Symmetries in field theories from an NQ-manifold perspective

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Motivation from symplectic/Poisson geometry

One-parameter groups of symplectomorphisms of a symplectic manifold $(M, \omega)$ are determined by *Hamiltonian vector fields*:

- For $f \in \mathcal{C}^\infty(M)$, the Hamiltonian vector field is $X_f$ with

  \[ \iota_{X_f} \omega = df. \]  

- On functions, an infinite symplectomorphism acts via Poisson brackets:

  \[ X_f(g) = \{ f, g \} = \left[ \pi, f \right], g \right] . \]  

  "The Poisson bracket is a derived bracket by the Poisson tensor $\pi""

- More modern: Take the graded Poisson manifold $T^* [1] M$ with homological function $Q$:

  \[ X_f(g) = \left\{ \{ Q, f \}, g \right\} . \]  

So the action of an infinite symplectomorphism is given by a derived bracket.
More examples

More examples of actions of infinitesimal (spacetime) symmetries:

- $M$ differentiable, inf. diffeomorphisms determined by vector fields $X \in \Gamma(TM)$:

$$X(f) = L_X(f), \quad X(Y) = L_X(Y) = [X, Y],$$

more general for Lie algebroids $(E, [\cdot, \cdot], \rho)$, a section in $\Gamma(E)$ acts with the anchor on functions on $M$:

$$s(f) = \rho(s)(f), \quad s(t) = [s, t]_E. \quad s, t \in \Gamma(E).$$

- For generalized geometry (Hitchin/Gualtieri), inf. elements of $\text{Diff}(M) \ltimes \Omega^2_{c1}(M)$ (locally) are sections

$$X = X + \omega, \quad Y = Y + \beta \quad \in \Gamma(TM \oplus T^*M)$$

and act via the Dorfman derivative:

$$X(f) = X(f), \quad X(Y) = [X, Y] = [X, Y] + L_X \beta - \iota_Y d\omega.$$
More examples

- In **extended field theories** (*Hull, Zwiebach, Hohm, Berman, Vaisman*) like double or exceptional field theory we have actions of the form

\[ S = \int_{M \times \tilde{M}} d\mu \, e^{-\phi} \mathcal{A}(\mathcal{H}) , \quad \mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} . \quad (7) \]

They have infinitesimal spacetime transformations parameterized by fundamentals of \( O(d, d) \), \( \mathcal{X}^M = (X^\mu(x, \tilde{x}), X_\mu(x, \tilde{x})) \) and acting on functions and vectors as:

\[ \mathcal{X}(f) = \mathcal{X}^M \partial_M(f) , \]
\[ \mathcal{X}(\mathcal{Y})^M = \mathcal{X}^N \partial_N \mathcal{Y}^M - \mathcal{Y}^N \partial_N \mathcal{X}^M + \mathcal{Y}^N \partial^M \mathcal{X}_N . \quad (8) \]

We will show that *all of them are of derived bracket form*. Especially in the last item, this gives insight into the algebraic structure of symmetries of DFT/EFT.
(Pre)-NQ-manifolds and derived brackets

Important definitions

**Definition.**
A symplectic pre-NQ-manifold of \( \mathbb{N} \)-degree \( n \) is an \( \mathbb{N} \)-graded manifold \( M \), together with symplectic form \( \omega \) of degree \( n \) and a vector field \( Q \) of degree 1, satisfying \( L_Q \omega = 0 \).

**Examples**
An important class where in addition \( Q^2 = 0 \), are the *Vinogradov Lie \( n \)-algebroids*:

\[
\mathcal{V}_n(M) := T^*[n] T[1]M.
\]

They have the following properties:

- Local coordinates \((x^\mu, \xi^{\mu}, \zeta_\mu, p_\mu)\) of degrees 0, 1, \( n - 1 \), \( n \).
- Symplectic form \( \omega = dx^\mu \wedge dp_\mu + d\xi^{\mu} \wedge d\zeta_\mu \)
- Nilpotent vector field \( Q \) with Hamiltonian \( Q = \xi^{\mu} p_\mu \), i.e. \( \{Q, Q\} = 0 \).
Let $\mathcal{M}$ be a symplectic pre-$NQ$-manifold of degree 2. On $C^\infty_1(\mathcal{M}) \oplus C^\infty_0(\mathcal{M})$ consider the maps:

\begin{align*}
\mu_1 &: \quad C^\infty_0(\mathcal{M}) \to C^\infty_1(\mathcal{M}) \\
\mu_2 &: \quad C^\infty_i(\mathcal{M}) \times C^\infty_j(\mathcal{M}) \to C^\infty_{i+j-1}(\mathcal{M}) \quad i + j > 0 \\
\mu_3 &: \quad C^\infty_i(\mathcal{M}) \times C^\infty_1(\mathcal{M}) \times C^\infty_1(\mathcal{M}) \to C^\infty_0(\mathcal{M}) \quad i = 0, 1
\end{align*}

