Logarithmic conformal field theory - an attempt at a status report

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based on work with Jürgen Fuchs Terry Gannon, Simon Lentner, Svea Mierach, Gregor Schaumann and Yorck Sommerhäuser.

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Overview

1. Introduction

2. A Lego-Teichmüller game with coends
   - The groupoid of extended surfaces
   - A Lego-Teichmüller game

3. Bulk correlators
   - Pinned block functors
   - Consistent systems of correlators
   - Main theorem

4. Boundary states

5. Towards derived modular functors

6. Open questions
Introduction: why logarithmic? – Ordinary primary fields

Chiral primary field $\phi(z)$ of conformal weight $h$: $L_0|\phi\rangle = h|\phi\rangle$

Virasoro modes act as

$$[L_{-1}, \phi(w)] = \partial \phi(w), \quad [L_0, \phi(w)] = h\phi(w) + w\partial \phi(w),$$

$$[L_1, \phi(w)] = 2hw\phi(w) + w^2 \partial \phi(w)$$
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Differential equations for the two-point blocks:

$$(\partial_z + \partial_w) \langle \phi(z) \phi(w) \rangle = 0, \quad (z\partial_z + w\partial_w + 2h) \langle \phi(z) \phi(w) \rangle = 0,$$

$$(z^2\partial_z + w^2\partial_w + 2h(z + w)) \langle \phi(z) \phi(w) \rangle = 0.$$
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General solution: scaling

$$
\langle \phi(z) \phi(w) \rangle = \frac{A}{(z - w)^{2h}},
$$
Introduction: why logarithmic? – Logarithms

Jordan partner: \( L_0 |\Phi\rangle = h |\Phi\rangle + |\phi\rangle \) leads to OPE

\[
T(z) \Phi(w) \sim \frac{h\Phi(w) + \phi(w)}{(z-w)^2} + \frac{\partial \Phi(w)}{z-w},
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Inhomogeneous differential equations for the two-point blocks:

$$(\partial_z + \partial_w) \langle \phi(z)\Phi(w) \rangle = 0, \quad (z\partial_z + w\partial_w + 2h) \langle \phi(z)\Phi(w) \rangle = -\langle \phi(z)\phi(w) \rangle,$$
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\]

Assume $\phi(z)$ and $\Phi(z)$ mutually bosonic $\Rightarrow$ two-point blocks are of the form

\[
\langle \phi(z) \phi(w) \rangle = 0, \quad \langle \phi(z) \Phi(w) \rangle = \frac{B}{(z-w)^{2h}},
\]

\[
\langle \Phi(z) \Phi(w) \rangle = \frac{C - 2B \log(z-w)}{(z-w)^{2h}},
\]

Global conformal invariance + non-diagonalisable $L_0$-action $\Rightarrow$ Logarithmic singularities in conformal blocks.
**Introduction: why logarithmic?**

**Fact:** much recent progress concerning classes of examples of chiral LCFTs
e.g. Feigin-Lentner-Semikhatov: screening charges $\rightarrow$ Lie theoretic structures

**This talk:** full local LCFT from chiral CFT

**Applications** of local LCFT: critical dense polymers, percolation, string(?)

**Strategy:**

- Conformal blocks: Vector bundle $\mathcal{V}$ of invariants over moduli space $\mathcal{M}_{g,n}$ of complex curves, genus $g$ and $n$ discs, with projectively flat connection

\[
\mathcal{M}_{g,n} \times (\mathcal{H}_{\lambda_1} \otimes \ldots \otimes \mathcal{H}_{\lambda_n})^* \leftarrow \mathcal{V}_{\lambda_1,\ldots,\lambda_n}
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- Horizontal sections multivalued, hence cannot be correlators
  Correlators are specific conformal blocks

- Monodromies encoded in representations of mapping class groups
  $\pi_1(\mathcal{M}_{g,n}) = \text{Map}_{g,n}$

- Modular functor (based on non-semisimple categories) keeps track of these representations
Remarks:
– work at a categorical level, using modular functors
– impose finiteness conditions (tractability)
– structural results that hopefully extend to more general situations (e.g. Liouville theory)
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Plan

2. A Lego-Teichmüller game ("chiral CFT", modular functor) for a factorizable ribbon category $\mathcal{D}$ beyond semisimplicty ("[L]+[BK]"
For applications to CFT: $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{rev} \cong \mathcal{Z}(\mathcal{C})$
("combining left and right movers")

