Markov chains

MC 1. Show that the usual Markov property

$$\mathsf{P}(\text{Future} \mid \text{Present}, \text{Past}) = \mathsf{P}(\text{Future} \mid \text{Present})$$

is equivalent to

$$\mathsf{P}(\text{Future}, \text{Past} \mid \text{Present}) = \mathsf{P}(\text{Future} \mid \text{Present})\, \mathsf{P}(\text{Past} \mid \text{Present})\,.$$

MC 2. Suppose that X_0, X_1, ... are independent, identically distributed random variables such that $\mathsf{P}(X_k = 1) = p$ and $\mathsf{P}(X_k = 0) = 1-p$. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geq 1$. In each of the following cases determine whether $(Y_n)_{n\geq 0}$ is a Markov chain:

a) $Y_n = X_n$; b) $Y_n = S_n$; c) $Y_n = S_0 + S_1 + \cdots + S_n$;
d) $Y_n = (S_n, S_0 + S_1 + \cdots + S_n)$.

In the cases where Y_n is a Markov chain find its state space and transition matrix, and in the cases where it is not a Markov chain give an example where the Markov property is violated, ie., when $\mathsf{P}(Y_{n+1} = k \,|\, Y_n = l, Y_{n-1} = m)$ is not independent of m.

MC 3. Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed non-negative random variables taking values in $\{0, 1, 2, \dots\}$. Define:

i) $S_n = X_1 + X_2 + \cdots + X_n$, ii) $T_n = \max\{X_1, X_2, \dots, X_n\}$,
iii) $U_n = \min\{X_1, X_2, \dots, X_n\}$, iv) $V_n = X_n + X_{n-1}$.

Which of these sequences are Markov chains? For those that are, find the transition probabilities; otherwise, give an example where the Markov property is violated.

MC 4. Let X_n, $n \geq 0$, be a Markov chain. Show that $Y_n \stackrel{\text{def}}{=} X_{2n}$ and $Z_n \stackrel{\text{def}}{=} (X_n, X_{n+1})$, $n \geq 0$, are Markov chains and find the corresponding transition probabilities. Is $U_n \stackrel{\text{def}}{=} |X_n|$ a Markov chain? Justify your answer.

MC 5. Let a Markov chain X have state space S and suppose $S = \cup_k A_k$, where $A_k \cap A_l = \varnothing$ for $k \neq l$. Let Y be a process that takes value y_k whenever the chain X lies in A_k. Show that Y is also a Markov chain provided $p_{j_1 m} = p_{j_2 m}$ for *all* $m \in S$ and *all* j_1 and j_2 in the *same* set A_k.

MC 6. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two independent Markov chains, each with the same discrete state space A and same transition probabilities. Define the process $Z_n = (X_n, Y_n)$ with state space $S \times S$. Show that $(Z_n)_{n\geq 0}$ is a Markov chain and find its transition matrix.

MC 7. Suppose that Z_n are iid representing outcomes of successive throws of a die. Define $X_n = \max\{Z_1, \dots, Z_n\}$. Show that X_n is a Markov chain and find its transition matrix $\mathbf{P}$. Calculate from structure of X_n higher powers of $\mathbf{P}$.

MC 8. Let Z_n, $-\infty < n < \infty$, be a sequence of iid random variables with $\mathsf{P}(Z = 0) = \mathsf{P}(Z = 1) = 1/2$. Define the stochastic process X_n with state space $\{0, 1, \dots, 6\}$ by $X_n = Z_{n-1} + 2Z_n + 3Z_{n+1}$, $-\infty < n < \infty$. Determine $\mathsf{P}(X_0 = 1, X_1 = 3, X_2 = 2)$ and $\mathsf{P}(X_1 = 3, X_2 = 2)$. Is X_n Markov? Why or why not?

MC 9. Show that if $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix $\mathbf{P}$ and $Y_n = X_{kn}$, then $(Y_n)_{n\geq 0}$ is a Markov chain with transition matrix $\mathbf{P}^k$.

MC 10. Let X be a Markov chain with state space S and transition probabilities p_{jk}. For any $n \geq 1$ and A_0, ..., $A_{n-1} \subset S$, show that

$$\mathsf{P}\big(X_{n+1} = k \mid X_0 \in A_0, \dots, X_{n-1} \in A_{n-1}, X_n = j\big) = p_{jk}\,.$$

By giving an example, show that the following statement is incorrect: *For subsets A_0, ..., $A_n \subset S$, where A_n is not a singleton, we have*

$$\mathsf{P}\big(X_{n+1} = k \mid X_0 \in A_0, \dots, X_n \in A_n\big) = \mathsf{P}\big(X_{n+1} = k \mid X_n \in A_n\big)\,.$$

MC 11. In a sequence of Bernoulli trials with outcomes S or F, at time n the state 1 is observed if the trials $n-1$ and n resulted in SS. Similarly, states 2, 3, and 4 stand for the patterns SF, FS, and FF. Find the transition matrix $\mathbf{P}$ and all of its powers.

MC 12. Find $\mathbf{P}^n$ for the transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ q & r & p \\ 0 & 0 & 1 \end{pmatrix}\,,$$

where p, q, r are positive numbers satisfying $p + q + r = 1$.

