We briefly introduce and discuss two two-player non-zero-sum games, mainly to illustrate that such games are substantially more complex to investigate than zero-sum games. Actually, the key aspect of such games is that the sum of the pay-offs to the two players is not constant over all pairs of strategies, as constant sum games are equivalent to zero-sum games (think about this; one can imagine two players having to jointly pay a constant value in order to play a zero-sum game, this would not affect the solution method). Both games we discuss below have triggered considerable further contributions to the literature, with many variations studied (material on this is easily found on-line). John Nash proposed an equilibrium solution for such games, which has the advantage that, if either player plays the corresponding strategy, the other player cannot do better than also play their corresponding strategy. Hence, as for two-person zero-sum games, this could possibly again be considered as a sensible solution if one wishes to eliminate superior skills your opponent may have. In these games, however, it may be possible for both players to do better if they take a ‘cooperative view’ - there is no discussion between the players, but sometimes doing so may be quite natural. (We have already seen this in the Prisoner’s Dilemma, so here just two further examples, and we formally formulate the Nash Equilibrium for such games.)

1. The Traveller’s Dilemma

Returning from a holiday, the airline has broken an item you had purchased. The airline manager is happy to compensate for the item, and informs you that, by coincidence, they also broke the same item of another traveller. To decide on the level of compensation, the manager asks you and the other traveller, independently, to write down the price, say any integer from 2 to 100. If both of you write down the same number, the manager will assume it is the true price and pay both of you this amount. If the numbers differ, the manager will take the lower number to be the true price and assume that the other person is lying. In this case, the manager will pay both of you the lower number, with an extra reward of 2 for the person who reported the lower number and a penalty of 2 for the person who reported the higher number. For example, if you wrote 87 and the other person 34, the other person will get 36 and you will get 32. Which number should you write down (assuming both players aim at maximizing their payoff)?

Analysis

Let $S_i (i = 1, 2)$ be a set of strategies for player $i$, and let $\pi_i(s_1, s_2)$ be a payoff for player $i$ if strategies $s_1 \in S_1$ and $s_2 \in S_2$ are played. A Nash Equilibrium is a pair of strategies $(s_1^*, s_2^*)$ such that

$$\pi_1(s_1^*, s_2^*) \geq \pi_1(s_1, s_2^*) \text{ for all } s_1 \in S_1 \quad \text{and} \quad \pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, s_2) \text{ for all } s_2 \in S_2$$

For the Traveller’s Dilemma, perhaps somewhat surprisingly, $(2, 2)$ is the unique Nash Equilibrium. To prove this claim, we only need to check the condition for player 1 as the game is fully symmetric. First, $(2, 2)$ is a Nash Equilibrium because, if player 2 plays strategy $s_2 = 2$, then player 1 can do no better than playing $s_1 = 2$:

$$\pi_1(2, 2) = 2 \geq 0 = \pi_1(n, 2) \text{ for all } n > 2$$

Next we show uniqueness, so there is no other pair of strategies $(s_1, s_2) = (m, n)$ which is a Nash Equilibrium. First suppose that $m = n > 2$, then

$$\pi_1(n - 1, n) = n + 1 > n = \pi_1(n, n) = \pi_1(m, n)$$
so player 1 can do better by playing $n - 1$ instead of $n$ if player 2 plays $n$. Secondly, suppose that $m > n > 2$, then

$$\pi_1(n-1,n) = n + 1 > n - 2 = \pi_1(m,n)$$

so player 1 can do better by playing $n - 1$ instead of $m$. Thirdly, suppose that $2 \leq m < n - 1$, then

$$\pi_1(n-1,n) = n + 1 > m + 2 = \pi_1(m,n)$$

Finally, from the argument to prove that $(2,2)$ is a Nash Equilibrium, it follows that for $m > n = 2$ player 1 will do better by playing 2 instead of $m$. Hence, from the perspective of player 1, and given player 2 plays $s_2 = n > 2$, the only candidate strategy to be part of a Nash Equilibrium is $s_1 = n - 1$. But if player 1 plays $s_1 = n - 1$, then for player 2 the only candidate strategy to be part of a Nash Equilibrium is $s_2 = n - 2$, et cetera. This completes the proof that no pair $(s_1, s_2) = (m,n) \neq (2,2)$ can be a Nash Equilibrium.