3. Bulk correlators:
   – Definition of consistent set of correlators
   – Theorem: are in bijection to modular Frobenius algebras in $\mathcal{D}$

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Section 2

A Lego-Teichmüller game with coends

Chiral CFT for logarithmic CFTs
Extended surfaces

Definition

**Extended surface**: $(E, \partial_{in}E, \partial_{out}E, \{p_\alpha\})$

- $E$ smooth oriented surface with boundaries
- $\partial_{in}E, \partial_{out}E$ oriented boundaries (to incorporate punctures)
- A marked point $\{p_\alpha\}$ on each boundary component

**Mapping class group** $\text{Map}(E)$: (classes of) orientation preserving diffeomorphisms, restricting to maps $\partial_{in}E \to \partial_{in}E$ and $\partial_{out}E \to \partial_{out}E$ and mapping marked points to marked points.
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- **Sewing**:
  - \((\alpha, \beta)\) ingoing and outoing boundary component
  - \(\sim\) new surface \(\cup_{\alpha,\beta}E\)
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**Sewing:**

\((\alpha, \beta)\) ingoing and outgoing boundary component
\(\leadsto\) new surface \(\bigcup_{\alpha, \beta} E\)

**Cut systems and (fine) markings**

- Pair of pants decomposition
- All components genus zero, \(\leq 3\) holes
- Graph with distinguished edge
Groupoid of fine markings [B-K]

\(E\) extended surface \(\sim\) groupoid \(\mathcal{F}M(E)\) of fine markings:
- Objects are fine markings of \(E\)
- Morphisms are sequences of moves, modulo relations

**Moves:**

(M1) \(Z\)-move
(M2) \(B\)-move
(M3) \(F\)-move
(M4) \(A\)-move
(M5) \(S\)-move (genus 1)

modulo 13 types of relations.
$E$ extended surface $\rightsquigarrow$ groupoid $\mathcal{FM}(E)$ of fine markings:
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![Diagram of moves](image)

Figure 20. A-move (“associativity constraint”).

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**Facts**

- Groupoid of fine markings $\mathcal{FM}(E)$ is a connected tree
- Unmarking functor
  \[ U : \mathcal{FM}(E) \sim \rightarrow E/\!\!/\!\!\text{Map}(E) \]
  determined by effect of mapping class on marking is equivalence of groupoids.
A specific coend and modularity

For $\mathcal{D}$ a finite ribbon category (in particular braided and pivotal), the coend

$$K := \int_{X \in \mathcal{D}} X^\vee \otimes X$$

is a Hopf algebra in $\mathcal{D}$ with Hopf pairing $\omega : K \otimes K \to 1$. [Lyubashenko, Majid 1995]
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$\iff$ Hopf pairing $\omega$ non-degenerate [Shimizu 2016]

Call then a finite ribbon category $\mathcal{D}$ modular
Facts

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- $\mathcal{D} \boxtimes \mathcal{D}^{\text{rev}} \xrightarrow{\sim} \mathcal{Z}(\mathcal{D})$ if Hopf pairing $\omega$ non-degenerate [Shimizu 2016]

  Call then a finite ribbon category $\mathcal{D}$ modular

- Coends in categories of left exact functors on finite tensor category are representable
  \[ \int_{X \in \mathcal{D}} \text{Hom}_\mathcal{D}(-, - \otimes X \otimes X^\vee) = \text{Hom}(-, - \otimes K) \]
A modular functor for non-semisimple categories:

1. Step: Construct for a given extended surface $E$ with fine marking

$$\tilde{Bl} : \mathcal{F}M(E) \rightarrow \mathcal{L}ex(D^{\boxtimes(p+q)}, \text{vect})$$
A modular functor for non-semisimple categories:

1. **Step**: Construct for a given extended surface $E$ with fine marking

   $\tilde{\mathcal{B}}l : \mathcal{F}M(E) \rightarrow \mathcal{L}ex(D^{\boxtimes(p+q)}, \text{vect})$

   - Given by invariants on spheres with less than 3 holes, e.g. $E = S_{3|0}$:

     $U_1 \boxtimes U_2 \boxtimes U_3 \mapsto \text{Hom}(1, U_1 \otimes U_2 \otimes U_3)$