MC 13. For $0 < a < 1$ and $0 < b < 1$, consider the stochastic matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}\,.$$

Show that for every $n \geq 0$,

$$\mathbf{P}^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}\,.$$

MC 14. A flea hops randomly on vertices of a triangle, hopping to each of the other vertices with equal probability. Find the probability that after n hops the flea is back where it started.

MC 15. A flea hops randomly on vertices of a triangle. It is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops the flea is back where it started?
[Hint: $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}} e^{\pm i\pi/6}$.]

MC 16. Suppose a virus can exist in N different strains and in each generation either stays the same, or with probability α mutates to another strain, which is chosen (uniformly) at random. Find the probability $p_{ii}^{(n)}$ that the strain in the nth generation is the same as that in the 0th. Compute $\lim\limits_{n\to\infty} p_{ii}^{(n)}$.

MC 17. Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix

$$\begin{pmatrix} 2/3 & 1/3 & 0 \\ 1-p & 0 & p \\ 1 & 0 & 0 \end{pmatrix}\,.$$

Calculate $\mathsf{P}(X_n = 3 \mid X_0 = 3)$ when

a) $p = 1/(16)\,,$ b) $p = 1/6\,,$ c) $p = 1/(12)\,.$

MC 18. In the Land of Oz, they never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance to having the same the next day. If there is change from snow or rain, only half of the time this is a change to a nice day. In other words, the corresponding Markov chain on the state space $S = \{R, N, S\}$ has the following transition matrix:

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} .$$

Find the transition probabilities $p_{RN}^{(n)}$, $p_{NN}^{(n)}$, and $p_{SN}^{(n)}$. Comment on what happens as $n \to \infty$.

MC 19. A die is 'fixed' so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probability $1/5$. If the first score is 6, what is the probability that the $(n+1)$st score is 6? What is the probability that the $(n+1)$st score is 1?

MC 20. Give an example to show that for a Markov chain to be irreducible, it is sufficient *but not necessary* that for some $n \geq 1$, $p_{jk}^{(n)} > 0$ for all states j, k.

MC 21. The birth and death chain is a Markov chain on the state space $S = \{0, 1, 2, \dots\}$ whose transition probabilities are defined by the restriction $p_{ij} = 0$ when $|i - j| > 1$ and

$$p_{k,k+1} = p_k \, , \qquad p_{k,k-1} = q_k \, , \qquad p_{kk} = r_k \qquad (k \geq 0)$$

with $q_0 = 0$. Assuming that *all* p_k and *all* q_k are strictly positive, show that every state $i \in S$ is aperiodic if and only if for some $k \in S$ one has $r_k > 0$. If the chain is periodic, find $d(i)$ for $i \in S$.

MC 22. Let a Markov chain have m states. Prove that if $j \to k$ then state k can be reached from j with positive probability in m steps or less.
This observation can be used to teach computers to classify states of a Markov chain: replace every positive entry in $\mathbf{P}$ with one and compute $\mathbf{P} + \mathbf{P}^2 + \cdots + \mathbf{P}^m$, where m is the number of states.

MC 23. Show carefully that if i and j are communicating states of a Markov chain, then the period $d(i)$ of i is a multiple of $d(j)$. Deduce that i and j have the same period.

MC 24. Show that every transition matrix on a finite state space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating classes.

MC 25. For a Markov chain on $\{1, 2, 3\}$ with the transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

a) describe the class decomposition and find the period of every state;
b) compute the hitting probabilities $h_2^{\{1\}}$ and $h_3^{\{1\}}$; explain your findings.

MC 26.$^{\circledast}$ Let $\mathbf{P} = \bigl(p_{ij}\bigr)_{i,j=1}^{m}$ be an irreducible aperiodic stochastic matrix. Show that there exists an integer $l \ge 1$ such that $\mathbf{P}^l$ is γ-positive with some $\gamma \in (0,1)$, ie., $\min_{i,j} p_{i,j}^{(l)} \ge \gamma > 0$.

Hint: show first that if state k is aperiodic, then the set $A_k \stackrel{\mathsf{def}}{=} \{n > 0 : p_{kk}^{(n)} > 0\}$ contains all large enough natural numbers, ie., for some natural $n_0 \ge 0$, A_k contains $\{n_0, n_0+1, \dots\}$.

MC 27. A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if the player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use the bold strategy, in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10. Let $X_0 = 2$ and let X_n be his capital after n throws.

a) Show that the gambler will achieve his aim with probability $1/5$.

b) What is the expected number of tosses until the gambler achieves his aim or loses his capital?

MC 28. Consider a Markov chain on $\{1,2,3,4,5\}$ with transition matrix

$$\begin{pmatrix} 1-r & r & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $r \in [0,1]$. Find the hitting probability $h_i^{\{5\}}$ and the mean hitting time $k_i^{\{5\}}$ as a function of r. Explain your findings.

MC 29. Consider a Markov chain on $\{0,1,2,\dots\}$ with transition probabilities $p_{0,1} = 1 - p_{0,0} = r$ and $p_{k,k+1} = p_{k,k-1} = \frac{1}{2}$ for $k \ge 1$. Find the expected number of moves to reach n starting from the initial position i.

MC 30. Assume that the experiment has m equally probable outcomes. Show that the expected number of independent trials before the first occurrence of k consecutive occurrences of one of these outcomes is $(m^k - 1)/(m-1)$.
It has been found that, in the decimal expansion of π, starting with $24,658,601$st digit, there is a run of nine 7's. What would your result say about the expected number of digits necessary to find such a run if the digits are produced randomly?

