1	1	1	1	1
1, 1, 0	$[0,1]$	$[0,1]$	$[1/4,1]$	1/4	1/4
1, 1, 1	$[0,1]$	$[0,1]$	$[1/4,1]$	$[1/4,32/36]$	6/16
1, 1, 2	$[0,1]$	$[0,1]$	$[1/4,1]$	$[1/4,32/36]$	16/24
1, 1, 3	1	1	1	1	1
1, 2, 0	$[0,1]$	$[0,1]$	$[1/4,1]$	4/6	4/6
1, 2, 1	$[0,1]$	$[0,1]$	$[1/4,1]$	$[4/6,32/36]$	18/24
1, 2, 2	$[0,1]$	$[0,1]$	$[1/4,1]$	32/36	32/36
1, 2, 3	1	1	1	1	1

Table 7: Bounds for the survival signature of the system in Figure 2 (Ex. 5)

6. Discussion

Computation of the survival signature of a system is difficult unless the number of components is small or the system has a relatively straightforward structure. For systems with only one type of component, the signature has been derived for some specific system structures [26, 27, 28, 36]. Deriving the survival signature for specific system structures in the case of multiple component types is an interesting topic for research. Recently, Aslett [7] has created a function in the statistical software R to compute the survival signature, given a graphical presentation of the system structure. This can, in principle, be used for systems of any size, but for systems with more than about 20 components which computation time rapidly becomes an issue.

Type 1		Type 2		Type 3	
0.004	0.629	0.290	1.006	0.321	0.876
0.112	0.752	0.412	1.029	0.348	0.877
0.177	0.839	0.531	1.057	0.375	0.920
0.196	0.974	0.579	1.113	0.613	0.955
0.223	1.234	0.603	1.127	0.650	0.973
0.239	1.311	0.617	1.146	0.689	1.064
0.260	1.325	0.677	1.252	0.743	1.102
0.356	1.436	0.883	1.350	0.747	1.110
0.486	3.097	0.901	1.586	0.788	1.129
0.490	4.150	0.942	1.885	0.856	1.150

Table 8: Component failure times (Ex. 5)

It may be possible to implement the results in Sections 3 and 4 in this R function. It is also interesting to investigate if it is possible to benefit from established methods to quantify system reliability, for example fault trees, Bayesian networks or binary decision diagrams, to derive the corresponding survival signature.

While the emphasis in this paper has been on system reliability, the closely related topic of reliability of networks is of great practical importance, for example in energy provision. In such networks there are typically many components of multiple types, with often quite large numbers of components of a specific type for which the assumption of exchangeable failure times is reasonable. Developing the survival signature approach for network reliability is therefore also an important research challenge, which includes computational challenges and could lead to results with great practical impact.

There are many further related research challenges related to theory and application of survival signatures. For example, one could consider the use of right-censored observations, which is likely to be possible with an adaptation of the NPI for Bernoulli data in line with the corresponding NPI theory for real-valued data with right-censored observations [21], which is related to the well-known Kaplan-Meier estimator for such data. It will also be of interest to consider possible system failure due to multiple failure modes [35], where the NPI approach provides interesting opportunities to consider unobserved or even unknown competing risks [22, 32]. Topics of optimal system design in order to provide suitable levels of redundancy [3, 10, 14, 23, 31], possibly including costs, also pose interesting questions for which

the use of the survival signature might provide new solutions. In the NPI framework some of such issues have been considered, but only for systems with relatively limited structures, for which the combinatorial aspects in computations already became quite complex [3].

The theory presented in this paper provides a framework in which these, and many other, problems can be studied for a wide variety of system structures. It will also be of interest to consider the use of the survival signature if failure data at the system level are available, possibly together with some component level data. For signatures, Bayesian inference for this situation was recently presented by Aslett [6], who also considered inferring the signature from failure data, which may be relevant for black-box systems. It will be interesting to develop similar methods for survival signatures, particularly because it widens applicability of such learning methods to systems with multiple types of components.

Acknowledgements

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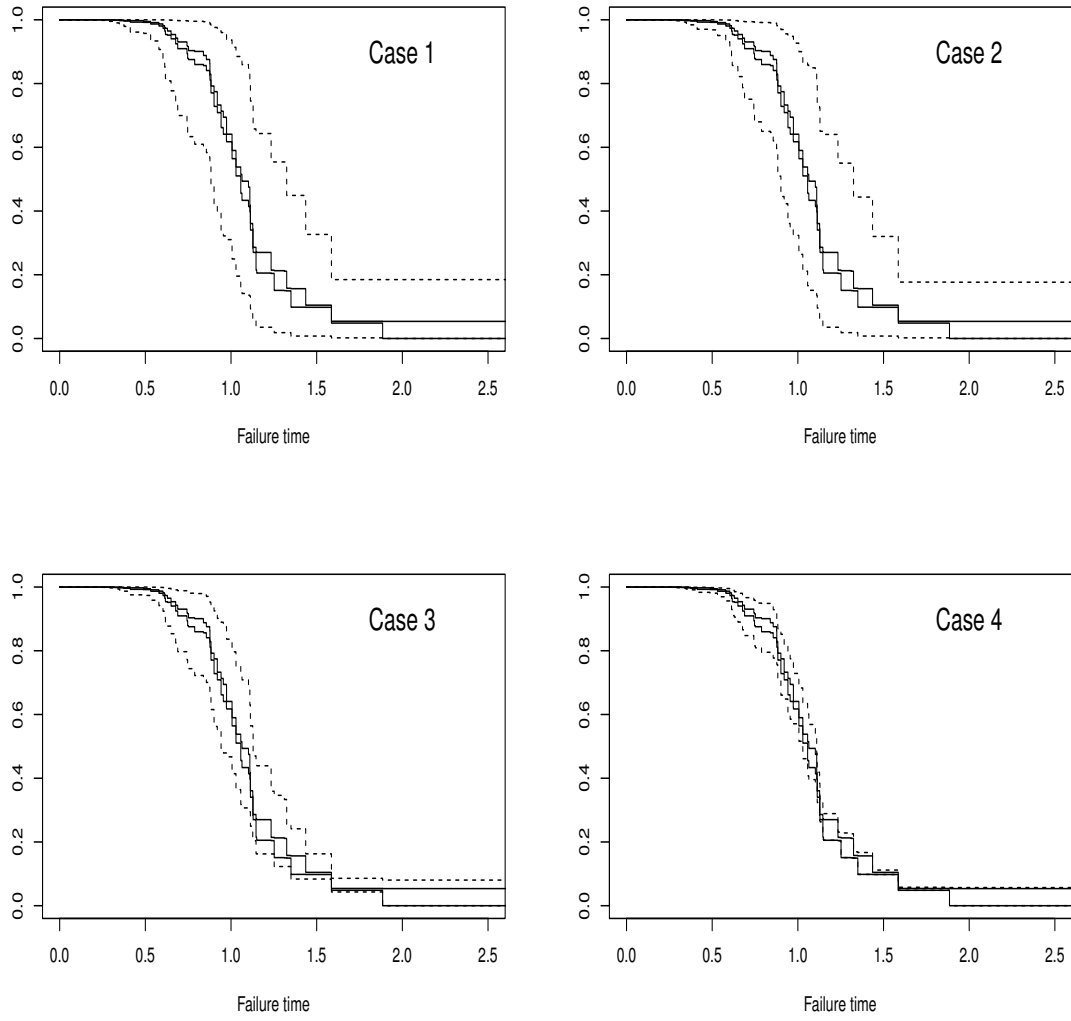
Figure 4: NPI lower and upper survival functions (Ex. 5)