1. The Gambler

(a) A gambler suggests that you play the following game. You each toss a coin. If you get heads (H) and he gets tails (T), you win £30. If you get T and he gets H, you win £10. If the coins match, you lose £20. You find that you have no coins. “Never mind”, says the gambler, “we can still play the game, instead of tossing coins, each time I’ll write down H or T and you do the same. We score as before.” Should you play? (i.e. find the value of the game.)

Answer. The pay-off table (with pay-off to you as shown, minus these values to the gambler) is:

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>-20</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-20</td>
</tr>
</tbody>
</table>

You choose H with probability \( p \), the gambler chose H with probability \( q \). Check that the value of the game is \( V = q(10 - 30p) + (1 - q)(50p - 20) \), so at the minimax solution we have (otherwise the gambler could benefit by changing \( q \)): \( 10 - 30p = 50p - 20 \), so \( V = \frac{-10}{8} \). As \( V < 0 \), it is better not to play! (Of course, if you are forced to play, then playing your minimax strategy may be a good idea.) □

(b) Suppose that you decline to play the game. The gambler offers you the following alternative. “You write down the outcome of one toss, I’ll write down the outcome of two tosses. Payoffs will be as follows”:

<table>
<thead>
<tr>
<th></th>
<th>HH</th>
<th>HT</th>
<th>TH</th>
<th>TT</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

(For example, if you play H and he plays TH then you pay him a pound.)

Analyse this game using a graphical method (i.e. find the value, strategies for both players, whether you should play, by each method).

Answer. After plotting \( f(p) \) (i.e. the minimum expected reward for you if you play H with probability \( 1 - p \)), the minimax solution is achieved at \( p^* = 1/2 \): you play H and T each with probability \( 1/2 \). The value of the game is \( -\frac{1}{2} \); again it is better not to play. The gambler’s minimax strategy is not unique; the gambler has three lines crossing at \( p = 1/2 \) so he can choose any mixture of these three which includes one line-segment which increases as function of \( p \) and one which decreases as function of \( p \). So he can play TH and either HH or HT. If he plays ‘TH or HH’, then he must play TH with probability \( \frac{3}{4} \). If he plays ‘TH or HT’, then he should play TH with probability \( \frac{1}{2} \). He can also allow all these three strategies, say with probabilities \( p_{TH}, p_{HT}, p_{HH} \) which sum up to one. Ensuring that you cannot benefit from superior skills, however, leads in this case to \( p_{TH} = \frac{1}{2}, p_{HH} = \frac{1}{4} \) and \( p_{HT} = 0 \), so it coincides with an earlier solution. This could have been different (it of course depends on the payoff table), e.g. it might happen that only the sum of
2 SOLUTION: ZERO SUM GAMES

two or more such probabilities is fixed but there is free-
don on how to choose the individual probabilities. The
most trivial case where this happens is if two columns in
the pay-off table are identical, and both cross with an-
other column, in the graphical method, at the optimal

\[ p^* \]

\[ \square \]

2. Minimax Solution: Graphical Solution

Suppose that in a particular zero sum game the payoffs
to R are as follows:

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0</td>
<td>-3</td>
<td>6</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>R2</td>
<td>1</td>
<td>4</td>
<td>-2</td>
<td>-1</td>
<td>-4</td>
</tr>
</tbody>
</table>

Use a graphical method to solve the game (include the strate-
gies for both players).

Answer. The minimax solution is: R plays his option 1 with
probability 0.4, so option 2 with probability 0.6. C plays his
option 1 with probability 0.4, and option 4 with probability
0.6. The value of the game is 2.4.

\[ \square \]

3. Another 2 \times n Game

Solve the two-person zero-sum game, with pay-offs given
below (with pay-offs as shown to R, and the negative of those
pay-offs to C), by use of a geometric method. Describe which
strategy each of the players should choose corresponding to
the solution.