MC 31.$^{\circledast}$ A transition matrix $\mathbf{P}$ is positive, if all its entries are strictly positive. A transition matrix is called regular, if for some $n \ge 1$ the n-step transitions matrix P^n is positive. Let X be a regular Markov chain with finite state space S. For states j, $k \in S$ and a subset $D \subseteq S$ consider the first passage times

$$T_{jk} \stackrel{\mathsf{def}}{=} \min\{n : X_n = k \mid X_0 = j\}, \qquad T_{jD} \stackrel{\mathsf{def}}{=} \min\{n : X_n \in D \mid X_0 = j\}.$$

Show that there exist positive constants c and $\rho < 1$ such that $\mathsf{P}(T_{jk} > n) < c\rho^n$; thus deduce that T_{jk} is finite with probability one and has a finite expectation. Derive similar properties for T_{jD}.

MC 32. A particle performs a random walk on the vertices of a cube in such a way that from every vertex it moves to any of its neighbours with equal probabilities. Find the mean recurrence time of each vertex and the mean passage time from a vertex to its diagonal opposite.

MC 33.⊛ A unit cube is placed such that one of its vertices, i, is at the origin $(0,0,0)$, and the opposite vertex o is at $(1,1,1)$. A particle performs a random walk on the vertices of the cube, starting from i and moving along the x, y, and z directions with probabilities p, q, and r respectively. Let T_{io} be the first passage time from i to o. Find the generating function of T_{io} and compute its mean. Do the same for the first passage time T_{ii}.

Hint: $\mathcal{F}_{io}(s)$ is the generating function of T_{io}.

MC 34. A process moves on integers 1, 2, 3, 4, and 5. It starts at 1 and, on each successive step, moves to an integer greater that its present position, selecting each of the available jumps with equal probabilities. State 5 is an absorbing state. Find the expected number of steps to reach state 5.

MC 35. Let X and X' be Markov chains on $\{1,2,3\}$ with transition matrices

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \qquad \text{and} \qquad \mathbf{P}' = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

respectively. Which of these Markov chains are irreducible? Which are periodic? In both cases compute the expected return time to state 1.

MC 36. A biased coin is tossed repeatedly. If X_n denotes the total number of 'heads' in the first n throws, consider $T_0 = \min\{n > 0 : X_n \text{ is a multiple of } 11\}$. Find $\mathsf{E}T_0$ quoting carefully any general result that you use.

MC 37. Consider a Markov chain on $\{1,2,\dots\}$ with transition probabilities $p_{k,1} = 1 - p_{k,k+1} = \frac{1}{k+1}$, $k \ge 1$. Find the first passage probabilities $f_{11}^{(n)}$ and the mean return time to state 1.

MC 38. For a Markov chain on $\{1,2,3,4,5\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 1/3 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) find the period $d(i)$ for every $i = 1,2,\dots,5$ and identify classes of communicating states; explain which classes are closed and which are not;

b) use the the n-step transition probabilities $p_{ii}^{(n)}$ to determine which states are recurrent and which are transient;

c) find $\mathsf{P}_4(H^{\{5\}} = k)$, $k \ge 1$, and deduce that $h_4^{\{5\}} \equiv \mathsf{P}(H^{\{5\}} < \infty \mid X_0 = 4)$ equals $3/4$;

d) show that $h_4^{\{1,2,3\}} = 1/4$.

e) find the following first passage probabilities: f_{11}, f_{44}, f_{55}, f_{43}, and f_{45};

f) use these values to check which states are transient and which are recurrent.

MC 39. For a Markov chain on $\{1,2,3,4,5,6\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/6 & 0 & 1/3 & 1/6 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) identify classes of communicating states; explain which classes are closed and which are not;

b) find the period $d(i)$ for every $i = 1, 2, \dots, 6$;

c) find the following first passage probabilities: f_{11}, f_{33}, f_{55}, f_{66}, f_{54}, and f_{56}; use these values to check which states are transient and which are recurrent.

d) use the the n-step transition probabilities $p_{ii}^{(n)}$ to check which states are transient and which are recurrent.

e) find $\mathsf{P}_5(H^{\{2,3,4\}} = k)$, $k \geq 1$, and deduce the value of the hitting probability $h_5^{\{2,3,4\}} \equiv \mathsf{P}(H^{\{2,3,4\}} < \infty \,|\, X_0 = 5)$;

f) find the probability $h_5^{\{1,6\}}$ and the expected hitting times $k_5^{\{1,6\}}$ and $k_5^{\{2,3,4\}}$;

g) find all stationary distributions.

MC 40. For a Markov chain on $S = \{1,2,3,4,5,6\}$ with transition matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

a) find the period $d(i)$ for every $i = 1, 2, \dots, 5$ and identify classes of communicating states; explain which classes are closed and which are not;

b) use the the n-step transition probabilities $p_{ii}^{(n)}$ to determine which states are recurrent and which are transient;

c) find the probability $\mathsf{P}_6(H^{\{5\}} = k)$, $k \geq 1$, and deduce the value of $h_6^{\{5\}}$;

d) find the value of $h_6^{\{1\}}$;

e) find the following first passage probabilities: f_{11}, f_{44}, f_{55}, f_{63}, and f_{65};

f) use these values to check which states are transient and which are recurrent.

g) find all stationary distributions for this chain.

