Thick morphisms and homotopy bracket structures

Theodore Voronov

University of Manchester, UK, and Tomsk State University, Russia

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“Microformal geometry” in brief

Key points: thick morphisms and nonlinear pullbacks

- There is a notion of *thick* (or *microformal*) *morphisms* of (super)manifolds generalizing ordinary smooth maps;
- Key difference: the pullback $\Phi^*$ by a thick morphism $\Phi: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$,
  $$\Phi^*: C^\infty(\mathcal{M}_2) \rightarrow C^\infty(\mathcal{M}_1),$$
  
is, in general, **nonlinear** (actually, formal) map of infinite-dimensional manifolds of even functions.
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  $$\Phi^*: C^\infty(M_2) \rightarrow C^\infty(M_1),$$
  is, in general, *nonlinear* (actually, formal) map of infinite-dimensional manifolds of even functions.

“Why we care”; in particular:

- Motivation: $L_\infty$-morphisms of homotopy Poisson brackets;
- Applications and development: homotopy structures; duality of vector spaces and bundles; fermionic and quantum versions;
- Hints at a “nonlinear extension” of algebra-geometry duality.
Recall that the following structures are equivalent:

- $L_\infty$-algebra (in antisymmetric version) $L$
- $L_\infty$-algebra (in symmetric version) $\Pi L = V$
- (Formal) $Q$-structure on $V$, i.e., $Q \in \text{Vect}(V)$, $\tilde{Q} = 1$, $Q^2 = 0$. 

$L_\infty$-morphisms of $L_\infty$-algebras as $Q$-maps

Problem: what for functions?!

If we have $L_\infty$-brackets on $C_\infty(M_1)$ and $C_\infty(M_2)$ (e.g. homotopy Poisson or homotopy Schouten), what is a "natural" construction for $L_\infty$-morphisms? (Should be NONLINEAR maps! Pullbacks will not work! )
Motivation

$L_\infty$-algebras (or SHLAs) as $Q$-manifolds

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$L_\infty$-morphism $L \leadsto K \iff$ (nonlinear) $Q$-morphism $\Pi L \to \Pi K$
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Example: higher Koszul brackets

**Classical fact:** for a Poisson $M$, there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{A}^k(M) & \xrightarrow{d_P} & \mathcal{A}^{k+1}(M) \\
\uparrow & & \uparrow \\
\Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M),
\end{array}
$$

and vertical arrows map Koszul bracket to the Schouten bracket.
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**Homotopy case:** for a homotopy Poisson $M$, one can still construct

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**SOLUTION:** pullback by a **thick morphism**!
Definition of a microformal (thick) morphism

Let $M_1$, $M_2$ be supermanifolds with local coordinates $x^a$, $y^i$. Let $p_a$ and $q_i$ be conjugate momenta (fiber coordinates in $T^*M_1$, $T^*M_2$) and $\omega_1 = dp_a dx^a$, $\omega_2 = dq_i dy^i$ be the symplectic forms.
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Definition

A microformal (aka thick) morphism $\Phi: M_1 \rightarrow M_2$ is a formal Lagrangian submanifold $\Phi \subset T^*M_2 \times T^*M_1$ w.r.t. $\omega_2 - \omega_1$ specified locally by a generating function of the form $S(x, q)$:

$$q_i dy^i - p_a dx^a = d(y^i q_i - S) \quad \text{on } \Phi,$$

where $S(x, q)$, regarded as a part of the structure, is a formal power series in momenta

$$S(x, q) = S_0(x) + S^i(x)q_i + \frac{1}{2} S^{ij}(x)q_jq_i + \frac{1}{3!} S^{ijk}(x)q_kq_jq_i + \ldots$$
Pullback by a microformal morphism

Construction of pullback

Let $\Phi : M_1 \rightarrow M_2$ be a thick morphism with the generating function $S$. The pullback $\Phi^*$ is a formal mapping $\Phi^* : \mathcal{C}^\infty(M_2) \rightarrow \mathcal{C}^\infty(M_1)$ of functional supermanifolds (of ‘bosonic’ functions) defined by

$$\Phi^*[g](x) = g(y) + S(x, q) - y^i q_i,$$

for $g \in \mathcal{C}^\infty(M_2)$, where $q_i$ and $y^i$ are determined from the equations

$$q_i = \frac{\partial g}{\partial y^i}(y), \quad y^i = (-1)^\tilde{i} \frac{\partial S}{\partial q_i}(x, q)$$

(giving $y^i = (-1)^\tilde{i} \frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y))$ solvable by iterations).

