Probability & Statistics III (Term 2) - Tutorial 3 Solutions

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Problem 1.

It is essential in this question to know that 'fraternal twins' means 'non-identical twins', i.e. from two eggs (I just have to give you the Dutch term for this: *'twee-eiig'*, which surely is too good a word not to share with you).

Let TB denote 'Elvis had a Twin Brother', and let I denote 'Identical twin' and F 'Fraternal twin'. Then, using Bayes' theorem without considering the denominator, which is often the quickest way, so in the form 'posterior $\propto$ likelihood $\times$ prior', we get:

$$P(I|TB) \propto P(TB|I)P(I) = 1 \times \frac{1}{300},$$
$$P(F|TB) \propto P(TB|F)P(F) = \frac{1}{2} \times \frac{1}{125}.$$

Introducing the proportionality constant c, we have

$$P(I|TB) = c \times \frac{1}{300} \text{ and } P(F|TB) = c \times \frac{1}{250},$$

where c is such that these two probabilities sum up to 1. Some basic maths will now lead to (check!):

$$P(I|TB) = \frac{5}{11} \text{ and } P(F|TB) = \frac{6}{11}.$$

Problem 2.

Note: This problem was supposed to be little bit 'beyond exam level', but it would illustrate the possible use of Bayes' theorem to update in light of new information, and indeed to predict further observations, if the observations are *not* conditionally independent.

However, you would need to know some basic probability results on 'sampling without replacement' (the so-called 'hypergeometric distribution'), which during the tutorial, to my great surprise, I learned have not been covered in Term 1.

I think therefore this is now 'way beyond' examinable material. Explaining it would require a lot of further material, which might be very confusing without explicitly lecturing on it. Hence, please neglect this exercise, sorry about this! (If you are very keen, and wish to learn more about probability, study the topic mentioned above from any basic probability textbook, and ask me for help if needed).

Problem 3.

In this problem, the natural way of reasoning is: (1) how does the information that John's birthday is later in the year than Paul's birthday affect our probability distribution for Paul's birthday (over the relevant periods of the year), and then (2) how to predict for Simon's birthday, given this information on Paul's birthday. We'll see that this is precisely the way that the Bayesian method works!

(i) Let random variable P denote the month in which Paul's birthday is, so our prior information is $P(P = i) = \frac{1}{12}$ for all months $i = 1, \ldots, 12$. Use similar notation and priors for John's birthday J and Simon's birthday S. The probability of interest is

$$P(S < P|P < J) = \sum_{i=1}^{12} P(S < P|P = i, P < J) \times P(P = i|P < J),$$

where we use the theorem of total probability to explicitly focus on the month of Paul's birthday, which is of course central to the whole problem. Note that, if we would know this month (the conditioning on $P = i$ here), then knowing in addition that John's birthday is later is irrelevant to the event $S < P$, hence

$$P(S < P|P = i, P < J) = P(S < P|P = i) = \frac{i-1}{12}.$$

So, we must now consider the second probability on the right-hand side, for which we use Bayes' theorem

$$P(P = i|P < J) = \frac{P(P < J|P = i)P(P = i)}{P(P < J)},$$

where we calculate the denominator, as usual, via the theorem of total probability

$$P(P < J) = \sum_{j=1}^{12} P(J > P|P = j)P(P = j) = \sum_{j=1}^{12} \frac{12-j}{12} \times \frac{1}{12}.$$

Combining all this, we get

$$P(S < P|P < J) = \sum_{i=1}^{12} \left[\frac{i-1}{12} \times \frac{\frac{12-i}{12} \times \frac{1}{12}}{\sum_{j=1}^{12} \frac{12-j}{12} \times \frac{1}{12}} \right] = \frac{1}{12 \times 66} \sum_{i=2}^{11} [(i-1) \times (12-i)] = 0.2778$$

(ii) We can generalize this by partitioning the year into n equal-length periods, and repeat the same exercise (e.g. $n = 365$ would make days, larger n would of course require e.g. knowing the exact hours of birth, etc, which may make it practically less relevant, but mathematically much nicer!). You should, of course, repeat the same analysis and reasoning as above, which will lead to (using the two sums given with the problem along the way)

$$P(P = i|P < J) = \frac{2(n-i)}{n(n-1)},$$

and

$$P(S < P|P < J) = \frac{1}{3} - \frac{2}{3n}.$$

(Check that for $n = 12$ this gives indeed the answers above.)

Notice that this last probability increases to $1/3$, for $n \to \infty$, which I think is intuitively clear, because effectively it is just asking what is the chance that Simon's birth-moment, considered within one year ('birth-moment' like we use the word 'birthday'), is the first of these three persons. As we know nothing about them, they all have the same chance of having the first birth-moment, hence $1/3$. The reason why, for any n, this probability is actually a little bit less than $1/3$ is because we exclude in our reasoning the probabilities of the birth-moments of two of them to be in the same period.

Gosh, do I need to know this?
Yes and no; you should understand the way the reasoning goes, and hopefully understand in particular part (i). If this had been an exam question, I would have first asked to consider $P(P = i|P < J)$, and then the predictive probability with regard to Simon's birthday.