With $\delta f := \{Q, f\}$ for $f \in C^\infty_1(\mathcal{M})$ and zero otherwise, they are

\begin{align*}
\mu_1(f) &= \{Q, f\} \\
\mu_2(f, g) &= \frac{1}{2} \left( \{\delta f, g\} \pm \{\delta g, f\} \right) \\
\mu_3(f, g, h) &= -\frac{1}{12} \left( \left\{ \{\delta f, g\}, h \right\} \pm \ldots \right).
\end{align*}
If \( \{Q, Q\} = 0 \), i.e. in the \( NQ \)-case, one can prove the relations:

\[
\begin{align*}
\mu_1(\mu_2(f, g)) &= \mu_2(f, \mu_1(g)) \\
\mu_2(\mu_1(g_1), g_2) &= \mu_2(g_1, \mu_1(g_2)) \\
\mu_1(\mu_3(f_1, f_2, f_3)) &= \mu_2(f_1, \mu_2(f_2, f_3)) + \mu_2(f_2, \mu_2(f_3, f_1)) \\
&\quad + \mu_2(f_3, \mu_2(f_1, f_2)) ,
\end{align*}
\]

i.e. \( C_1^\infty(\mathcal{M}) \oplus C_0^\infty(\mathcal{M}) \) with the \( \mu_i \) is a 2-term \( L_\infty \) algebra which is equivalent to a **Lie 2-algebra**.
If \( \{Q, Q\} \neq 0 \), one has to choose an appropriate subset of functions to ensure to still have the Lie 2-algebra relations:

**Theorem.**

Consider a subset of \( C_1^\infty(\mathcal{M}) \oplus C_0^\infty(\mathcal{M}) \), such that the Poisson brackets and the maps \( \mu_i \) close on this subset. Then the latter is a 2-term \( L_\infty \)-algebra if and only if

\[
\{Q^2f, g\} + \{Q^2g, f\} = 0, \\
\{Q^2X, f\} + \{Q^2f, X\} = 0, \\
\{\{Q^2X, Y\}, Z\}_{[X,Y,Z]} = 0,
\]

for all functions \( f, g \in C_0^\infty(\mathcal{M}) \) and \( X, Y, Z \in C_1^\infty(\mathcal{M}) \) and the subscript \([X,Y,Z]\) means the alternating sum over \( X, Y, Z \).
The Lie/Dorfman bracket as a derived bracket

ROYTENBERG, WEINSTEIN

- $M$ manifold, take $\mathcal{V}_1(M)$. Then

\[ \mu_1(f) = df , \]
\[ \mu_2(X, Y) = [X, Y] = \{\{Q, X\}, Y\} , \]

i.e. the Lie bracket is a derived bracket.

- Take $\mathcal{V}_2(M)$. Then objects in $C_1^\infty(M)$ correspond to generalized vectors and

\[ \mu_1(f) = df , \]
\[ \mu_2(X + \alpha, Y + \beta) = [X, Y] + L_X\beta - L_Y\alpha - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha) , \]
\[ \mu_3(X, Y, Z) = \langle[X, Y], Z\rangle + \text{cycl.} . \]

(14)

i.e. the Dorfman bracket is a derived bracket.
We take the same setting as before, but instead of $M$ as base, we take $T^*M$, i.e. we take $\mathcal{V}_2(T^*M)$. Locally we have $(x^M, \xi^M, \zeta_M, p_M)$ of degree $(0, 1, 1, 2)$, i.e. $2d + 2d$ vectors. To get $2d$ we project:

$$\theta^M = \frac{1}{\sqrt{2}}(\xi^M + \eta^{MN}\zeta_N), \quad \beta^M = \frac{1}{\sqrt{2}}(\xi^M - \eta^{MN}\zeta_N). \quad (15)$$