Heuristically, if $f = \Phi^*[g]$, then $\Lambda_f = \Lambda_g \circ \Phi$, where $\Lambda_f = \text{gr}(df)$.
General form of pullback

**Example**

Let $S(x, q) = S^0(x) + \varphi^i(x)q_i$. Then: $\Phi^*[g] = S^0 + \varphi^*g$.

(NB: ordinary maps have generating functions $S = \varphi^i(x)q_i$.)
General form of pullback

Example

Let \( S(x, q) = S^0(x) + \varphi^i(x)q_i \). Then: \( \Phi^*[g] = S^0 + \varphi^*g \).

(NB: ordinary maps have generating functions \( S = \varphi^i(x)q_i \).)

For a general \( S(x, q) = S^0(x) + \varphi^i(x)q_i + \ldots \), the equation
\[
y^i = (-1)^i \partial S \partial q_i (x, \partial g \partial y (y))
\]
defines a map \( \varphi_g : M_1 \to M_2 \) as a formal perturbation of \( \varphi = \varphi_0 : M_1 \to M_2 \):
\[
\varphi^i_g(x) = \varphi^i(x) + S^{ij}(x)\partial_j g(\varphi(x)) + \ldots,
\]
and \( \Phi^*[g](x) = (g(y) + S(x, q) - y^i q_i) \big|_{y=\varphi_g(x), q=\partial g/\partial y(\varphi_g(x))} \),
which gives \( \Phi^* \) as a formal nonlinear differential operator:

General form of \( \Phi^*: \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1) \)

\[
\Phi^*[g](x) = S^0(x) + g(\varphi(x)) + \frac{1}{2} S^{ij}(x)\partial_i g(\varphi(x))\partial_j g(\varphi(x)) + \ldots
\]
Coordinate invariance

Transformation law for generating functions of thick morphisms

A generating function $S(x, q)$ as a geometric object on $M_1 \times M_2$ transforms by

$$S'(x', q') = S(x, q) - y^i q_i + y^{i'} q_{i'}.$$  

Here $S(x, q)$ is the expression for $S$ in ‘old’ coordinates and $S'(x', q')$ is the expression for $S$ in ‘new’ coordinates. At the r.h.s., the variables $x^a$ and $y^{i''}$ are given by substitutions: $x^a = x^a(x')$ and $y^{i''} = y^{i''}(y)$, while $q_i$ and $y^i$ are determined from

$$q_i = \frac{\partial y^{i''}}{\partial y^i}(y) q_{i''}, \quad y^i = (-1)^\tilde{i} \frac{\partial S}{\partial q_i}(x, q).$$

The transformation law satisfies the cocycle condition. The canonical relation $\Phi \subset T^* M_2 \times (-T^* M_1)$ specified by $S$ is well-defined. Pullbacks do not depend on a choice of coordinates.
Theorem

Let $\Phi : M_1 \to M_2$ be a thick morphism. Consider the pullback

$$\Phi^* : \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1).$$

Then for every $g \in \mathcal{C}^\infty(M_2)$, the derivative $T\Phi^*[g]$ is given by

$$T\Phi^*[g] = \varphi_g^*,$$

where $\varphi_g^* : \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1)$ is the usual pullback with respect to the map $\varphi_g : M_1 \to M_2$ defined by $y^i = \left( -1 \right)^i \partial S / \partial q_i (x, \partial g / \partial y(y))$ (depending perturbatively on $g$, $\varphi_g = \varphi_0 + \varphi_1 + \varphi_2 + \ldots$).