MC 41. Let X be a Markov chain with state space S and transition probabilities p_{jk}. If $X_0 = j$ and $p_{jj} > 0$, find the distribution of the exit time

$$\eta_j = \min\{n \geq 1 : X_n \neq j\}.$$

MC 42. Consider a Markov chain on $\{0, 1, 2, \dots\}$ with transition probabilities $p_{0,1} = 1$ and $p_{k,k+1} = 1 - p_{k,k-1} = p$ for $k \geq 1$. Show that:

a) if $p < 1/2$, each state is positive recurrent;

b) if $p = 1/2$, each state is null recurrent;

c) if $p > 1/2$, each state is transient.

MC 43. Let X be a Markov chain on $S = \{0, 1, 2, \dots\}$ with transition probabilities $p_{j,j+1} = 1 - p_{j,0} = p_j \in (0, 1)$. Use the first passage probabilities $f_{jj}^{(n)}$ to derive a criterion of recurrence of X in terms of the probabilities p_j.

MC 44. If the state space S of a Markov chain X is finite, show that not all states can be transient.

MC 45.$^{\circledast}$ A lazy random walk $(X_n)_{n\ge0}$ on $K_m = \{1, 2, \dots, m\}$ jumps to each other state with probability $\frac{1}{m}$ (and thus stays at the current state with probability $\frac{1}{m}$). Let $(Y_n)_{n\ge0}$ be another lazy random walk on K_m. We define a Markov chain $Z_n = (X_n, Y_n)$ on $K_m \times K_m$ using the following jump probabilities:

$$\tilde{p}_{(x,y)(x',y')} = \begin{cases} \frac{1}{m^2}\,, & \text{if } x \neq y\,, \\ \frac{1}{m}\,, & \text{if } x = y \text{ and } x' = y'\,, \\ 0\,, & \text{if } x = y \text{ and } x' \neq y'\,. \end{cases}$$

a) Show that the marginal distributions of Z_n coincide with those of X_n and Y_n respectively, ie., show that

$$\sum_{y'} \tilde{p}_{(x,y)(x',y')} = p_{x,x'}\,, \qquad \sum_{y'} \tilde{p}_{(x,x)(x',y')} = \tilde{p}_{(x,x)(x',x')} = p_{x,x'}\,.$$

b) For $X_0 = x \neq y = Y_0$, let $T \stackrel{\text{def}}{=} \min\{n \ge 1 : X_n = Y_n\}$ be the time when both Markov chains $(X_n)_{n\ge0}$ and $(Y_n)_{n\ge0}$ meet together; find the distribution of T and deduce that $\mathsf{P}(T < \infty) = 1$, ie., these Markov chains meet with probability one.
c) Show that for all $n \ge 0$, we have $\mathsf{P}(X_n = j, T \le n) = \mathsf{P}(Y_n = j, T \le n)$ for all vertices j in K_m.
d) Deduce that

$$\sum_{j\in K_m} \bigl|\mathsf{P}(X_n = j) - \mathsf{P}(Y_n = j)\bigr| \le 2\mathsf{P}(T > n) \to 0 \qquad \text{as } n \to \infty\,.$$

e) If $\mathsf{P}(Y_0 = j) = \frac{1}{m}$, ie., chain $(Y_n)_{n\ge0}$ starts from equilibrium, deduce that $\bigl|\mathsf{P}(X_n = j) - \frac{1}{m}\bigr| \to 0$ as $n \to \infty$. In other words, the Markov chain $(X_n)_{n\ge0}$ approaches its stationary distribution as $n \to \infty$.
f) Show that $\mathsf{P}(T > n) \le e^{-n/m}$, and deduce that the convergence towards the equilibrium in part e) is at least exponentially fast. Can you find the true speed of this convergence?

MC 46.$^{\circledast}$ Generalize the argument in Problem MC48 to any aperiodic irreducible Markov chain with finite state space.
[**Hint:** Recall **MC26**.]

MC 47. In a finite Markov chain, state j is transient iff there exists some state k such that $j \to k$ but $k \not\to j$. Give an example to show that this is false if the Markov chain has an infinite number of states.

MC 48. Find an example of a Markov chain with a transient closed communicating class.

MC 49. Recall that for a Markov chain, $f_{jk} = \sum_n f_{jk}^{(n)}$. Show that

$$\sup_{n\ge1}\{p_{jk}^{(n)}\} \le f_{jk} \le \sum_{n=1}^{\infty} p_{jk}^{(n)}\,.$$

Deduce that: i) $j \to k$ iff $f_{jk} > 0$; ii) $j \leftrightarrow k$ iff $f_{jk} f_{kj} > 0$.

MC 50. A Markov chain X_n on $\{0, 1, 2 \dots\}$ has transition probabilities

$$\mathsf{P}\big(X_{n+1} = k+1 \mid X_n = k\big) = 1 - \mathsf{P}\big(X_{n+1} = k-2 \mid X_n = k\big) = p\,, \qquad k \ge 2\,,$$

$$\mathsf{P}\big(X_{n+1} = k+1 \mid X_n = k\big) = 1 - \mathsf{P}\big(X_{n+1} = 0 \mid X_n = k\big) = p\,, \qquad k = 0, 1\,,$$

where $p \in (0, 1)$. Establish a necessary and sufficient condition in terms of p for positive recurrence of X_n, and find the stationary distribution when it exists. Show that the process is null recurrent when $p = 2/3$.

