Developments in 6d SCFTs

Seok Kim
(Seoul National University)

“Higher structures in M-theory” Durham
Aug 14, 2018
1. **String theory constructions**
   - Discovery
   - Constructions from string, M-, F- theories

2. **Conventional QFT approaches**
   - Tensor branch & self-dual strings
   - 5d Yang-Mills at weak/strong couplings
   - 4d Yang-Mills & S-duality

3. **6d strings, S-duality & “non-Abelian” physics**
   - 2d QFT approaches
   - Elliptic genera, modularity, etc.
   - Relations to S-duality & non-Abelian degrees of freedom

4. **Summary and some future directions**

I’ll focus on several key themes, related to one another:

  - self-dual strings, self-dual tensors, compactifications, S-duality, …
Discovery of 6d QFTs

- 2nd string revolution had a corollary: new interacting 6d QFTs “discovered”
- type IIB on $R^{5+1} \times C^2/\Gamma_g$ singularity, w/ $g=$ADE: [Witten] (1995)
- 6d system at singularity, decoupled from gravity at low E.
  - D3’s wrapped on 2-cycles ~ strings: tension ~ volume of 2-cycles (~ VEV of a 6d field)
  - 6d QFT has strings “charged” with 2-form tensor fields.
  - Like W-bosons charged with 1-form gauge fields. (mass ~ VEV)
- Singular limit: 6d maximal superconformal theory. $N = (2,0)$ SUSY.
  - Should (abstractly) view this as a local QFT [Seiberg]
  - Even without fully known quantum description (e.g. Lagrangian).
6d $N=(2,0)$ theories

- Obeys an ADE classification
  - $A_{N-1}$ type may also be viewed as $N$ M5s’ worldvolume theory
  - Strings given by open M2’s. [Strominger] [Howe, Lambert, West]
  - In a sense, QFT for these strings, not for particles…

- Number of degrees of freedom scales like $\sim N^3$: much larger than Yang-Mills on D-branes

  $N \gg 1$ M5’s of M-theory make AdS$_7$:
  - black 5-branes at temperature $T$ [Klebanov, Tseytlin]

\[
\frac{S_{BH}}{(\text{volume})_5} \sim N^3 T^5
\]

- $D_N$ type may also be viewed as $N$ M5s’ probing $R^{5+1} \times R^5 / \mathbb{Z}_2$ orbifold of M-theory
- $E_N$ type realized only as type IIB on $R^{5+1} \times C^2 / \Gamma_{E_N}$
6d $N = (1,0)$ theories

- $N = (1,0)$ SCFTs: $R^{5+1} \times M_4$ w/ nontrivial axion-dilaton fields $\sim$ “F-theory”
- F-theory on elliptically fibered $CY_3$
- Earlier examples [Morrison, Vafa] [Witten] 1996
- Recently, appeared a “classification”
  [Heckman, Morrison, Rudelius, Vafa] 2015
  [Del Zotto, Heckman, Tomasiello, Vafa] 2014 , ………

- This is basically a classification of $CY_3$ with suitable properties to have 6d SCFTs.
- Like the (2,0) theories, D3-branes wrapping 2-cycles yield strings.
Aspects

- There is a universal “branch” in all known 6d SCFTs. “tensor branch”
  - 6d self-dual tensor supermultiplet: comes with a real scalar in the multiplet
    \[ B_{\mu\nu} \text{ with } H = dB = *dA \text{, } \Psi^A \text{, } \Phi \]
  - Give VEVs to the real scalars: “tensor branch”
  - Geometrically, in F-theory, VEV’s are the volumes of resolved 2-cycles
  - In M-theory w/ M5-branes, VEV’s are the separations of branes.

- Effective action in the tensor branch is known, which only contains the “Abelian part” of the tensor multiplet.
- 6d supermultiplets in the tensor branch:
  - Abelian tensor multiplets \[ B_{\mu\nu} \text{ with } H = dB = *dA \text{, } \Psi^A \text{, } \Phi \]
  - non-Abelian vector multiplets: 6d vector + chiral fermions
  - Hypermultiplets: Two (complex) scalars + anti-chiral fermions

- Vectors/hypers are optional, depending on models, but tensors are universal.
Some approaches from “conventional” QFTs

• Various conventional QFTs serve as either
  - effective field theory (EFT) in certain limits,
  - Exact description of the 6d QFT by taking limits (like “large N”… more later)
  - Etc.

• We shall first discuss insights from these descriptions, to understand various “phenomenological” constraints on the true QFT to be found in the future.

