$L^r$ convergence

**Def.2.1:** Let $r > 0$ be fixed. A sequence $(X_n)_{n \geq 1}$, of random variables *converges* to a random variable $X$ *in $L^r$* as $n \to \infty$ (write $X_n \xrightarrow{L^r} X$), if

$$
E |X_n - X|^r \to 0 \text{ as } n \to \infty.
$$

**Example 2.2:** Let $(X_n)_{n \geq 1}$ be a sequence of random variables such that for some real numbers $(a_n)_{n \geq 1}$, we have

$$
P(X_n = a_n) = p_n, \quad P(X_n = 0) = 1 - p_n. \quad (2.1)
$$

Then $X_n \xrightarrow{L^r} 0$ iff $E|X_n|^r = |a_n|^r p_n \to 0$ as $n \to \infty$. 
Theorem 2.3: Let \( X_j, j \geq 1 \), be a sequence of uncorrelated random variables with

\[
\mathbb{E}X_j = \mu \quad \text{and} \quad \text{Var}(X_j) \leq C < \infty.
\]

Denote \( S_n = X_1 + \cdots + X_n \). Then \( \frac{1}{n} S_n \xrightarrow{L^2} \mu \) as \( n \to \infty \).

Proof: Indeed,

\[
\mathbb{E}\left( \frac{1}{n} S_n - \mu \right)^2 = \mathbb{E}\left( \frac{S_n - n\mu}{n^2} \right)^2 = \frac{\text{Var}(S_n)}{n^2} \leq \frac{Cn}{n^2} \to 0
\]

as \( n \to \infty \).
Convergence in probability

**Def. 2.4:** A sequence \((X_n)_{n \geq 1}\), of random variables *converges in probability* as \(n \to \infty\) to a random variable \(X\) (write \(X_n \xrightarrow{P} X\)), if for every fixed \(\varepsilon > 0\)

\[
P(|X_n - X| \geq \varepsilon) \to 0 \quad \text{as } n \to \infty.
\]

**Example 2.5:** Let the sequence \((X_n)_{n \geq 1}\) be as in (2.1). Then for every \(\varepsilon > 0\)

\[
P(|X_n| \geq \varepsilon) \leq P(X_n \neq 0) = p_n,
\]

so that \(X_n \xrightarrow{P} 0\) if \(p_n \to 0\) as \(n \to \infty\).
**Theorem 2.6:** Let $X_j, j \geq 1$, be a sequence of uncorrelated random variables with

$$EX_j = \mu \quad \text{and} \quad \text{Var}(X_j) \leq C < \infty.$$ 

Denote $S_n = X_1 + \cdots + X_n$. Then

$$\frac{1}{n} S_n \xrightarrow{p} \mu \quad \text{as} \quad n \to \infty.$$ 

**Exercise 2.7:** Derive this theorem from the Chebyshev inequality!
Theorem 2.6 follows directly from the following observation:

**Lemma 2.8:** Let \((X_n)_{n \geq 1}\), be a sequence of random variables. If \(X_n \xrightarrow{L^r} X\) for some fixed \(r > 0\), then \(X_n \xrightarrow{P} X\) as \(n \to \infty\).

**Proof:** Indeed, for every fixed \(\varepsilon > 0\),

\[
P(\{|X_n - X| \geq \varepsilon\}) \equiv P(\{|X_n - X|^r \geq \varepsilon^r\}) \leq \frac{E|X_n - X|^r}{\varepsilon^r} \to 0
\]

as \(n \to \infty\).
A high dimensional cube $\approx$ a sphere

One can use convergence in probability to argue that, for large $n$, most points of the $n$-dimensional cube $[-1, 1]^n$ are located near the sphere of radius $\sqrt{n/3}$!

If interested, see Example 2.9 in the notes.
Theorem 2.10: Let random variables \((S_n)_{n \geq 1}\), have two finite moments, 
\[
\mu_n \equiv E S_n , \quad \sigma_n^2 \equiv \text{Var}(S_n) < \infty .
\]
If, for some sequence \(b_n\), we have \(\sigma_n/b_n \to 0\) as \(n \to \infty\), then 
\[
\frac{S_n - \mu_n}{b_n} \to 0 \quad \text{as } n \to \infty
\]
both in \(L^2\) and in probability.