Corollary

For every $g$, the derivative $T\Phi^*[g]$ of $\Phi^*$ is an algebra homomorphism $\mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1)$.
Composition law

Consider thick morphisms $\Phi_{21} : M_1 \rightarrow M_2$ and $\Phi_{31} : M_2 \rightarrow M_3$ with generating functions $S_{21} = S_{21}(x, q)$ and $S_{32} = S_{32}(y, r)$.

**Theorem**

The composition $\Phi_{32} \circ \Phi_{21}$ is well-defined as a thick morphism $\Phi_{31} : M_1 \rightarrow M_3$ with the generating function $S_{31} = S_{31}(x, r)$, where

$$S_{31}(x, r) = S_{32}(y, r) + S_{21}(x, q) - y^i q_i$$

and $y^i$ and $q_i$ are expressed through $(x^a, r_\mu)$ from the system

$$q_i = \frac{\partial S_{32}}{\partial y^i}(y, r), \quad y^i = (-1)^\tilde{i} \frac{\partial S_{21}}{\partial q_i}(x, q),$$

which has a unique solution as a power series in $r_\mu$ and a functional power series in $S_{32}$. 
Further facts

**Formal category**

Composition of thick morphisms is associative and and

\[(\Phi_{32} \circ \Phi_{21})^* = \Phi_{21}^* \circ \Phi_{32}^*\]. In the lowest order, the composition is as in \(S\text{Man} \ltimes \mathbb{C}^\infty\), whose arrows are pairs \((\varphi_{21}, f_{21})\) with the composition \((\varphi_{32}, f_{32}) \circ (\varphi_{21}, f_{21}) = (\varphi_{32} \circ \varphi_{21}, \varphi_{21}^* f_{32} + f_{21})\).

Thick morphisms form a *formal category* ("formal thickening" of \(S\text{Man} \ltimes \mathbb{C}^\infty\)). Denote it \(\mathcal{E}\text{Thick}\).

**“Fermionic version”**

There is a *fermionic version* based on anticotangent bundles \(\Pi T^* M\) and odd generating functions \(S(x, y^*)\): “odd thick morphisms” \(\Psi: M_1 \Rightarrow M_2\) induce nonlinear pullbacks \(\Psi^*: \Pi C^\infty(M_2) \rightarrow \Pi C^\infty(M_1)\) of odd functions ("fermionic fields") and their composition gives another formal category, \(\mathcal{O}\text{Thick}\), which contains \(S\text{Man} \ltimes \Pi C^\infty\).
Recollection: $L_\infty$-algebras and $L_\infty$-morphisms – 1

We consider $\mathbb{Z}_2$-graded version. (One can include a $\mathbb{Z}$-grading.)

There are two parallel notions: “symmetric” and “antisymmetric”.

**Definition ($L_\infty$-algebra: antisymmetric version)**

A vector space $L = L_0 \oplus L_1$ with a collection of multilinear operations

$$[-, \ldots, -]: L \times \ldots \times L \rightarrow L \quad (\text{for } k = 0, 1, 2, \ldots)$$

such that

- the parity of the $k$th bracket is $k \mod 2$;
- all brackets are antisymmetric (in $\mathbb{Z}_2$-graded sense);
- $\sum_{r+s=n} \sum_{\text{shuffles}} (-1)^{\alpha+\sigma} [[x_{\sigma(1)}, \ldots, x_{\sigma(r)}], \ldots, x_{\sigma(r+s)}] = 0$, for all $n = 0, 1, 2, 3, \ldots$

(here $(-1)^\alpha$ comes from parities and $(-1)^\sigma = \text{sign } \sigma$).
A parallel notion is as follows.