It seems quite obvious that both players reporting value 2 is probably not satisfactorily, as they wish to maximize their payoff, and it is unlikely to cover the real value of the item that was broken. Actually, this game has been played (without breaking actual items..) and analyzed in detail. It turns out that most people choose to report a value in the range 90 to 99, which seems quite sensible if one assumes that the other player reports the value 100 or also a value in this range. This assumes implicitly some cooperation among the players. To include this in the analysis, the concept of cooperative strategies has been introduced as the set of strategies which result in equal payoff for both players, and the concept of cooperative equilibrium strategies as the set of optimal cooperative strategies in the sense of maximum payoff for both players among all cooperative strategies. Clearly, in the Traveller’s Dilemma $(100,100)$ is the unique cooperative equilibrium strategy, which differs enormously from the unique Nash equilibrium $(2,2)$. Hence, it does make sense in such games to assume that the other player is also willing to cooperate, albeit not formally so (e.g. as no discussion is allowed), and to play $s_1 = 100$. Researchers have also investigated this further, by you assigning a probability to the event that the other player will cooperate and decide your best strategy accordingly, we do not address this further, but it should be clear that already for quite a small probability that the other player will cooperate you are better off, in expectation, playing $s_1 = 100$ than $s_1 = 2$.

2. The Hawk - Dove Game

This is a basic game which plays an important role in research into (aspects of game theory underlying) behaviour in biology (including humans). Many more complex scenarios are variations of this basic game. Consider two animals (often males..) who may compete for a ‘reward’, e.g. territory, a female, or perhaps just to show off. Each can choose to adopt one of two possible strategies, namely to ‘display’ or to ‘escalate’, the former is referred to as ‘Dove’, where the individual will not enter violent confrontation, and the latter as ‘Hawk’, where the individual will start violent altercation. When a Dove (D) is confronted by a Hawk (H), D loses immediately. When two D confront each other, the display will go on for some time after which one will give up, leaving the reward to the other. When two H confront each other, they will fight until one is injured (which may be severely, even death) and gives up leaving the reward to the other. It is assumed that, in the D-D and H-H cases, each individual may win with probability 0.5. Let $W > 0$ denote the reward and $L > 0$ the loss due to injury for one H in case two H-H confront each other, then the (expected) pay-off table is as follows, where pay-off $(R,C)$ denotes the pay-off to the Row and Column individuals; the game is completely symmetrical so we can analyze it from either’s perspective.
The two individuals, assumed to be of similar size and apparent strength, can each opt to take the D or H strategy, and we only consider a single contest. Variations to this game involve multiple contests with possible learning, differences to the pay-off table corresponding to different assumptions (e.g. there may be a loss involved for a D losing out to another D, but not as high as $L$) and more possible strategies could be considered. But, restricting attention to this basic game, we consider two cases.

If $W \geq L$, then strategy H dominates strategy D for each player (that is at least equally good for all strategies chosen by the opponent, and strictly better for at least one such strategy). Hence, $(H, H)$ is a Nash equilibrium, but both players are worse off (in expectation) than for strategy $(D, D)$. This is similar to the Prisoner’s Dilemma.

The case $W < L$ is more interesting, and possibly more realistic if e.g. losing in the H-H confrontation may imply severe injury or death. Suppose C plays H with probability $p$ and D with probability $1 - p$, then the expected pay-off to R when playing H or D are

\[
E_R(H) = \left(\frac{W - L}{2}\right)p + W(1 - p) \quad \text{and} \quad E_R(D) = \frac{W}{2}(1 - p)
\]

If C wishes to eliminate any advantage R may have due to supreme reasoning skills, he can do this by ensuring that $E_R(H) = E_R(D)$. This holds if and only if he chooses $p^* = W/L$. Note that, in situations where possible death may follow losing in an H-H confrontation, $L$ may be very much greater than $W$, hence the probability of playing the H strategy would be very small. (This has been mentioned as an explanation why conflicts like the cold war tend not to escalate.)

It is easy to check that, if one replaces the pay-off for the D-D confrontation to $(0, 0)$, as may be appropriate for some scenarios with $W < L$, then the corresponding $p^* = 2W/(W + L)$, which is also similarly small as the value of $p^*$ just discussed, so this does not make a major difference.

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