Proof: Indeed, 
\[
E \left( \frac{(S_n - \mu_n)^2}{b_n^2} \right) = \frac{\text{Var}(S_n)}{b_n^2} \to 0 \quad \text{as } n \to \infty.
\]
Example 2.11: In the coupon collector’s problem (problem R 4), let $T_n$ be the time to collect all $n$ coupons. We know that

$$E T_n = n \sum_{m=1}^{n} \frac{1}{m} \sim n \log n, \quad \text{Var}(T_n) \leq n^2 \sum_{m=1}^{n} \frac{1}{m^2} \leq \frac{\pi^2 n^2}{6},$$

so that

$$\frac{T_n - E T_n}{n \log n} \to 0 \quad \text{ie.}, \quad \frac{T_n}{n \log n} \to 1$$

as $n \to \infty$ both in $L^2$ and in probability.
Almost sure convergence

Recall: **WLLN** is just \( \rightarrow \)

Let \( X_1, X_2, \ldots \) be i.i.d. r.v. with \( E X_1 = \mu \) and \( \text{Var} X_1 < \infty \).
Denote \( S_n \overset{\text{def}}{=} \sum_{k=1}^{n} X_k \).

By the usual \textit{(weak) law of large numbers} (**WLLN**): for every \( \delta > 0 \)
\[
P\left( \left| n^{-1} S_n - \mu \right| > \delta \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.2)
\]

I.e., **WLLN** states: \( n^{-1} S_n \overset{P}{\rightarrow} X \equiv \mu = E(X_1) \), as \( n \rightarrow \infty \).
**Notice:** In general

\[ X_n \xrightarrow{P} X \text{ is not related to } X_n(\omega) \rightarrow X(\omega) \text{ for fixed } \omega \in \Omega \]

ie., the **pointwise** convergence:

**Example 2.13:** Let \( \Omega = [0, 1] \), let \( \mathcal{F} = \sigma((a, b): 0 \leq a \leq b \leq 1) \), and let, for \( A = [a, b] \subseteq [0, 1] \), \( P(A) = b - a \).

\[
\forall A \in \mathcal{F} \quad \sim \quad \mathbb{1}_A(\omega) \overset{\text{def}}{=} \begin{cases} 
1, & \text{if } \omega \in A, \\
0, & \text{if } \omega \notin A.
\end{cases}
\] (2.3)

Consider \( X_n \overset{\text{def}}{=} \mathbb{1}_{A_n} \), where, for \( n \geq 1 \) such that \( 2^m \leq n < 2^{m+1} \),

\[ A_n = \left[ \frac{n-2^m}{2^m}, \frac{n+1-2^m}{2^m} \right] \subseteq [0, 1]. \]

Then \( X_n \xrightarrow{P} X \equiv 0 \), but

\[
\{ \omega \in \Omega : X_n(\omega) \rightarrow X(\omega) \equiv 0 \text{ as } n \rightarrow \infty \} = \emptyset, 
\]

ie., there is **no point** \( \omega \in \Omega \) for which \( X_n(\omega) \rightarrow X(\omega) \).
Almost sure convergence

Def. 2.14: A sequence $X_1, X_2, \ldots$ of r.v. in $(\Omega, \mathcal{F}, P)$ converges, as $n \to \infty$, to a random variable $X$ with probability one or almost surely (write $X_n \stackrel{a.s.}{\to} X$) if

$$P\left( \left\{ \omega \in \Omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty \right\} \right) = 1. \quad (2.4)$$

Remark 2.14.1: For $\varepsilon > 0$, let $A_n(\varepsilon) = \{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}$. Then the definition (2.4) is equivalent to saying that for every $\varepsilon > 0$

$$P\left( \{ A_n(\varepsilon) \text{ finitely often} \} \right) = 1. \quad (2.5)$$

This is why the Borel-Cantelli lemma is so useful in studying almost sure limits.
Example 2.13 [cont’d]: Let $(\Omega, \mathcal{F}, P)$ be as before; consider $Y_n \overset{\text{def}}{=} \mathbb{1}_{(0,1/n]}$ and $Z_n \overset{\text{def}}{=} \mathbb{1}_{[0,1/n]}$. Then

$$\{ \omega \in \Omega : Y_n(\omega) \to 0 \text{ as } n \to \infty \} \equiv [0, 1],$$

$$\{ \omega \in \Omega : Z_n(\omega) \to 0 \text{ as } n \to \infty \} \equiv (0, 1),$$

so that $Y_n \overset{\text{a.s.}}{\to} 0$ and $Z_n \overset{\text{a.s.}}{\to} 0$ as $n \to \infty$. 
Example 1.5 [cont’d]:

Let $X$ be a finite random variable, $P(|X| < \infty) = 1$.

Then the sequence $(X_k)_{k \geq 1}$ defined via $X_k \overset{\text{def}}{=} \frac{1}{k}X$ converges to zero with probability one.

Indeed, the event $\{\omega : X_k(\omega) \not\to 0\} = \{\omega : |X(\omega)| = \infty\}$ has probability zero.
Lemma 2.15: Let $X_1, X_2, \ldots$ and $X$ be r.v's. If, for every $\varepsilon > 0$, 

$$
\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty, \quad (2.6)
$$

then $X_n$ converges to $X$ almost surely.