Results in Poisson structure and hamilton function:

$$\omega = dx^M \wedge dp_M + \frac{1}{2} \eta_{MN} d\theta^M \wedge d\theta^N, \quad Q = \theta^M p_M. \quad (16)$$

We still have $L_{\chi_Q} \omega = 0$ but $\{Q, Q\} \neq 0$, so we get a **pre-NQ manifold**.
The C-bracket as derived bracket

A.D., Sämann, Stasheff

With local coordinates \((x^M, \theta^M, p_M)\) of degree \((0, 1, 2)\), and most general degree one objects \(X = x_M \theta^M\) (using \(\eta_{MN}\) to raise and lower indices) we get:

\[
\begin{align*}
\mu_1(f) &= \partial_M f \theta^M, \\
\mu_2(X, Y) &= \frac{1}{2} (X(Y) - Y(X))_M \theta^M, \\
\mu_3(X, Y, Z) &= X^M Z^N \partial_M Y_N - Y^M Z^N \partial_M X_N + \text{cycl.}
\end{align*}
\]

I.e. the C-bracket is a derived bracket.

Recall that \(X(Y)^M = x^N \partial_N Y^M - Y^N \partial_N x^M + Y^N \partial^M x^N\).
Evaluating the theorem on the pre-$NQ$-manifold locally in a patch $U \subset T^*M$ with coordinates $(x^M, \theta^M, p_M)$ with $f \in C^\infty(U)$ and $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in C^\infty_1(T^*[2] T[1] U)$, we get

1. $\{Q^2 f, g\} + \{Q^2 g, f\} = 2\partial_M f \eta^{MN} \partial_N g = 0$
   This is the strong constraint.

2. $\{Q^2 \mathcal{X}, f\} + \{Q^2 f, \mathcal{X}\} = 2(\partial_M \mathcal{X}_K \theta^K)\eta^{MN} \partial_N f = 0$
   This is the strong constraint for vectors and functions.

3. $\{\{Q^2 \mathcal{X}, \mathcal{Y}\}, \mathcal{Z}\}[\mathcal{X}, \mathcal{Y}, \mathcal{Z}] = 2\theta^K ((\partial^M \mathcal{X}_K) (\partial_M \mathcal{Y}^N) \mathcal{Z}_N)[\mathcal{X}, \mathcal{Y}, \mathcal{Z}] = 0$
   Additional constraint for vectors? ...Shows up in properties of the Riemann tensor of double field theory.
We have seen that for geometry, generalized geometry and local double field theory infinitesimal spacetime or configuration space transformations are of derived bracket form. Same can be shown for heterotic generalized geometry and heterotic double field theory. This suggests to use it as a guiding principle:

Infinitesimal spacetime/configuration space transformations are of derived bracket form.

To be further investigated, many other examples:
- Exceptional generalized geometries, exceptional field theories
- Noncommutative field and gravity theories
- Counter examples?
A look on the roadmap

- The same constructions done for $\mathcal{V}_n(M)$ give the Vinogradov brackets on

\[ X = X^\mu \dot{\xi}_\mu + X_{\mu_1...\mu_{n-1}} \dot{\xi}^1 \cdots \dot{\xi}^{n-1} \in \Gamma(TM \oplus \wedge^{n-1} T^*M). \]

In particular $SL(5)$ exceptional generalized geometry in case of $n = 3$.

- In case of $E_5, E_6$ exceptional generalized geometry the NQ manifolds are more complicated (Arvanitakis). No NQ manifolds for $E_7$ or higher so far.

- AKSZ constructions with such NQ-manifolds and deformation quantization (Chatzistavrakidis, Jonke, Khoo, Kokenyesi, Mylonas, Schupp, Sinkovics, Szabo).

- Exceptional field theories (Berman, Cederwall, Godazgar, Kleinschmidt, Palmkvist, Perry, Samtleben, Thompson), I guess their symmetries are of derived form.