     (order of tensor factors from graph on $E$, duals involved)
A modular functor for non-semisimple categories:

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  (order of tensor factors from graph on $E$, duals involved)

- Combine with sewing:
  - coend implements summing over all intermediate states from cut system:
  $$\tilde{\mathcal{B}}l_{E,\Gamma}(X_1, \ldots, X_n) = \bigotimes_i \tilde{\mathcal{B}}l_{E_i,\gamma_i}(\ldots)$$

  Rules for coend $\implies \tilde{\mathcal{B}}l_{E,\Gamma}(-) \cong \text{Hom}_D(1, - \otimes K^{\otimes g})$
2. Step: Kan extension

\[
\begin{array}{c}
\mathcal{F}M(E) \xrightarrow{\tilde{B}l} \mathcal{L}ex(D \boxtimes (p+q), \text{vect}) \\
\downarrow U \\
E//\text{Map}(E)
\end{array}
\]

\[R_U \tilde{B}l =: Bl\]
2. Step: Kan extension

\[
\begin{align*}
\mathcal{F} M(E) & \xrightarrow{\mathcal{B}l} \mathcal{L}ex(\mathcal{D} \boxtimes (p+q), \text{vect}) \\
\downarrow U & \quad \rightleftharpoons \\
E \sqcup \text{Map}(E) & \xrightarrow{\bar{R}U \mathcal{B}l} \mathcal{B}l
\end{align*}
\]

Theorem (FS 2017)

*The right Kan extension \( \mathcal{B}l \) exists and has a natural monoidal structure.*
Chapter 3

Bulk correlators for (non-)semisimple conformal field theories
We are now ready to describe bulk fields and their correlators.

**Idea**

**Ingredient:** bulk object. Recall RCFT, $\mathcal{C}$ modular and semisimple.

$$F := \bigoplus_{i,j \in \pi_0(\mathcal{C})} Z_{ij} \, S_i \boxtimes \overline{S}_j \in \mathcal{C} \boxtimes \mathcal{C}^{rev} =: \mathcal{D} \quad \text{with} \quad Z_{ij} \in \mathbb{Z}_{\geq 0}$$

Now $F \in \mathcal{D}$ bulk object, in general not $\boxtimes$-factorizable.
Bulk objects

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Now $F \in \mathcal{D}$ bulk object, in general not $\boxtimes$-factorizable.

Fix external objects to be $F \in \mathcal{D}$ and define the following monoidal categories:
The pinned block functor

For a given bulk object $F$, construct monoidal functor, the pinned block functor, by inserting $F$ into $\mathcal{B}_1$.
First consider

$$\tilde{\mathcal{B}}_1^{(F)} : mSurf \rightarrow \text{vect}$$

with $mSurf$: objects marked surfaces
morphisms: admissible moves, sewings non-invertible morphisms
(central extensions suppressed)
– on objects $\tilde{\mathcal{B}}_1, \Gamma(F, \ldots, F)$
– on moves: e.g. on $Z$-move

For sewing: use coend morphism $\iota_F : F \otimes F^\vee \rightarrow K$, e.g.

$$\text{Hom}_D(1, K^g \otimes F \otimes F^\vee) \xrightarrow{(\iota_F)_*} \text{Hom}_D(1, K^\otimes(g+1))$$
Proposition

This assignment of morphisms in $\mathcal{D}$ respects all 13 relations of moves and thus defines a functor $\tilde{\text{Bl}}^{(F)} : m\text{Surf} \to \text{vect}$.

The right Kan extension along unmarking functor $U : m\text{Surf} \to \text{Surf}$ with $\text{Surf}$: objects: extended surfaces
morphisms: $\varphi \in \text{Map}(E)$, sewings non-invertible morphisms

exists and is monoidal.
Consistent systems of correlators

Definition

Let \( \mathcal{D} \) be a modular finite ribbon category, \( F \in \mathcal{D} \) and \( \Delta_k \) trivial blocks.

A consistent system of bulk field correlators is a monoidal natural transformation s.t.

\[
\nu_F(E^{a}_{b|c}) = \nu_F(E^0_{1|1}) \in \text{End}_{\mathcal{D}}(F)
\]

is invertible.