This is **NOT** the definition of the almost sure convergence, but only a **sufficient condition** for it!

**Notice:** Let $X_1, X_2, \ldots$ and $X$ be r.v's. If $X_n \xrightarrow{P} X$, then there exists a **non-random** sequence of integers $n_1, n_2, \ldots$ such that 

$$
X_{n_k} \xrightarrow{\text{a.s.}} X \quad \text{as } n \to \infty.
$$

see Problem C11!
\textbf{L}^4 \textbf{S}trong \textbf{L}aw of \textbf{L}arge \textbf{N}umbers

\textbf{Theorem 2.16 (SLLN, Borel):} Let $X_1, X_2, \ldots$ be i.i.d. r.v. with

$$E(X_k) = \mu \quad \text{and} \quad E((X_k)^4) < \infty.$$ 

If $S_n \overset{\text{def}}{=} X_1 + X_2 + \cdots + X_n$, then

$$S_n/n \overset{\text{a.s.}}{\rightarrow} \mu \quad \text{as} \quad n \rightarrow \infty.$$
**L¹ Strong Law of Large Numbers**

**Theorem 2.17 (SLLN, Kolmogorov):** Let \( X_1, X_2, \ldots \) be i.i.d. r.v. with

\[
E|X_k| < \infty.
\]

If \( E(X_k) = \mu \) and \( S_n \overset{\text{def}}{=} X_1 + X_2 + \cdots + X_n \), then

\[
\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mu \quad \text{as} \quad n \to \infty.
\]
Relations between $\xrightarrow{L^r}$, $\xrightarrow{P}$, and $\xrightarrow{\text{a.s.}}$

We know that (Lemma 2.8)

$$X_n \xrightarrow{L^r} X \iff X_n \xrightarrow{P} X;$$

one can also show (we shall not do it here!)

$$X_n \xrightarrow{\text{a.s.}} X \iff X_n \xrightarrow{P} X.$$ 

In addition, according to Example 2.13,

$$X_n \xrightarrow{P} X \not\iff X_n \xrightarrow{\text{a.s.}} X,$$

and the same construction shows that

$$X_n \xrightarrow{L^r} X \not\iff X_n \xrightarrow{\text{a.s.}} X.$$
Example 2.18: Let $X_n$ be a sequence of independent random variables such that

$$P(X_n = 1) = p_n, \quad P(X_n = 0) = 1 - p_n.$$ 

Then, with $X \equiv 0$,

$$X_n \overset{P}{\to} X \iff p_n \to 0 \iff X_n \overset{L^r}{\to} X \quad \text{as } n \to \infty,$$

whereas

$$X_n \overset{\text{a.s.}}{\to} X \iff \sum_{n} p_n < \infty.$$ 

Taking $p_n = 1/n$, we get $X_n \overset{L^r}{\to} X$ but not $X_n \overset{\text{a.s.}}{\to} X$.

Notice that this example also shows that $X_n \overset{P}{\to} X \nRightarrow X_n \overset{\text{a.s.}}{\to} X$. 

$$X_n \overset{L^r}{\to} X \nRightarrow X_n \overset{\text{a.s.}}{\to} X.$$
\[
X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{L^r} X
\]

**Example 2.19:** Let \((\Omega, \mathcal{F}, P)\) be the canonical probability space (recall Example 2.13). For \(n \geq 1\), define

\[
X_n(\omega) \overset{\text{def}}{=} e^n \cdot 1_{[0,1/n]}(\omega) \equiv \begin{cases} 
  e^n, & 0 \leq \omega \leq 1/n \\
  0, & \omega > 1/n.
\end{cases}
\]

Clearly, \(X_n \xrightarrow{\text{a.s.}} 0\), and \(X_n \xrightarrow{P} 0\) as \(n \to \infty\); however, for every \(r > 0\)

\[
E|X_n|^r = \frac{e^{nr}}{n} \to \infty, \quad \text{as } n \to \infty, \quad \text{ie.,} \quad X_n \xrightarrow{L^r} 0.
\]

Notice that this example also shows that \(X_n \xrightarrow{\text{a.s.}} X \not\Rightarrow X_n \xrightarrow{L^r} X\).
Convergence of Random Variables

By the end of this section you should be able to:

- define convergence in $L^r$, verify whether a given sequence of random variables converges in $L^r$;
- define convergence in probability, verify whether a given sequence of random variables converges in probability;
- explain the relation between convergence in $L^r$ and convergence in probability (Lem 2.8);
- state and apply the sufficient condition for convergence in $L^2$ (Thm 2.10);
- define almost sure convergence, verify whether a given sequence of random variables converges almost surely;
- state and apply the sufficient condition for almost sure convergence (Lem 2.15);
- state and apply the Strong Laws of Large Numbers.