MC 51. A Markov chain has state space $\{0, 1, 2, \dots\}$ and transition probabilities

$$p_{k,k+1} = 1 - p_{k,0} = \frac{\lambda}{k + \nu + 1}\,,$$

where $\lambda > 0$ and $\nu \ge 0$ are constants. State any other necessary restrictions on the values of λ and ν. Show that the chain is irreducible, aperiodic and positive recurrent. Find explicitly the stationary distribution for $\nu = 0$ and $\nu = 1$.

MC 52. The rooted binary tree is an infinite graph T with one distinguished vertex R from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

MC 53. Find all stationary distributions for a Markov chain with transition matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-a & a \\ b & 0 & 1-b \end{pmatrix}.$$

MC 54.$^{\circledast}$ Let $(X_n)_{n \ge 0}$ be a Markov chain on $\{0, 1, 2, \dots\}$ with transition probabilities given by

$$p_{0,1} = 1\,, \qquad p_{k,k+1} + p_{k,k-1} = 1\,, \quad p_{k,k+1} = \Big(\frac{k+1}{k}\Big)^2 p_{k,k-1}\,, \quad k \ge 1\,.$$

a) Show that if $X_0 = 0$, the probability that $X_n \ge 1$ for all $n \ge 1$ is $6/\pi^2$.

b) Show that $\mathsf{P}(X_n \to \infty \text{ as } n \to \infty) = 1$.

c) Suppose that the transition probabilities satisfy instead

$$p_{k,k+1} = \Big(\frac{k+1}{k}\Big)^\alpha p_{k,k-1}$$

for some $\alpha \in (0, \infty)$. What is then the value of $\mathsf{P}(X_n \to \infty \text{ as } n \to \infty)$?

MC 55. A knight confined to a 5×5 chessboard instantaneously makes standard knight's moves each second in such a way that it is equally likely to move to any of the squares one move away from it. What long-run fraction of the time does it occupy the centre square?

MC 56. a) A random knight moves on the standard chessboard and makes each permissible move with equal probability. If it starts in a corner, how long on average will it take to return?

b) Suppose that the same knight, being in a melancholic mood, flips a fair coin before each attempt and moves only if the coin shows tails. Find the expected duration for the same journey.

c) Now assume that the coin in b) is biased, and find the averaged return time to the initial corner. Explain your findings.

MC 57. A random walker on the standard chessboard makes each permissible move with equal probability. If it starts in a corner, how long on average will it take to return, if:

a) only horizontal and vertical moves are allowed (ie, in the middle of the chessboard there are four permissible moves)?

b) the diagonal moves are also allowed (ie, in the middle of the chessboard there are eight permissible moves)?

MC 58. Suppose that each box of cereal contains one of n different coupons. If the coupon in each box is chosen independently and uniformly at random from the n possibilities, how many boxes of cereal must you buy before you obtain at least one of every type of coupon?
[**Hint:** if X_k is the number of different coupons found in the first k boxes, describe $(X_k)_{k\geq 0}$ as a Markov chain, and find the corresponding expected hitting time.]

MC 59. a) Suppose that a Markov chain has m states and is *doubly stochastic*, if all its column sums equal one, that is, for every state j we have $\sum_i p_{ij} = 1$. Show that the vector $\boldsymbol{\pi} = (1/m, \dots, 1/m)$ is a stationary distribution for this chain.
b) A fair dice is thrown repeatedly. Let X_n denote the sum of the first n throws. Find $\lim_{n\to\infty} \mathsf{P}(X_n$ is a multiple of $9)$ quoting carefully any general theorem that you use.

[**Hint:** You may think of X_n as a finite state Markov chain on the set $\{0, 1, \dots, 8\}$ of residuals modulo 9. Guess the answer and prove the result.]

MC 60. Find all stationary distributions for a Markov chain on $\{1, 2, 3, 4, 5\}$ with the transition matrix (where $0 < p = 1 - q < 1$ and $0 \leq r \leq 1$)

$$\begin{pmatrix} 1-r & r & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) by solving the standard equations $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$; b) by solving the detailed balance equations. Explain your findings.

MC 61.[⊛] Show that the simple symmetric random walk in $\mathbb{Z}^4$ is transient.

MC 62. a) An electric light that has survived for n seconds fails during the $(n+1)$st second with probability q $(0 < q < 1)$. Let $X_n = 1$ if the light is functioning at time n seconds, otherwise let $X_n = 0$. Let T be the time to failure (in seconds) of the light, ie., $T = \min\{n > 0 : X_n = 0\}$. Find $\mathsf{E}T$.

b) A building contains m lights of the type described above, which behave independently. At time 0 they all are functioning. Let Y_n denote the number of lights functioning at time n. Specify the transition matrix of Y_n. Find the generating function $\varphi_n(s) \equiv \mathsf{E}s^{Y_n}$ of Y_n and use it to find $\mathsf{P}(Y_n = 0)$ and $\mathsf{E}Y_n$.