• Key concepts to be explored today:
  - 4d S-duality
  - its 6d geometrization & its manifestations in various 6d, 5d, 4d descriptions
  - $N^3$
  - self-dual strings, and how they encode information on the other issues above
 Tensor branch

- For simplicity, consider SCFTs with 1 dimensional tensor branch (i.e. 1 Abelian tensor multiplet in this branch)
- For example, $A_1$ type $N = (2,0)$ has 1 tensor multiplet + 1 hypermultiplet, which makes $N = (2,0)$ tensor multiplet.

- $N = (1,0)$ theories having vector multiplets have subtler tensor branch action

$$S^\text{bos}_{v+t} = \int \left[ \frac{1}{2} d\Phi \wedge \ast d\Phi + \frac{1}{2} H \wedge \ast H \right] + \sqrt{c} \int [-\Phi tr(F \wedge \ast F) + B \wedge tr(F \wedge F)]$$

$$H \equiv dB + \sqrt{c} \ tr \left( AdA - \frac{2i}{3} A^3 \right)$$

- “action” involving self-dual tensor: impose self-duality on e.o.m. later by hand.
- $c$ is a model-dependent positive constant.
- Vector & tensor should couple according to the second term, to yield the classical gauge anomaly: 6d Green-Schwarz mechanism [Green, Schwarz, West] [Sagnotti]
Tensor branch & self-dual strings

- There are strings which are charged under $B_{\mu\nu}$ with equal electric/magnetic charge, thus called self-dual strings. [Strominger] [Howe, Lambert, West]

- $N = (2,0)$ theory, or more generally for tensors not coupled to vectors

$$dH_3 = d \star H_3 \sim \delta^{(4)}$$

BPS equation for the tensor multiplet fields

$$H = \mp \star_4 d\Phi$$

- Generic $N = (1,0)$ theory: Yang-Mills instanton charge = string charge

$$d \star H = dH = \sqrt{c} \text{tr}(F \wedge F)$$

BPS equations (valid only at the leading order in $1/\langle \Phi \rangle$ expansion)

$$F = \pm \star_4 F \quad k = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{tr}(F \wedge F) \text{ is quantized}$$

$$H = \mp \star_4 d\Phi$$

- These strings are analogous to the $W$-bosons of Yang-Mills theory: traces of non-Abelian d.o.f. in the broken phase. String tension $\propto \langle \Phi \rangle$.

- These strings contain information on “non-Abelian” natures of the 6d QFT. I’ll later explain how to (partly) see them.
Self-dual strings & electromagnetic duality

• Consider the (2,0) theory, compactified on a torus.
• It is known that small torus reduction yields the 4d maximal SYM.
• Since the 6d strings are self-dual, 4d electric/magnetic particles come from same self-dual string, wrapped on different sides of $T^2$.

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \rightarrow -1/\tau$$

• Naturally explains S-duality, or more generally $SL(2, Z)$ duality [Montonen, Olive]
• This aspect is both a benefit and a challenge.
  - Better geometric understanding of 4d S-duality (& its generalization to other 4d QFTs)
  - Challenge for formulating 6d QFT, to realize S-duality manifestly upon compactification.

• Similar, and perhaps even more drastic, roles should be played by these strings for the $N = (1,0)$ theories compactified on torus. Also $N = (2,0), (1,0)$ theories on Riemann surfaces.
• I will mostly restrict the discussions to the rather better studied (2,0).
5d Yang-Mills theory

- It is illustrative to make $T^2$ compactification in steps.
- First make $S^1$ compactification with radius $R$. Obtains maximal super-Yang-Mills.

5d Yang-Mills at low energy: e.g. for $A_{N-1}$ type, reducing N M5’s yield N D4’s:

$$ S = \frac{1}{g^2_{YM}} \int d^5x \text{tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi^I D^\mu \phi^I - \frac{i}{2} \bar{\lambda}_i \gamma^\mu D_\mu \lambda^i + \frac{1}{4} [\phi^I, \phi^J]^2 - \frac{i}{2} \bar{\lambda}_i (\hat{\gamma}_I)^i_j [\phi^I, \lambda^j] \right] $$

- Parameters: $g^2_{YM} \sim R$.
- Weakly-coupled when $E \ll 1/g^2_{YM} \sim 1/R$, i.e. when 6d KK modes are heavy.

- Naïve reduction of Lagrangian QFT at small circle yields strong coupling.
- Morally, electromagnetic duality is involved, from tensor to vector, to get 5d YM.