**Definition (\(L_\infty\)-algebra: symmetric version)**

A vector space \( V = V_0 \oplus V_1 \) with a collection of multilinear operations

\[
\{-, \ldots, -\} : V \times \ldots \times V \rightarrow V \quad \text{(for } k = 0, 1, 2, \ldots)\]

such that

- all brackets are odd;
- all brackets are symmetric (in \( \mathbb{Z}_2 \)-graded sense);
- \[\sum_{r+s=n} \sum_{\text{shuffles}} (-1)^\alpha \{v_{\sigma(1)}, \ldots, v_{\sigma(r)}\}, \ldots, v_{\sigma(r+s)}\} = 0,\]
  for all \( n = 0, 1, 2, 3, \ldots \)

(here \((-1)^\alpha\) comes from parities only).
Recollection: $L_\infty$-algebras and $L_\infty$-morphisms – 3

**Equivalent structures:**

- Antisymmetric $L_\infty$-algebra structure on $L$
- Symmetric $L_\infty$-algebra structure on $\Pi L$
- Homological vector field $Q \in \text{Vect}(\Pi L)$, i.e., $\tilde{Q} = 1$, $Q^2 = 0$

(Also: $P_\infty$- on $L^*$ and $S_\infty$- on $\Pi L^*$, to be discussed later.)

NB: we identify super vector spaces with supermanifolds.
Recollection: $L_\infty$-algebras and $L_\infty$-morphisms – 4

**Relation between brackets in $L$ and $\Pi L$**

\[
\{\Pi x_1, \ldots, \Pi x_k\} = (-1)^{(k-1)}\tilde{x}_1 + \cdots + \tilde{x}_{k-1} \Pi [x_1, \ldots, x_k].
\]

**Relation with $Q$**

- $Q(\xi) = \sum \frac{1}{n!} \{\xi, \ldots, \xi\}$, where $\xi \in V = \Pi L$

- Higher derived bracket formula:

  \[\iota([x_1, \ldots, x_k]) = \pm [\ldots [Q, \iota(x_1)], \ldots, \iota(x_k)](0),\]

  for $x = x^i e_i \in L$, and $\iota(x) := (-1)^k \tilde{x}^i x^j \partial/\partial \xi^i \in \text{Vect}(\Pi L)$ (sign fixed by linearity condition).

**Description of $L_\infty$-morphisms**

An $L_\infty$-morphism $L_1 \rightsquigarrow L_2$ is given by a sequence $\Lambda^n L_1 \to L_2$ or $S^n(\Pi L_1) \to \Pi L_2$ satisfying a sequences of identities ("higher homotopies"). It is equivalent to a $Q$-map $\Pi L_1 \to \Pi L_2$. 
Digression: $P_\infty$- and $S_\infty$-structures

Let $M$ be a (super)manifold. Then a $P_\infty$- ($S_\infty$-) structure on $M$ is an antisymmetric (resp., symmetric) $L_\infty$-structure on $C^\infty(M)$ such that the brackets are multiderivations.

- A $P_\infty$-structure on $M$ is specified by an even \( P \in C^\infty(\Pi T^*M) \) satisfying \([P, P] = 0\), by the formula:

\[
\{ f_1, \ldots, f_k \}_P = \left[ \ldots [P, f_1], \ldots, f_k \right]_M.
\]

- An $S_\infty$-structure on $M$ is specified by an odd \( H \in C^\infty(T^*M) \) satisfying \((H, H) = 0\), by the formula:

\[
\{ f_1, \ldots, f_k \}_H = (\ldots (H, f_1), \ldots, f_k)_M.
\]

Homological vector fields ("Hamilton–Jacobi"): 

- \( Q_P = \int_M Dx P(x, \frac{\partial \psi}{\partial x}) \frac{\delta}{\delta \psi(x)} \in \text{Vect}(\Pi C^\infty(M)) \)

- \( Q_H = \int_M Dx H(x, \frac{\partial f}{\partial x}) \frac{\delta}{\delta f(x)} \in \text{Vect}(C^\infty(M)) \)
Key theorem: pullback as an $L_\infty$-morphism

Let $M_1$ and $M_2$ be $S_\infty$-manifolds, with $H_i \in C^\infty(T^*M_i)$, $i = 1, 2$.