Hint: Show that $\varphi_n(s) = \varphi_{n-1}(q + ps)$.

MC 63. A professor has m umbrellas, which he keeps either at home or in his office. He walks to and from his office each day, and takes an umbrella with him if and only if it is raining. Throughout each journey, it either rains, with probability p, or remains fine, with probability $1 - p$, independently of the past weather. What is the long run proportion of journeys on which he gets wet?

MC 64.$^{\circledast}$ A queue is a line where the customers wait for service. We consider a model for a bounded queue where time is divided into steps of equal length. At each step exactly one of the following occurs:

- if the queue has fewer than n customers, a new customer joins the queue with probability α;
- if the queue is not empty, the head of the queue is served (and leaves the queue) with probability β;
- with the remaining probability the queue is unchanged.

This queue can be described by a Markov chain on $\{0, 1, \dots, n\}$ with transition probabilities

$$p_{0,1} = 1 - p_{0,0} = \alpha\,, \qquad p_{n,n-1} = 1 - p_{n,n} = \beta$$
$$p_{k,k+1} = \alpha\,, \quad p_{k,k-1} = \beta\,, \quad p_{k,k} = 1 - \alpha - \beta\,, \qquad k = 1, \dots, n-1\,.$$

Find the stationary distribution of this Markov chain. What happens when $n \to \infty$? Justify your answer.

MC 65. Let $\mathcal{P}_{ij}(s)$ be the generating function of the sequence $p_{ij}^{(n)}$.
a) Show that for every fixed state $l \in S$, and all $|s| < 1$,

$$\mathcal{P}_{lj}(s) = \delta_{lj} + s \sum_{i \in S} \mathcal{P}_{li}(s) p_{ij}\,.$$

b) Let an irreducible recurrent Markov chain X_n with transition matrix $\mathbf{P}$ have the property $\mathsf{E}_j T_j < \infty$ for all $j \in S$. Deduce that $\boldsymbol{\rho} = (\rho_i, i \in S)$ defined via $\rho_i = 1/\mathsf{E}_j T_j$ is an invariant measure for $\mathbf{P}$, ie., $\boldsymbol{\rho} = \boldsymbol{\rho}\mathbf{P}$.

Hint: Show that $(1-s)\mathcal{P}_{lj}(s) \to 1/\mathcal{F}'_{jj}(1) \equiv 1/\mathsf{E}_j T_j$ as $s \to 1$, for every state $j \in S$.

MC 66. At each time $n = 0, 1, 2, \dots$ a number Y_n of particles is injected into a chamber, where $(Y_n)_{n \ge 0}$ are independent Poisson random variables with parameter λ. The lifetimes of particles are independent and geometric with parameter p. Let X_n be the number of particles in the chamber at time n. Show that X_n is a Markov chain; find its transition probabilities and the stationary distribution.

MC 67.$^{\circledast}$ Let S_n be a random walk such that $\mathsf{P}(S_{n+1} - S_n = 2) = p$ and $\mathsf{P}(S_{n+1} - S_n = -1) = q$, where $p + q = 1$. If the origin is a retaining barrier (that is to say, we assume that $s_N \ge 0$ and the negative jump out of the origin is suppressed, $\mathsf{P}(S_{n+1} = 0 \mid S_n = 0) = q$), show that the equilibrium is possible if $p < 1/3$; also show that in this case the stationary distribution has probability generating function

$$G_{\boldsymbol{\pi}}(s) = \frac{1 - 3p}{q - ps(1+s)}\,.$$

MC 68. Consider a Markov chain on $\{0, 1, \dots, n\}$, with transition probabilities $p_{n,n} = p_{n,0} = \frac{1}{2}$ and $p_{k,k+1} = p_{k,0} = \frac{1}{2}$ for $k < n$. Find the stationary distribution of this chain.

MC 69. Suppose an irreducible Markov chain with a not necessarily finite state space has a transition matrix with the property that $\mathbf{P}^2 = \mathbf{P}$.

a) Prove that the chain is aperiodic;

b) Prove that $p_{jk} = p_{kk}$ for all j, k in the state space. Find a stationary distribution in terms of $\mathbf{P}$.

MC 70. If X_n is a Markov chain with stationary distribution $\boldsymbol{\pi}$, show that the process $(X_n, X_{n+1})_{n\geq 0}$ is also a Markov chain and find its stationary distribution.

MC 71. Consider a Markov chain on $S = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

Find its stationary distribution, its expected return times $m_j = \mathsf{E}_j \tau_j$ for all $j \in S$, and compute $\lim_{n\to\infty} \mathbf{P}^n$.

MC 72. Suppose a virus can exist in N different strains and in each generation either stays the same, or with probability α mutates to another strain, which is chosen (uniformly) at random.

a) Find the stationary distribution of this chain.
[**Hint**: you may first guess the result and then justify it rigorously.]

b) Use your answer to find $p_{11}^{(n)}$ and $p_{12}^{(n)}$.

MC 73. Can a reversible Markov chain be periodic?

MC 74. A flea hops randomly on vertices of a triangle. It is twice as likely to jump clockwise as anticlockwise.

a) Find the stationary distribution of the chain by solving the system $\pi = \pi\,\mathbf{P}$.

b) Explain how you could have guessed the stationary distribution of this chain, justify its uniqueness.

c) Use the knowledge about the stationary distribution to deduce the value of the probability that after n hops the flea is back where it started.
[Hint: recall that $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}} e^{\pm i\pi/6}$.]