- This gauge theory description provides great intuitions on this system.
5d instanton solitons & KK modes

- D0-branes bound to D4’s: “instanton” solitons see the KK tower.

\[ F_{\mu\nu} = \pm \star_4 F_{\mu\nu} \quad \text{on } \mathbb{R}^4 \]

\[ k = \frac{1}{8\pi^2} \int \text{tr}(F \wedge F) \in \mathbb{Z} \]

- The radius of 6th circle:

\[ \frac{4\pi^2}{g_{YM}^2} = \frac{1}{r_1} \]

- It makes the studies of 5d (maximal) SYM very interesting. [Douglas] [Lambert, Papageorgakis, Schmidt-Sommerfeld] (2010)

- At least apparently, quantum calculus appears to see UV incompleteness, despite not always in the usual violent form of UV divergences.

  - E.g. in SUSY observables, one should consider the moduli space dynamics of instantons, which has small instanton singularities, creating possible ambiguities of calculus.

  - Can be cured by simple UV completion, e.g. by uplifting to D0-D4 system. (But I didn’t really use maximal SUSY, which might eliminate ambiguity within moduli space dynamics.)
Some instanton calculus

- The Witten indices of D0-branes (Nekrasov partition function)
  - R-symmetry: $SO(5)$, or $SU(2)_R \times SU(2)_L$ in Coulomb branch (separate M5’s along a line)
    \[
    Z_k(\epsilon_{1,2}, a_i, m) = \text{Tr}_k \left[ (-1)^F e^{-\epsilon_1(J_1+J_R)-\epsilon_2(J_2+J_R)} e^{-a^i q_i} e^{-2m J_L} \right]
    \]
  - Can be computed using 5d instanton calculus, or D0-D4 quantum mechanics
    \[
    Z_k = \sum_{\sum_i |Y_i|=k} \prod_{i=1}^{N} \prod_{s \in Y_i, j=1}^{N} \frac{2 \sinh \frac{E_{ij}(s)+m-\epsilon_+}{2} \cdot 2 \sinh \frac{E_{ij}(s)-m-\epsilon_+}{2}}{2 \sinh \frac{E_{ij}(s)}{2} \cdot 2 \sinh \frac{E_{ij}(s)-2\epsilon_+}{2}}
    \]
    \[
    E_{ij}(s) = a_i - a_j - \epsilon_1 h_i(s) + \epsilon_2 (v_j(s) + 1)
    \]
  - Its grand partition function, $Z(q = e^{2\pi i \tau}) = \sum_{k=0}^\infty Z_k q^k$, is a partition function on $R^4 \times T^2$
  - For instance, single M5-brane theory, shows the expected 6d spectrum
    \[
    Z(q, \epsilon_{1,2}, m) = PE \left[ Z_1(\epsilon_{1,2}, m) \frac{q}{1-q} \right] \equiv \exp \left[ \sum_{n=1}^\infty \frac{1}{n} Z_1(n\epsilon_{1,2}, nm) \frac{q^n}{1-q^n} \right]
    \]
  - For multiple M5-branes, one needs more technical controls. (later)
    - d.o.f. carrying momenta, bound to 6d strings: $N^2 \rightarrow N^3$ enhancement as $q \rightarrow 1^-$ ?
4d Yang-Mills theory

- Further reduction on the second circle, to 4d.
- We have important implications from 6d self-dual QFT

- Electric/magnetic particles/fields should have same origin.
- Montonen-Olive electromagnetic duality: [Montonen, Olive] [Osborne]

- For special 4d gauge theories, such as $N = 4$, there is a chance of realizing S-duality because both W-boson & monopoles are in the same type of supermultiplet.

- Actual studies of quantum spectrum of dyons in Coulomb branch justified S-duality. [Sen], …
- Many other observables studied in the unbroken phase (such as curved space partition function [Vafa, Witten], … ) justifies S-duality more generally.
S-duality from gauge theory

• However, in the current formulation in terms of gauge theory, electromagnetic duality is not manifest at all.

• E.g., to show that the dyon spectra respect $SL(2,\mathbb{Z})$ in SU(2) Yang-Mills, one has to show that the spectrum is the same for all $(p, q)$ magnetic/electric charges, for coprime $p, q$.

• This is because these $(p, q)$ are related to the W-boson at $(0,1)$ charge.

• A. Sen & others had to study the moduli space of magnetic monopoles, making a hard study of their bound state wave functions. On the other hand, elementary W-boson’s spectrum is so easy to get.

• The results support $SL(2,\mathbb{Z})$, but it is “emergent” in the gauge theory.