- Everything local in the talk so far. What about global issues, examples?
A look on the roadmap

- The same constructions done for $\mathcal{V}_n(M)$ give the Vinogradov brackets on

$$X = X^\mu \xi_\mu + X_{\mu_1 \ldots \mu_{n-1}} \xi^1 \cdots \xi^{n-1} \in \Gamma(TM \oplus \wedge^{n-1} T^*M).$$

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Heisenberg nilmanifold: Periodicity

\[ S^1 \hookrightarrow N_{j,k} \]

\[
\begin{align*}
(x^1, x^2, x^3) &\sim (x^1, x^2 + 1, x^3) \sim (x^1, x^2, x^3 + 1) \\
\sim (x^1 + 1, x^2, x^3 - jx^2)
\end{align*}
\]

- Invariant one-forms: \( dx^1, \ dx^2, \ dx^3 + jx^1 \ dx^2 \).
- Metric: \( g = (dx^1)^2 + (dx^2)^2 + (dx^3 + jx^1 \ dx^2)^2 \).
- First Chern class:
  \[
c_1(N_{j,k}) = j \ dx^1 \wedge dx^2, \quad \int_{T^2} c_1(N_j) = j. \quad (18)
\]
- \( H \)-flux representing Dixmier Douady class of \( U(1) \)-gerbe:
  \[
  H = k \ dx^1 \wedge dx^2 \wedge dx^3, \quad \int_{N_{j,k}} H = k. \quad (19)
  \]
U(1)-gerbe and periodicity

Generalized geometry on $N_{j,k}$:

\[
0 \longrightarrow T^* N_{j,k} \longrightarrow \mathcal{E}_{j,k} \longrightarrow TN_{j,k} \longrightarrow 0
\]

\[
X_\beta + \gamma_\beta = X_\alpha + \gamma_\alpha + \iota_{X_\alpha} (B_\beta - B_\alpha), \quad \text{on } U_\beta \cap U_\alpha \subset N_{j,k}.
\]

In terms of periodicity on $N_{j,k}$, e.g.:

\[
X + \gamma \mid_{(x^1+1,x^2,x^3-jx^2)} = X + \gamma \mid_{(x^1,x^2,x^3)} + \iota_X (k \, dx^2 \wedge dx^3), \quad (20)
\]

for $B$-field $B = kx^1 \, dx^2 \wedge dx^3$. Invariant sections are $e^{-B}(X + \gamma)$, with inverse $B$-transform $e^{-B}$.
In terms of $N\!Q$—manifolds: $\mathcal{T}^*[2]\mathcal{T}[1]\mathcal{N}_{j,k}$

- Degree-1 coordinates corresponding to the invariant forms:
  \[
  \bar{\xi}^1 \sim dx^1, \quad \bar{\xi}^2 \sim dx^2, \quad \bar{\xi}^3 \sim dx^3 + jx^1 dx^2, \quad (21)
  \]
  and the corresponding vector fields ($\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3$).

- The corresponding coordinates invariant under patchings of $\mathcal{C}_{j,k}$ are
  \[
  \xi^i = \bar{\xi}^i, \quad i = 1, 2, 3, \\
  \zeta_1 = \bar{\zeta}_1, \quad \zeta_2 = \bar{\zeta}_2 - kx^1 \bar{\xi}^3, \quad \zeta_3 = \bar{\zeta}_3 + kx^1 \bar{\xi}^2. \quad (22)
  \]

- Degree-2 momenta: Periodicity due to Roytenberg
  \[
  (x^1, x^2, x^3, p_1, p_2, p_3) \sim (x^1 + 1, x^2, x^3 - jx^2, p_1, p_2 + jp_3, p_3). \quad (23)
  \]

- Periodicity-invariant symplectic form, homological function:
  \[
  \omega = dx^i \wedge dp_i + d\xi^1 \wedge d\zeta_i, \quad Q = \xi^i a^j_i p_j, \quad (24)
  \]
  with anchor $a^j_i$ relating $\zeta_i$ to the standard basis on $TN_{j,k}$. 
T-duality between subalgebras

\[ N_j, k \times T^2 N_k, j \]

\[ p \quad \tilde{p} \]

\[ N_j, k \quad \pi \quad \tilde{\pi} \quad N_k, j \]

\[ T^2 \]

\[ \tilde{p}^* \tilde{H} - p^* H = d(p^* A \wedge \tilde{p}^* \tilde{A}) \]

(Bouwknegt, Evslin, Mathai, Hannabuss, Rosenberg)

\[ \text{Interchanging } j \text{ and } k \text{ (Buscher)} \]

\[ g = g_0 + (dx^3 + jx^1 dx^2)^2, \quad B = kx^1 dx^{23} \]

\[ \tilde{g} = g_0 + (dx^4 + kx^1 dx^2)^2, \quad \tilde{B} = jx^1 dx^{24} \]

Construct pre-NQ manifold \( \mathcal{E}_{j,k} = \mathcal{E}_{k,j} \) using periodicity invariant \((x, \xi, \zeta, p)\) by reducing a Courant algebroid on \( N_j, k \times T^2 N_k, j \).