MC 75. Each morning a student takes one of the three books she owns from her shelf. The probability that she chooses book k is α_k, $0 < \alpha_k < 1$, $k = 1, 2, 3$, and choices on successive days are independent. In the evening she replaces the book at the left-hand end of the shelf. If p_n denotes the probability that on day n the student finds the books in the order $1, 2, 3$ from left to right, show that, irrespective of the initial arrangement of the books, p_n converges as $n \to \infty$, and determine the limit.

MC 76.$^{\circledast}$ Let Y_1, Y_2, ... be independent identically distributed random variables with values in $\{1, 2, \dots\}$. Suppose that the set of integers $\{n : \mathsf{P}(Y = n) > 0\}$ has greatest common divisor 1. Set $\mu = \mathsf{E}Y_1$. Show that the following process is a Markov chain:

$$X_n = \inf\bigl\{m \geq n : m = Y_1 + Y_2 + \cdots + Y_k \text{ for some } k \geq 0\bigr\} - n\,.$$

Determine $\lim_{n\to\infty} \mathsf{P}(X_n = 0)$ and hence show that

$$\lim_{n\to\infty} \mathsf{P}\Bigl(\bigl\{n = Y_1 + Y_2 + \cdots + Y_k \text{ for some } k \geq 0\bigr\}\Bigr) = \frac{1}{\mu}\,.$$

[**Hint:** This solves the general renewal problem, cf. **GF-14**.]

MC 77.$^{\circledast}$ An opera singer is due to perform a long series of concerts. Having a fine artistic temperament, she is liable to pull out each night with probability $1/2$. Once this has happened she will not sing again until the promoter convinces her of his high regard. This he does by sending flowers every day until she returns. Flowers costing x thousand pounds, $0 \le x \le 1$, bring about a reconciliation with probability $\sqrt{x}$. The promoter stands to make $\pounds 750$ from each successful concert. How much should he spend on flowers?

MC 78. In each of the following cases determine whether the stochastic matrix $\mathbf{P} = (p_{jk})$, which you may assume is irreducible, is reversible:

a) $\begin{pmatrix} 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \\ 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \end{pmatrix}$;

b) $S = \{0, 1, \dots, N\}$ and $p_{jk} = 0$ if $|j - k| \ge 2$;

c) $S = \{0, 1, \dots\}$ and $p_{01} = 1$, $p_{j,j+1} = p$, $p_{j,j-1} = 1 - p$ for $j \ge 1$;

d) $p_{jk} = p_{kj}$ for all j, $k \in S$.

In the reversible cases, find the corresponding reversible measures and stationary distributions.

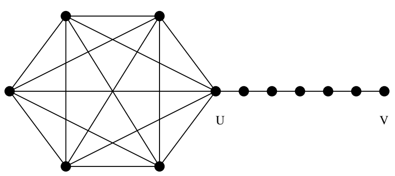
MC 79. Let $(X_n)_{n\ge0}$ and $(Y_n)_{n\ge0}$ be independent irreducible Markov chains, and set $Z_n = (X_n, Y_n)$, $n \ge 0$. Is Z_n irreducible? If X and Y are reversible and also aperiodic, show that Z is reversible.

MC 80. Let X_n be a Markov chain on $\{0, 1, 2, \dots\}$ with transition probabilities

$$p_{01} = 1\,, \qquad p_{i,i+1} + p_{i,i-1} = 1\,, \quad p_{i,i+1} = \Bigl(\frac{i+1}{i}\Bigr)^{\alpha} p_{i,i-1}\,, \quad i \ge 1\,,$$

for some $\alpha > 0$. Is this chain irreducible? Is it reversible? Whenever they exist, find the corresponding reversible measures and stationary distributions. Explain which states are transient and which are recurrent.

MC 81.$^{\circledast}$ A lollipop graph L_{2n} on $2n$ vertices is a clique on n vertices (ie, a complete graph K_n) connected to a path on n vertices, see a picture of L_{12} below. The node u is a part of both the clique and the path; we use v to denote the other end of the path.



If s and t are vertices of a graph, and $k_s^{\{t\}}$ is the expected hitting time of the set $\{t\}$ starting from s, the expected *covering time* θ_s of the graph (starting from s) is $\max_t k_s^{\{t\}}$.

a) Show that for L_{2n} as described above, θ_v is of order n^2 and θ_u is of order n^3.

b) Evaluate the expected covering times θ_v and θ_u for a lollipop graph $L_{k,m}$ consisting of a k-clique and an m-path attached to it at the vertex u.

MC 82.$^{\circledast}$ Let $(X_N)_{n\ge0}$ be an irreducible Markov chain with state space S, transition probabilities p_{jk}, and stationary distribution $\boldsymbol{\pi} = (\pi_j)_{j\in S}$. Let A be some subset of S, and suppose that a new chain Y is formed by banning transitions out of A. That is to say, Y has transition probabilities q_{jk}, where

$$q_{jk} = p_{jk} \quad \text{if } j \ne k \text{ satisfy } j, k \in A\,, \qquad \text{and} \qquad q_{jj} = p_{jj} + \sum_{k \notin A} p_{jk}\,.$$

Show that if X is reversible in equilibrium, then so is Y, and write down the stationary distribution of Y.