• If one can make a covariant formulation of the 6d theory, one can naturally imagine that this duality should be more manifest.
6d self-dual strings

- So far, I talked about:
  1) universality of tensor branch & self-dual strings,
  2) Spectrum calculus using 5d approach (whose structures, e.g. $N^3$, are left unaddressed yet)
  3) 4d S-duality

- Now, I’ll explain that 2) and 3) can be better addressed by studying 1).
  - Tensionless string at conformal point? Worldsheet descriptions of them? (difficult)
  - I shall explain aspects of tensionful strings in the tensor branch. Even here, many aspects of these strings are different from fundamental strings.

- The approach I’ll explain uses 2d QFTs living on these strings, which are expected to flow to 2d SCFTs on these strings.
  - Somewhat like studying massive 4d dyons using quiver quantum mechanics. [Denef], ...
  - Depending on how one embeds the 2d CFTs in UV, various calculations become easier/difficult. So it involves a bit of “art” of constructing good UV descriptions.
Approaches from 2d QFT & RG flow

• The case of $N = (2,0)$ theory: uplift to IIA. M2-M5 $\to$ D2-NS5

$\text{NS5} \quad \text{NS5} \quad \text{NS5} \quad \text{NS5}$

yields 2d N=(4,4) quiver

tricky UV theory use: only sees $SO(3)_R \subset SO(4)_R$
(similar to 3d $N = 8$ SYM vs. M2: $SO(7) \subset SO(8)$)

$\text{NS5}$  
$\text{D6}$  
$\text{D2}$

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS5</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>D6</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>D2</td>
<td>•</td>
<td>•</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>•</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
</tbody>
</table>

Sees $SU(2) \times U(1) \subset SO(4)_R$: similar to “mirror dual” of 3d $N = 8$
SYM $U(1)^2 \times SU(2)^2 \subset SO(8)$ [Haghighat, Iqbal, Kozcaz, Lockhart, Vafa]

• $k_1, \ldots, k_{N-1}$ are string numbers in various $U(1)^{N-1}$ tensors.

• String theory asserts that these QFTs flow to those on self-dual string worldsheet
  in IR.
Strings of $N = (1,0)$

- Similar 2d constructions can be made for the strings of various $N = (1,0)$ theory.
- E.g. the “E-string theory” (w/ rank 1 tensor branch)

- The 2d quiver with (0,4) SUSY

- Many other (1,0) strings are explored from different approaches, related to the dynamics of instanton string worldsheet dynamics. [Haghighat, Klemm, Lockhart, Vafa] [J.Kim,SK,K.Lee, Park, Vafa] [Gadde,H,JK,SK,L,V] [Del Zotto, Lockhart] [Del Zotto, Gu, Huang, Kashani-Poor, Klemm, Lockhart] [H.-C.Kim, SK, Park] …
Elliptic genera

- Some invariant observables under RG flow: Can be computed from UV.
- Elliptic genus: $T^2$ partition function w/ SUSY

$$Z_k(\tau, \epsilon_{1,2}, m_a) = \text{Tr} \left[ (-1)^F e^{2\pi i \tau H + e^{2\pi i \tau H} - e^{2\pi i \epsilon_1 (J_1 + J_R)}} e^{2\pi i \epsilon_2 (J_2 + J_R)} \prod_{a \in \text{flavor}} e^{2\pi i m_a F_a} \right]$$

$$H_\pm = \frac{H \pm P}{2} \quad \text{and} \quad H_- \sim \{Q, \overline{Q}\}$$

- The elliptic genus at given string winding numbers, $n_1, \ldots, n_{N-1}$ on spatial $S^1$

$$Z(n_i) = \sum_{Y_1, \ldots, Y_{N-1} | Y_i = n_i} \prod_{i=1}^{N} \prod_{s \in Y_i} \theta_1(\tau \mid \frac{E_{i,i+1}(s)-m+\epsilon_-}{2\pi i}) \theta_1(\tau \mid \frac{E_{i,i-1}(s)+m+\epsilon_-}{2\pi i}) \frac{\theta_1(\tau \mid \frac{E_{i,i}(s)+\epsilon_1}{2\pi i}) \theta_1(\tau \mid \frac{E_{i,i}(s)-\epsilon_2}{2\pi i})}{\theta_1(\tau \mid \frac{E_{i,i}(s)+\epsilon_1}{2\pi i}) \theta_1(\tau \mid \frac{E_{i,i}(s)-\epsilon_2}{2\pi i})}$$

$$E_{i,j}(s = (a, b)) = (Y_{i,a} - b)\epsilon_1 - (Y_{j,b} - a)\epsilon_2$$

- Coefficient of the grand partition function, which sums over the string numbers

$$Z(v_i, \tau, \epsilon_{1,2}, m) = e^{-\epsilon_0} Z_{U(1)}^N \sum_{n_1, \ldots, n_N} e^{-n_i(v_i-v_{i+1})} Z(n_i)(\tau, m, \epsilon_{1,2})$$