(Normalization on cylinder.)
Consistent systems of correlators

**Definition**

Let $\mathcal{D}$ be a modular finite ribbon category, $F \in \mathcal{D}$ and $\Delta_k$ trivial blocks. A consistent system of bulk field correlators is a monoidal natural transformation $\nu_F$ s.t.

$$\nu_F(E^0_1|_1) = \in \text{End}_\mathcal{D}(F)$$

is invertible. (Normalization on cylinder.)

**Remark**

Definition implies covariance of correlators under sewing and invariance under the mapping class group:

where the lower arrow is the action of any element of $\text{Map}(E)$ or any sewing.
Idea of construction

Construct first precorrelators: \[ m_{\text{Surf}} \xrightarrow{\tilde{B}_1(F)} \text{vect} \]

and then use universal property of Kan extension

\[ m_{\text{Surf}} \xrightarrow{\tilde{B}_1(F)} \text{vect} \]

\[ \Delta_k \]

\[ U \]

\[ \tilde{\nu}_F \]

\[ \text{Surf} \]

\[ \frac{}{} \]

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Idea

Use the three morphisms
Main theorem

Theorem (FS 2017)

\[ \Delta_F := \quad m_F := \]

\[ \text{determine the natural transformation } \nu_F : \Delta_k \Rightarrow \text{Bl}^{(F)}. \]
\[ \nu_F \text{ is consistent at all genera } \iff (F, m_F, \Delta_F, \epsilon_F, \eta_F) \text{ (co-)commutative} \]
\[ \text{symmetric Frobenius algebra and } \text{modular}: \]

Note: Categories with dualities are Frobenius pseudo-monoids in the bicategory of categories (microcosm principle)
Remarks

- Existence of modular Frobenius algebras:
  \( H \) factorizable ribbon Hopf algebra, \( \omega : H \to H \) ribbon automorphism

\[
\int_{m \in H\text{-mod}} \omega(m) \boxtimes m \in H\text{-bimod}
\]

(Fuchs-S-Stigner 2014)

- In particular \( H^* \) (coadjoint bimodule) is a modular Frobenius algebra. ("Cardy case")

- A simple formula for bulk correlators
The Cardy-Cartan invariant

A algebra in a pivotal category $\mathcal{C}$, $M$ an $A$-module. Partial trace $\Rightarrow$ **character** $\chi_M^A \in \text{Hom}_C(A, 1)$
The Cardy-Cartan invariant

A algebra in a pivotal category \( \mathcal{C} \), \( M \) an \( A \)-module.
Partial trace \( \Rightarrow \) character \( \chi_A^M \in \text{Hom}_\mathcal{C}(A, 1) \)

Recall \( K = H^* \) with coadjoint action
Relation to algebraic characters \( \chi_H^M \in \text{Hom}(H, k) \)

\[ \chi^K_M = \chi_H^M \circ m(f_Q \otimes uv^{-1}) \]

with \( f_Q : H^* \to H \) Drinfeld map, \( u \) Drinfeld element and \( v \) ribbon element.
The Cardy-Cartan invariant

A algebra in a pivotal category $C$, $M$ an $A$-module.
Partial trace $\Rightarrow$ character $\chi^A_M \in \text{Hom}_C(A,1)$

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Partition function $Z = \chi^K_F^{\mathcal{K}} \in \text{Hom}_{\mathcal{C} \otimes \mathcal{C}}(K,1)$ obeys

$$Z = \sum_{i,j \in \mathcal{I}} c_{i,j} \chi^K_i \otimes \chi^K_j$$

$c$ the Cartan matrix of the category $H$-mod:

$$c_{ij} = [P_j, S_i] = \dim_k \text{Hom}_H(P_i, P_j)$$
Chapter 4

Boundary states
Results on boundary states

Three postulates:

**BC** Boundary conditions are objects of a category.
Cardy case: take $C$ itself.

**BS** The “boundary state” is an element in the center of $C$:

$$\text{End}(\text{id}_C) = \int_{c \in C} \text{Hom}_C(c, c) \cong \int_{c \in C} \text{Hom}(c^\vee \otimes c, 1) \cong \text{Hom}_C(L, 1)$$