### Definition of an $S_\infty$ ("homotopy Schouten") thick morphism

A thick morphism $\Phi: M_1 \rightarrow M_2$ is $S_\infty$ if $\pi_1^*H_1 = \pi_2^*H_2$ on $\Phi$.

Note: this is expressed by the Hamilton–Jacobi equation

$$H_1(x, \frac{\partial S}{\partial x}) = H_2\left(\frac{\partial S}{\partial q}, q\right).$$

### Theorem

If a thick morphism of $S_\infty$-manifolds $\Phi: M_1 \rightarrow M_2$ is $S_\infty$, then the pullback

$$\Phi^*: C^\infty(M_2) \rightarrow C^\infty(M_1)$$

is an $L_\infty$-morphism of the homotopy Schouten brackets.

In greater detail: $\Phi^*$ intertwines the homological vector fields $Q_{H_2} \in \text{Vect}(C^\infty(M_2))$ and $Q_{H_1} \in \text{Vect}(C^\infty(M_1))$. 
Another application: adjoint for a nonlinear transformation

Theorem

1. For a fiberwise map of vector bundles $\Phi: E_1 \to E_2$, there is a fiberwise thick morphism

$$\Phi^*: E_2^* \to E_1^*, \quad (1)$$

with the same properties as the usual adjoint and coinciding with it if $\Phi$ is fiberwise-linear. Construction:

$$\Phi^* := (\kappa \times \kappa)(\Phi)^{op} \subset T^*E_1^* \times (-T^*E_2^*),$$

where $\kappa: T^*E \to T^*E^*$ is the Mackenzie–Xu diffeomorphism.

2. The obtained pushforward of functions on the dual bundles

$$\Phi_* := (\Phi^*)^*: C^\infty(E_1^*) \to C^\infty(E_2^*)$$

if restricted on the space of sections $C^\infty(M, E_1)$ takes it to $C^\infty(M, E_2)$ and coincides on sections with $\Phi_* (\nu) = \Phi \circ \nu$. 

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Thick morphisms and homotopy bracket structures 18 / 33
An $L_\infty$-algebroid is a vector bundle $E \to M$ with an antisymmetric $L_\infty$-algebra structure on sections and a sequence of $n$-ary anchors $E \times_M \ldots \times_M E \to TM$ so that the Leibniz identities hold:

$$[u_1, \ldots, u_{n-1}, fu_n] = a(u_1, \ldots, u_{n-1})(f)u_n + (-1)^\alpha f[u_1, \ldots, u_n],$$

where $(-1)^\alpha = (-1)^{(\tilde{u}_1 + \ldots + \tilde{u}_{n-1} + n)\tilde{f}}$.

An $L_\infty$-algebroid structure on $E \to M$ is equivalent to a (formal) homological vector field on the supermanifold $\Pi E$.

An $L_\infty$-morphism of $L_\infty$-algebroids $\Phi: E_1 \to E_2$ is specified by a map (in general, nonlinear) $\Phi: \Pi E_1 \to \Pi E_2$ such that the corresponding homological vector fields are $\Phi$-related.

Example: all anchors assemble into an $L_\infty$-morphism $\Pi E \to \Pi TM$. 
An $L_\infty$-morphism of $L_\infty$-algebroids over a base $M$ induces $L_\infty$-morphisms of the homotopy Poisson and homotopy Schouten algebras of functions on the dual and antidual bundles respectively.