MC 83. $^{\circledast}$ Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain with state space S, transition probabilities p_{jk}, and stationary distribution $\boldsymbol{\pi} = (\pi_j)_{j\in S}$. Let A be some subset of S, and suppose that $(Z_n)_{n\geq 0}$ is a Markov chain with state space A and transition probabilities

$$q_{jk} = \frac{p_{jk}}{p_{jA}}\,, \quad j,k \in A\,, \qquad \text{where} \quad p_{jA} = \sum_{k\in A} p_{jk}\,.$$

If X is reversible, show that Z is reversible with stationary distribution $\boldsymbol{\rho}$ given by

$$\rho_j = \frac{\pi_j p_{jA}}{\sum_{j\in A} \pi_j p_{jA}}\,.$$

MC 84. $^{\circledast}$ The following is a variation of a simple children's board game. A player starts at position 0, and on every turn, she rolls a standard six-sided die. If her position was $x \geq 0$ and her roll is y, then her new position is $x+y$, except in two cases:

- if $x+y$ is divisible by 6 and less than 36, her new position is $x+y-6$;
- if $x+y$ is greater than 36, the player remains at x.

The game ends when a player reaches the goal position 36.
a) Let X_i be the (random) number of rolls needed to get to 36 from position i, where $0 \leq i \leq 35$. Give a set of equations that characterize $\mathsf{E}X_i$.
b) Solve this system numerically.

MC 85. We model the DNA sequence by a Markov chain. The state space $\mathcal{S}$ is a four-element set, built up with four base nucleotides A, C, G, T that form DNA. Let p_{jk} be the probability that base j mutates to become base k. We assume that all possible nucleotides substitutions are equally likely. Let α be the rate of substitution. The transition probabilities thus become $3p_{jk} = \alpha = 1 - p_{jj}$, where j, $k \in \{1,2,3,4\}$ and $j \neq k$.

a) Show that the n-step transition probabilities satisfy $p_{jj}^{(n)} = a_n$, $p_{jk}^{(n)} = b_n$ with $a_n + 3b_n = 1$, and find the values a_n and b_n.

b) Find the long-term frequency of each of the base nucleotides. What is the invariant measure of this chain?

c) Let X_k and Y_k be two independent sequences of a single base mutation process with a common ancestor $X_0 = Y_0 = s_0 \in \mathcal{S}$. Show that the probability

$$\mathbf{p} = \mathbf{p}_n = \mathsf{P}\big(X_n \neq Y_n \mid X_0 = Y_0 = s_0\big)$$

that they disagree at time n satisfies $\mathbf{p} = \frac{3}{4}\big(1 - \big(1 - \frac{4\alpha}{3}\big)^{2n}\big)$.

d) Consider two DNA sequences having the same ancestor, whose nucleotides independently evolve as described above. The time n since this ancestor existed is unknown. Let $\mathbf{p}$ be the fraction of sites that differ between the two sequences. Justify the following estimate of n: $\hat{n} = \frac{1}{2}\log(1-4\mathbf{p}/3)/\log(1-4\alpha/3)$.

This model is know as the Jukes-Cantor model of DNA mutation.

MC 86. $^{\circledast}$ A cat and a mouse each independently take a random walk of a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of order $m^2 n$ on the expected time before the cat eats the mouse.
[**Hint:** Consider a Markov chain whose states are ordered pairs (a,b), where a is the position of the cat and b is the position of the mouse.]

MC 87.[⊛] Let n equidistant points (labelled clockwise from 0 to $n-1$) be marked on a circle. Initially, a wolf is at 0 and there is one sheep at each of the remaining $n-1$ points. The wolf takes a random walk on the circle. For each step, it moves to one of its two neighbouring points with the same probability $1/2$. At the first visit to a point, the wolf eats a sheep if there is still one there. Which sheep is most likely to be the last eaten?

MC 88.[⊛] Let $(u_n)_{n\geq 0}$ be a sequence defined by $u_0 = 1$ and, for $n > 0$, by $u_n = \sum_{k=1}^{n} f_k u_{n-k}$, where $f_k > 0$ and $\sum_{k=1}^{\infty} f_k \leq 1$.

a) Show that if $\sum_{k=1}^{\infty} \rho^k f_k = 1$, then $v_n = \rho^n u_n$, $n \geq 0$, is a renewal sequence.

b) Show that as $n \to \infty$, we have $\rho^n u_n \to c$, for some constant $c > 0$.

[**Hint:** This is a more general version of the renewal problem, cf. Problem MC76 and GF-14.]

MC 89.[⊛] A *colouring* of a graph is an assignment of a colour to each of its vertices. A graph is *k-colourable* if there is a colouring of the graph with k colours such that no two adjacent vertices have the same colour. Let G be a 3-colourable graph.

a) Show that there exists a colouring of the graph with two colours such that no triangle is monochromatic. (A *triangle* of a graph G is a subgraph of G with three vertices, which are all adjacent to each other.)

b) Consider the following algorithm for colouring the vertices of G with two colours so that no triangle is monochromatic. The algorithm begins with an arbitrary 2-colouring of G. While there are any monochromatic triangles in G, the algorithm chooses one such triangle and changes the colour of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recolouring steps before the algorithm finds a 2-colouring with the desired property.