<table>
<thead>
<tr>
<th></th>
<th>C:</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R:</td>
<td>1</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Answer. C3 and C4 are dominated by C5 so they can be
omitted right away. Plotting the lines of this game corre-
sponding to the remaining strategies of C, shows that \( p^* \) is

solution of (intersection of C2 and C5)

\[ 4p - 3(1 - p) = -4p + (1 - p) \]

So \( p^* = 1/3 \) and the value of the game is \( V = 4p^* - 3(1 -
\[ p^* \]) = -2/3. \]

We end up with a 2 \times 2 game:

\[ \begin{bmatrix} 4 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} q^* \\ 1 - q^* \end{bmatrix} = V \]

C will play a strategy so that his payoff does not depend on
\( p. \) Hence, solving

\[ \begin{bmatrix} 4 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} q^* \\ 1 - q^* \end{bmatrix} = V \]

for \( q^* \), we find \( q^* = 5/12 \), so the second player plays C2 with
probability 5/12 and C5 with probability 7/12.

\[ \square \]

4. Two Burglars (*)

(This problem explores what happens in the repeated
prisoners’ dilemma when the number of games is random.
This is not directly an exam-style question, as we have only
paid little attention to such games in the lectures. If such a
question were to be asked in the exam, more guidance would
be given instead of just asking you to analyse the game.)

Burglars Bill and Betty are arrested. The police have
enough evidence to lock them up for a Small Job. They
know, but cannot prove, that they also committed a Big
Job. Bill and Betty are separated and each is offered the
following deal. If neither confesses, then each will serve
\( u \) years for the small crime. If one confesses, and the other
does not, then the burglar who confesses will serve \( v \) years,
the other \( w \) years. If both confess, they will each do \( x \) years.
The values are such that \( v < u < x < w. \)

Suppose further that the police have evidence to convict
Bill and Betty of several such jobs. For each job, there is
a small version for which they can convict both burglars, and a big version which they can only convict on if at least one burglar confesses. Each crime is handled in sequence with `scoring’ as above. Bill and Betty are not allowed to communicate, but they are told, after each crime is treated, whether or not their partner confessed.

Suppose also that Bill and Betty do not know how many crimes they will be accused of. Suppose in particular that after each crime has been settled the police toss a coin which has probability $p$ of landing heads. If the coin lands heads, then Bill and Betty are accused of the next crime, while if the coin falls tails the process stops. [Not realistic— but it gives us a stopping rule which is easy to analyse.]

Analyse the game. In particular, show that for any $v < u < x < w$ there are certain values of $p$ for which never confessing is an equilibrium strategy for Bill and Betty (i.e. devise a strategy such that if either burglar follows the strategy then both burglars can do no better in expected payoff than never confessing).

If Bill does not confess, nor does Betty, then payoffs are $u$ years and Bill’s expected payoff is $c_2 = u + pu + p^2u + p^3u + \ldots$.

It is optimal for Bill not to confess (and hence never to confess) if $c_2 < c_1$, so if $p > \frac{u-v}{x-v}$. Therefore, we can always find values of $p \in [0, 1]$ for which not to confess is optimal. It is an equilibrium strategy as no player can improve things for him/her by moving away from this strategy, for such values of $p$. For other values of $p$ it is harder to analyse ‘correct play’. □

Answer. Suppose Betty plays the following strategy: She will never confess to a crime until she hears that Bill has confessed to a crime. As soon as she hears that Bill has confessed, she confesses to all crimes from that point onwards. This strategy makes it least advantageous for Bill to confess, as he is punished most severely if he does.

Now, supposing Betty plays that strategy, should Bill ever confess? If he confesses, then on that play he gets $v$ years (as Betty does not confess), and on every further play he gets $x$ years (as Betty confesses, and the best Bill can do is to confess). So, because the probability of playing game $k + 1$ is $p^k$, Bill’s expected payoff from the moment of confessing is: $c_1 = v + px + p^2x + p^3x + \ldots$.