The anchor for an $L_\infty$-algebroid $E \to M$ induces $L_\infty$-morphisms

$$\mathcal{C}^\infty(\Pi E^*) \to \mathcal{C}^\infty(\Pi T^* M)$$

for the homotopy Schouten brackets, and

$$\Pi \mathcal{C}^\infty(E^*) \to \Pi \mathcal{C}^\infty(T^* M).$$

for the homotopy Poisson brackets.
Application to a homotopy Poisson manifold

In particular, we have the following:

**Corollary**

*On a homotopy Poisson manifold $M$, there is an $L_\infty$-morphism*

$$
\Omega(M) = C^\infty(\prod TM) \to C^\infty(\prod T^* M) = \mathfrak{A}(M),
$$

*between the higher Koszul brackets on forms (induced by a homotopy Poisson structure) and the canonical Schouten bracket on multivector fields.*

(This was our initial problem discussed in the beginning.)
Quantum pullbacks and quantum thick morphisms

Definition

A quantum pullback $\hat{\Phi}^* : \text{OC}_{\hbar}^\infty(M_2) \to \text{OC}_{\hbar}^\infty(M_1)$ is defined by

$$(\hat{\Phi}^*[w])(x) = \int_{T^* M_2} DyDq \ e^{i\hbar(S_{\hbar}(x,q) - y^i q_i)} w(y).$$

A quantum thick (or microformal) morphism $\hat{\Phi} : M_1 \to \hbar M_2$ is the corresponding arrow in the dual category.

Here $S_{\hbar}(x, q)$ is a quantum generating function:

$$S_{\hbar}(x, q) = S_{\hbar}^0(x) + \varphi_{\hbar}^i(x) q_i + \frac{1}{2} S_{\hbar}^{ij}(x) q_j q_i + \frac{1}{3!} S_{\hbar}^{ijk}(x) q_k q_j q_i + \ldots$$

$\text{OC}_{\hbar}^\infty(M)$ is the algebra of oscillatory wave functions, i.e. sums of formal expressions $w(x) = a_{\hbar}(x)e^{i\hbar \rho}(x)$, where $a_{\hbar}(x)$ and $b_{\hbar}(x)$ are formal power series in $\hbar$.

$(Dq := (2\pi\hbar)^{-n}(i\hbar)^m Dq$ in dimension $n|m.$)
Let $\hat{\Phi}: M_1 \xrightarrow{\hat{\Phi}} M_2$ be a quantum thick morphism with a quantum generating function $S_\hbar$. Consider $S_0(x, q) := \lim_{\hbar \to 0} S_\hbar(x, q)$ as the (classical) generating function of a (classical) thick morphism $\Phi: M_1 \xrightarrow{\Phi} M_2$. Then for any oscillatory wave function of the form $w(y) = e^{i/\hbar} g(y)$ on $M_2$, the quantum pullback given by

$$\hat{\Phi}^* \left[ e^{i/\hbar} g \right] = e^{i/\hbar} f_\hbar(x),$$

where $f_\hbar = \Phi^*[g] + O(\hbar)$, and $\Phi^*$ is the pullback by the classical microformal morphism $\Phi: M_1 \xrightarrow{\Phi} M_2$ defined by $S_0(x, q)$.

We say that $\Phi = \lim_{\hbar \to 0} \hat{\Phi}$. 
Suppose

\[ S_\hbar(x, q) = S^0_\hbar(x) + \varphi^i_\hbar(x)q_i + S^+_\hbar(x, q), \]

where \( S^+_\hbar(x, q) \) is the sum of all terms of order \( \geq 2 \) in \( q_i \).

**Theorem**

*The action of \( \hat{\Phi}^* \) defined by \( S_\hbar(x, q) \) can be expressed as follows:*

\[ (\hat{\Phi}^*w)(x) = e^{i \hbar S^0_\hbar(x)} \left( e^{i \hbar S^+_\hbar(x, \frac{\hbar}{i} \frac{\partial}{\partial y})} w(y) \right) \bigg|_{y^i = \varphi^i_\hbar(x)}. \]

