Probability & Statistics III (Term 2) - Homework 7

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Problem 1.

This exercise illustrates that the posterior distribution is a compromise between prior information and data: let Y be the number of heads in n spins of a coin, whose probability of heads is θ (unknown).

(a) Derive the prior predictive distribution for Y corresponding to a uniform prior distribution for θ on the interval $[0, 1]$. (So no data taken into account here.)

(b) Now assume a Beta(α, β) prior for θ, and assume y heads observed out of n spins. Show that the posterior mean of θ always lies between the prior mean and the observed relative frequency of heads in the data.

(c) Show that, if the prior is uniform, the posterior variance of θ is always less than the prior variance.

(d) Give an example of a Beta(α, β) prior, and data y, n, for which the posterior variance of θ is greater than the prior variance of θ. Give your comments on such situations.

Problem 2.

Suppose one has a Beta$(4, 4)$ prior for the probability θ that a coin will yield head when spun in a specific manner. The coin is spun ten times, and head appears fewer than 3 times, i.e. the exact number is not given, only that it is less than 3. Derive the corresponding posterior probability density function for θ, up to a proportionality constant.