The conditions of theorem give embeddings:

\[ \mathcal{C}^\infty(\mathcal{E}_{j,k}) \]

\[ \iota \quad \tilde{\iota} \]

\[ \mathcal{C}^\infty(T^* [2] T^*[1] N_j, k) \quad \mathcal{C}^\infty(T^* [2] T^*[1] N_k, j) \]
T-duality between subalgebras
A.D., Sämann, work in progress

Studying T-dual subalgebras means solving the following problem:

Find embeddings $\iota, \tilde{\iota}$ such that $\text{Im}(\iota), \text{Im}(\tilde{\iota}) \subset C^\infty(\mathcal{E}_{j,k})$ satisfy the conditions:

\[
\{Q^2 f, g\} + \{Q^2 g, f\} = 0, \\
\{Q^2 X, f\} + \{Q^2 f, X\} = 0, \\
\{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} = 0,
\]

for $f$ of degree-0 and $X, Y, Z$ of degree-1 in the image of $\iota, \tilde{\iota}$.

Interesting: $Q^2$ now differs from the flat case:

\[
Q^2 = (p_3 - k\xi^1\xi^2)(p_4 - j\xi^1\xi^2).
\]

Compare to “solving the section condition” in DFT?
Is there a more geometric interpretation of $Q^2$?

- Recall the local version $p_3 p_4$ and $\{p_i, f\} = \partial_i f$.
- Here e.g. $\{D, \cdot\} = \{p_3 - k \xi^1 \xi^2, \cdot\}$
- Meaning e.g. on degree-1:

  $$\{D, X\} = \partial_3 X + k (X^1 \xi^2 - X^2 \xi^1)$$

- NOT the connective structure (would be $\xi^2 \xi^3$)
- Courant algebroid connection? (Alekseev, Gualtieri, Xu), see Jan’s talk.
Next steps and outlook

- Find and classify embeddings $\iota$, study properties (Lagrangian etc.).
- Can we compare to DFT? Solving the section condition $\leftrightarrow$ finding dual subalgebras. If so, interpret the additional terms containing the Dixmier Douady classes $k\xi_1\xi_2$ and $j\xi_1\xi_2$.
- For $NQ$-manifolds, we studied aspects of Riemannian geometry. Apply here.
- What about the exotic cases where there are $Q$- and $R$-fluxes?
- The formalism extends readily to (so far some) exceptional generalized geometries.
- Important operations are given in terms of Poisson brackets. What happens if we replace $\{\cdot, \cdot\}$ by $[\cdot, \cdot]_*$?
Appendix: Construction of $E_{j,k}$

As we have coordinates with periodicity, we imitate the local case:

- First construct a Courant algebroid on $N_{j,k} \times T^2 N_{k,j}$, containing $p^* H \pm \tilde{p}^* \tilde{H}$ as twist. In particular, the degree-2 momenta have periodicity:

\[
\left( x^1, x^2, x^3, x^4, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4 \right) \\
\sim \left( x^1 + 1, x^2, x^3 - jx^2, x^4 - kx^2, \bar{p}_1, \bar{p}_2 + j\bar{p}_3 + k\bar{p}_4, \bar{p}_3, \bar{p}_4 \right)
\]

Possible to choose periodicity invariant coordinates $(x^i, \xi^i, \zeta_i, p_i)$.

- Reduce to the pre-$NQ$-case by setting

\[
\theta_{\pm} := \frac{1}{2} \left( \xi^3 \pm \zeta_4 \right), \quad \theta_{\pm}^2 := \frac{1}{2} \left( \xi^4 \pm \zeta_3 \right)
\]

and only keeping $\theta_{\pm}^1$ and $\theta_{\pm}^2$.

- Reduced invariant symplectic form and hamiltonian:

\[
\omega = dx^i \wedge dp_i + d\xi^1 \wedge d\zeta_1 + d\xi^2 \wedge d\zeta_2 + d\theta_{\pm}^1 \wedge d\theta_{\pm}^2, \\
Q = \xi^1 p_1 + \xi^2 p_2 + \theta_{\pm}^1 p_3 + \theta_{\pm}^2 p_4 - jx^1 \xi^2 p_3 - kx^1 \xi^2 p_4.
\]