Hence the quantum pullback \( \hat{\Phi}^* \) is a special type formal linear differential operator over a ‘quantum-perturbed’ map \( \varphi_\hbar: M_1 \to M_2 \). Here \( S^0_\hbar(x) \) gives the phase factor, \( \varphi^i_\hbar(x)q_i \) gives the map, and \( S^+_\hbar(x, q) \) is responsible for “quantum corrections”.
Let $A$ be a commutative algebra with 1 over $\mathbb{C}[[\hbar]]$. Let $\Delta$ be a linear operator on $A$. Consider two sequences of multilinear operations (of parity $\tilde{\Delta}$ and symmetric in the supersense):

**Definition (a modification of Koszul’s)**

Quantum brackets generated by $\Delta$:

$$\{a_1, \ldots, a_k\}_{\Delta, \hbar} := (-i\hbar)^{-k} \ldots [\Delta, a_1], \ldots, a_k](1);$$

Classical brackets generated by $\Delta$:

$$\{a_1, \ldots, a_k\}_{\Delta, 0} := \lim_{\hbar \to 0} (-i\hbar)^{-k} \ldots [\Delta, a_1], \ldots, a_k](1)$$

- $\Delta$ is a formal $\hbar$-differential operator if all quantum brackets are defined;
- $\Delta$ is an $\hbar$-differential operator of order $\leq n$ if all quantum brackets vanish for $k > n$. 
Remark ( Explicit formulas)

- For $k = 0$, $\{\emptyset\}_{\Delta, \hbar} = \Delta(1)$;
- for $k = 1$, $\{a\}_{\Delta, \hbar} = (-i\hbar)^{-1} (\Delta(a) - \Delta(1)a)$;
- for $k = 2$, $\{a, b\}_{\Delta, \hbar} = (-i\hbar)^{-2} (\Delta(ab) - \Delta(a)b - (-1)^{\tilde{a}\tilde{b}} \Delta(b)a + \Delta(1)ab)$;
- for general $k$, $\{a_1, \ldots, a_k\}_{\Delta, \hbar} = (-i\hbar)^{-k} \sum_{s=0}^{k} \sum_{(k-s,s)\text{-shuffles}} (-1)^{\alpha} \Delta(a_{\tau(1)} \cdots a_{\tau(k-s)}) a_{\tau(k-s+1)} \cdots a_{\tau(k)}$,

where $(-1)^{\alpha} = (-1)^{\alpha(\tau; \tilde{a}_1, \ldots, \tilde{a}_k)}$ is the Koszul sign.

$\hbar$-differential operators

$\text{ord}_\hbar \Delta \leq k$ iff for all $a \in A$, $[\Delta, a] = i\hbar B$ where $\text{ord}_\hbar B \leq k - 1$. 
**Let $\Delta$ on $A$ be odd.** If $\Delta^2 = 0$, then the quantum brackets define an $L_\infty$-algebra (in the odd symmetric version). They additionally satisfy the modified Leibniz identity

$$
\{a_1, \ldots, a_{k-1}, ab\}_{\Delta, \hbar} = \{a_1, \ldots, a_{k-1}, a\}_{\Delta, \hbar} b \pm a\{a_1, \ldots, a_{k-1}, b\}_{\Delta, \hbar} + (-i\hbar)\{a_1, \ldots, a_{k-1}, a, b\}_{\Delta, \hbar}.
$$

We call such an algebraic structure an $S_\infty, \hbar$-algebra.

Note: the operator $\Delta$ and the whole $S_\infty, \hbar$-structure are fully defined by 0- and 1-brackets.

**Lemma**

The quantum brackets generated by $\Delta$ correspond to a “Batalin-Vilkovisky homological vector field” on $A$ (regarded as a supermanifold)

$$
Q = e^{-i\hbar a}\Delta(e^{i\hbar a}) \frac{\delta}{\delta a}.
$$
BV-manifolds and BV quantum morphisms

Definition

(1) A BV-manifold: a supermanifold $M$ equipped with an odd formal $\hbar$-differential operator $\Delta$, $\Delta^2 = 0$. The operator $\Delta$ is the BV-operator.

