Probability & Statistics III (Term 2) - Tutorial 2, short solutions

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During the tutorials in week 15 we did not completely cover the problems on the sheet 'Tutorial 2'. Below are very short solutions to parts that we did cover, and extended solutions to the parts we did not (i.e. sensitivity analysis in 1(a), and 2).

Problem 1.

(a) The investor's utilities for both options (invest or not) are identical, so the investor is indifferent between investing or not in this situation. For the sensitivity analysis, in the tutorials we calculated the preferred choices in case his assets were £9,000 (not invest) or £11,000 (invest). We can do this more generally as follows:
Assume the investor's assets are £$\gamma 10K$, with γ close to 1. Not investing gives $U(\gamma 10K) = \ln(\gamma 10K) = \ln\gamma + \ln 10K$. Investing £$5K$ leads to the gamble

$$g_\gamma = \frac{1}{2}(\gamma 10K + 10K) +_g \frac{1}{2}(\gamma 10K - 5K),$$

with utility

$$U(g_\gamma) = \frac{1}{2}\ln((\gamma+1)10K) + \frac{1}{2}\ln((2\gamma-1)5K) = \frac{1}{2}\left[\ln(\gamma+1) + \ln 10K + \ln(2\gamma-1) + \ln 5K\right].$$

Hence, the investor will not invest if $U(\gamma 10K) > U(g_\gamma)$, which is (check!) if

$$\frac{1}{2}\left[\ln(\gamma+1) + \ln(2\gamma-1)\right] - \ln\gamma < \frac{1}{2}(\ln 10K - \ln 5K) = \frac{1}{2}\ln 2.$$

Call the left-hand side of this inequality $f(\gamma)$, and analyse this function. At $\gamma = 1$, $f(1) = \frac{1}{2}\ln 2$, which we of course knew as this is just the case considered above, so the investor is indifferent. Take the first derivative of $f(\gamma)$:

$$\frac{df(\gamma)}{d\gamma} = \frac{1}{2}\left[\frac{1}{\gamma+1} + \frac{2}{2\gamma-1}\right] - \frac{1}{\gamma},$$

so $f'(1) = 0.25$. Hence, at $\gamma = 1$, this function $f(\gamma)$ is increasing, and due to its continuity, and its first derivative being continuous, at γ close to 1, it follows that the investor would not invest if his assets were less than $10K$, but would do so if his assets were more than $10K$. Note that we have really only shown this for a small region around $10K$ with this argument, but this analysis can be extended straightforwardly, by considering $f'(\gamma)$ a bit further. It is easy to show that $f'(\gamma)$ is positive for all $0.5 < \gamma < 2$ (where we start from 0.5 to prevent a denominator becoming zero in the expression above). Hence, $f(\gamma)$ is less than $f(1)$ for all $\gamma \in (0.5, 1)$, which implies no investment, and $f(\gamma)$ is greater than $f(1)$ for all $\gamma \in (1, 2)$, which implies investment. Actually, the investor would also invest for larger assets than $20K$, and not for any smaller than $5K$, but that is easier to show directly than using this argument based on $f'(\gamma)$.

(b) $U_1(x) = U(x) - \ln 1{,}000$, so just a linear transformation of the utility function, hence these two utility functions represent the same preferences (you may want to do part (a) again, using U_1, to see that this is indeed true).

(c) The investor will invest for any $m < 5{,}000$, but not for $m > 5{,}000$. This, as well as our study in part (a), relates directly to his risk-averseness, with decreasing local risk aversion for larger assets (check!).

(d) Confirm the following optimal strategy: the investor will accept the bet with his friend. If he wins the bet, he will invest, if he loses the bet he will not invest. This gives maximum utility of 9.2155.

Problem 2.
For current assets £2.5K, the person prefers A. For assets £5K, the person prefers B. Interestingly, for assets £10K the person prefers A again. Note that this person is risk averse, but decreasingly so for larger assets (same as in Problem 1). With small assets, option A is 'safer' as it will certainly lead to a profit, but most probably a small profit. Then, when assets increase somewhat, option B becomes more interesting as it has a large probability to win a more substantial amount, with a small risk of losing a bit. When the assets increase even more, the person becomes ever less risk averse, and his preference will turn towards A, which after all has the highest Expected Money Value of these two alternatives (2.8K vs 2.5K), and then it is the chance of winning a substantial amount that seems to have most influence. (Think about this in detail: do you think such preferences are reasonable?)