**F** Bulk states for Cardy case are given by

$$F = \int_{c \in C} c^\vee \boxtimes c \in \overline{C} \boxtimes C$$
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Boundary state is a map $\mathcal{C} \rightarrow \text{End}(\text{id}_\mathcal{C})$, thus a decategorification.
Factors through (co-)characters of $L$:

$$\chi^L_m \in \text{Hom}_\mathcal{C}(L, 1) \quad \hat{\chi}^L \in \text{Hom}_\mathcal{C}(1, L)$$
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$$\chi^L_m \in \text{Hom}_\mathcal{C}(L, 1) \quad \hat{\chi}^L \in \text{Hom}_\mathcal{C}(1, L)$$

Theorem (Gannon, Fuchs, Schaumann, CS, 2018)

Factorization gives annuli $A_{mn} = \sum N^k_{mn} \hat{\chi}^L_k$, as befits the Cardy case.
Chapter 5

Towards derived modular functors
Towards a derived modular functor

$C$ modular, not necessarily semisimple

**Modular functor:** to surface $\Sigma_{g,n}$ associate left exact functor

$$\text{Bl} : \quad \nu_1 \boxtimes \ldots \boxtimes \nu_n \mapsto \text{Hom}_C(\nu_1 \otimes \ldots \otimes \nu_n, K^\otimes g)$$

Actions of the mapping class group $\text{Map}_{g,n}$ on these functors.
Towards a derived modular functor

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Actions of the mapping class group \( \text{Map}_{g,n} \) on these functors.

Fact: \( \text{Hom} \) is only left exact and can be derived.
(Conformal blocks are invariants, and taking invariants is not exact)
Towards a derived modular functor

\[ \mathcal{C} \text{ modular, not necessarily semisimple} \]

**Modular functor:** to surface \( \Sigma_{g,n} \) associate left exact functor

\[
\text{Bl} : \quad v_1 \boxtimes \ldots \boxtimes v_n \mapsto \text{Hom}_\mathcal{C}(v_1 \otimes \ldots \otimes v_n, K^{\boxtimes g})
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Actions of the mapping class group \( \text{Map}_{g,n} \) on these functors.

Fact: \( \text{Hom} \) is only left exact and can be derived.
(Conformal blocks are invariants, and taking invariants is not exact)

**Derive** \( \text{Hom}_\mathcal{C} \) to obtain \( \text{Ext}^n_\mathcal{C} \)

**Theorem (Lentner, Mierach, CS, Sommerhäuser, 2018)**

- The mapping class group \( \text{Map}_{g,n} \) naturally acts on \( \text{Ext}^n_\mathcal{C}(v_1 \otimes \ldots \otimes v_n, K^{\boxtimes g}) \).
- In particular, the modular group \( SL(2, \mathbb{Z}) \)
  acts on the Hochschild complex of a factorizable ribbon Hopf algebra.

Subtle interplay of monoidal structure (including duality) and homological algebra
Idea of construction

- Fix surface $\Sigma_{g,n}$, genus $g$ and $n$ disjoint discs.
- Fix a projective resolution $P_* \to 1$ of the monoidal unit
  insert it at auxiliary point of $\Sigma_{g,n}$
- Functoriality of $(n + 1)$-point blocks
  
  $$F(\Sigma_{g,n+1}) : C^{n+1} \to \text{vect}$$
  
  gives a complex of left exact functors $C^n \to \text{vect}$
  with (projective) action of $\text{Map}_{g,n+1}$
- Short exact sequence
  
  $$1 \to K \to \text{Map}_{g,n+1} \to \text{Map}_{g,n} \to 1$$
  
  with explicit description of the kernel $\Rightarrow$ Action of $\text{Map}_{g,n}$ on Ext-groups.
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with explicit description of the kernel $\Rightarrow$ Action of $\text{Map}_{g,n}$ on Ext-groups.

Examples

$\mathcal{C} = H - \text{mod}$ with $H$ commutative factorizable Hopf algebra:
- tensor product of Hochschild-cohomologies with $K \otimes g$
- mapping class group actions and homological algebra decouple

Group algebra of $S_3$ in characteristic 2,3: permutation representations
Chapter 6

Open questions
Open problems

Constructions, challenges

- Correlators involving boundary fields and defect fields
- Holographic construction from a \((2+\epsilon)\)-dimensional TFT

Conceptual questions

- How stable are the results conceptually, i.e. do they apply to more general (categorical) frameworks?
- In particular, how crucial is duality?
- Non-semisimplicity / LCFT in string theory?
- Do “derived conformal blocks” have physical applications, e.g. in string theory or statistical mechanics?