(2) A (quantum) BV-morphism of BV-manifolds $(M_1, \Delta_1)$ and $(M_2, \Delta_2)$: a quantum thick morphism $\hat{\Phi}: M_1 \rightarrow \hbar M_2$ such that

$$\Delta_1 \circ \hat{\Phi}^* = \hat{\Phi}^* \circ \Delta_2.$$ 

Since $\Delta$ induces a sequence of quantum brackets, and is defined by the 0- and 1-brackets, a BV-structure and an $S_{\infty, \hbar}$-structure on $M$ are equivalent.

Question

How to obtain an $L_{\infty}$-morphism of quantum brackets generated by BV-operators? (Note: the operator $\hat{\Phi}^*$ is linear, so cannot be the answer.)
Define \( \hat{\Phi}^! : C^\infty_{\hbar}(M_2) \to C^\infty_{\hbar}(M_1) \) by

\[
\hat{\Phi}^! := \frac{\hbar}{i} \ln \circ \hat{\Phi}^* \circ \exp \frac{i}{\hbar},
\]

or \( \hat{\Phi}^!(g) = \frac{\hbar}{i} \ln \hat{\Phi}^*(e^{i\hbar g}) \), for a \( g \in C^\infty_{\hbar}(M_2) \).

**Theorem**

If \( \hat{\Phi} : M_1 \to \hbar M_2 \) is a BV quantum morphism, then \( \hat{\Phi}^! \) is an \( L_\infty \)-morphism of the \( S^\infty_{\hbar} \)-algebras of functions.

Or, in greater detail: \( \hat{\Phi}^! \) is a morphism of infinite-dimensional \( Q \)-manifolds \( C^\infty_{\hbar}(M_2) \to C^\infty_{\hbar}(M_1) \), where

\[
Q_\Delta = \int Dx \ e^{-i\hbar f} \Delta(e^{i\hbar f}) \left( \frac{\delta}{\delta f(x)} \right).
\]
From a quantum BV morphism to a classical $S_\infty$ thick morphism

Let $M$ be a BV-manifold with a BV-operator $\Delta$. In the limit $\hbar \to 0$, $\Delta$ gives an $S_\infty$-structure. Its “master Hamiltonian” is

$$H(x, p) = \lim_{\hbar \to 0} e^{-\frac{i}{\hbar} x^a p_a} \Delta(e^{\frac{i}{\hbar} x^a p_a}).$$

Theorem (“analog of Egorov’s theorem”)

Let $M_1$ and $M_2$ be BV-manifolds and let $\hat{\Phi}: M_1 \to \hbar \cdot M_2$ be a BV quantum thick morphism. Then its classical limit $\Phi: M_1 \to M_2$ is a homotopy Schouten morphism for the induced $S_\infty$-structures.

Explicitly: the intertwining relation $\Delta_1 \circ \hat{\Phi}^* = \hat{\Phi}^* \circ \Delta_2$ implies the Hamilton-Jacobi equation for the classical thick morphism $\Phi = \lim_{\hbar \to 0} \hat{\Phi}$:

$$H_1(x, \frac{\partial S}{\partial x}) = H_2\left(\frac{\partial S}{\partial q}, q\right).$$
Some open questions

“Non-linear algebra-geometry duality”

- Define a *non-linear homomorphism* of algebras to be a map $A_1 \to A_2$ such that its derivative at every element $a \in A_1$ is an algebra homomorphism. (Variant: a formal map $A_1 \to A_2$.) Question: how to describe such maps?
- In particular, is it true that all such non-linear homomorphisms between algebras $C^\infty(M)$ are pullbacks by thick morphisms?

Other

- “Thick manifolds”: if we have thick diffeomorphisms, what can be obtained by gluing?
- Action of thick morphisms on forms, cohomology, etc. ...
References

- “Nonlinear pullbacks” of functions and $L_\infty$-morphisms for homotopy Poisson structures. 

- Thick morphisms of supermanifolds and oscillatory integral operators. 

- Microformal geometry and homotopy algebras. 

- Tangent functor on microformal morphisms. 
  arXiv:1710.04335
Thank you for attention!