Probability & Statistics III (Term 2) - Homework 8

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A discrete random variable X, which can take as value all non-negative integers, has a Poisson distribution with parameter $\lambda > 0$ if

$$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}.$$

(a) Show that this is a member of the Exponential family of distributions. Show that the conjugate priors for this distribution are the Gamma distributions, where the Gamma(α, β) distribution for non-negative continuous random quantity Y has probability density function (with $\alpha > 0$ and $\beta > 0$)

$$p(y|\alpha, \beta) = \frac{\beta(\beta y)^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)},$$

where $\Gamma(\alpha)$ is the so-called 'Gamma function' (for $z > 0$):

$$\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du,$$

which, for integer z, is $\Gamma(z) = (z-1)!$. This Gamma distribution has mean α/β and variance α/β^2 (no proof needed).

(b) Suppose that the numbers of typing errors I make in first drafts of homework sheets can be modelled as random quantities with a Poisson distribution, with unknown parameter $\lambda > 0$. My prior for λ is a Gamma distribution with mean value 1 and variance 0.5. Suppose that, in the first drafts of the first six homework sheets for a particular course I made 3, 6, 2, 1, 2 and 3 typing errors. Derive the corresponding posterior distribution for λ. Are the data and my prior beliefs about my typing skills reasonably in agreement?