- This is the same partition function as those computed from 5d approach
Aspects of elliptic genus

- These elliptic genera behave very differently from the partition functions of fundamental strings, which obey the Hecke transformation formula
  - Single fundamental string’s partition function $Z_1(\tau, z)$.
  - Multiple strings’ partition function $Z_n(\tau, z)$ is given by the Hecke transformation

$$Z(w, \tau, z) \equiv \sum_{n=0}^{\infty} w^n Z_n(\tau, z) = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} w^n \sum_{ad=n; a, d \in \mathbb{Z} b(\text{mod} d)} Z_1 \left( \frac{a\tau + b}{d}, az \right) \right]$$

- A consequence of multi-particle statistics & twisted sectors.
- Self-dual strings’ elliptic genera don’t obey this formula. More nontrivial bound states.

- S-modular property: modular anomaly factor $\propto n^2$ is incompatible w/ Hecke formula

$$Z_{(n_i)} \left( -\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau} \right) = \exp \left[ \frac{1}{4\pi i \tau} \left( \epsilon_1 \epsilon_2 \sum_{i,j=1}^{N-1} \Omega^{ij} n_i n_j + 2(m^2 - \epsilon_+^2) \sum_{i=1}^{N-1} n_i \right) \right] Z_{(n_i)}(\tau, m, \epsilon_{1,2})$$

$$\Omega^{ii} = 2, \quad \Omega^{i,i+1} = \Omega^{i,i-1} = -1$$

- The last modular property is connected to both 4d S-duality & 6d $\mathcal{N}^3$
Modular property & S-duality

- The grand partition function: Nekrasov’s partition function of QFT on $R^4 \times T^2$

$$Z(v_i, \tau, \epsilon_{1,2}, m) = e^{-\epsilon_0} Z_{U(1)}^N \sum_{n_1, \ldots, n_N} e^{-n_i(v_i-v_{i+1})} Z_{(n_i)}(\tau, m, \epsilon_{1,2})$$

- Reduces to 4d or 5d Coulomb branch partition function of mass-deformed maximal super-Yang-Mills theory by taking suitable limits

- In the limit $\epsilon_{1,2} \to 0$, yields Seiberg-Witten prepotential $-\log Z \to f(v, m, \tau)/\epsilon_1 \epsilon_2$

- $f$ is “quantum prepotential” related to full prepotential by $F = \pi i \tau v^2 + f$.

- S-modular property of $Z_{(n_i)}$ determines S-duality of (mass-deformed) N=4 SYM.

- In 4d limit, expects Legendre transformation of E/M prepotentials under S-duality.

  $$F(-1/\tau, a_D) = F(\tau, a) - a_D = F(\tau, a) - a \frac{dF}{da}$$

- In full 6d, from physics reasoning we don’t expect such a simple S-duality.

  - $\tau \to i\infty$: small spatial $S^1$ of $T^2$. Or low T. Expect $\log Z \sim N^2$ from 5d Yang-Mills
  - $\tau \to i0^+$: large spatial $S^1$. Or high T. Expect $\log Z \sim N^3$ from 6d (2,0) physics

- Therefore, we expect certain anomaly of S-duality of full 6d system on $T^2$. 


Details

- \( \tau \) dependence via quasi-modular forms:

\[
\theta_1(\tau|z) = 2\pi iz \eta(\tau) \exp \left[ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi iz)^{2k} \right]
\]

- 3 generators:

\[
E_2(-1/\tau) = \tau^2 \left( E_2 + \frac{6}{\pi i \tau} \right), \quad E_4(-1/\tau) = \tau^4 E_4(\tau), \quad E_6(-1/\tau) = \tau^6 E_6(\tau)
\]

causes modular anomaly of \( Z_{(n_i)}(\tau, m, \epsilon_{1,2}) \)

- modular anomaly equation:

\[
\frac{\partial}{\partial E_2} Z_{(n_i)}(\tau, m, \epsilon_{1,2} : E_2) = \frac{1}{24} \left[ \epsilon_1 \epsilon_2 \Omega^{ij} n_i n_j - 2\Omega^{ij} (m^2 - \epsilon_+^2) \rho_i n_j \right] Z_{(n_i)}
\]

\[
\hat{Z}(\tau, v, m, \epsilon_{1,2}) = \sum_{n_i, \ldots, n_r = 0}^{\infty} e^{-\sum_{i=1}^{r} n_i \alpha_i(v)} Z_{(n_i)}
\]

\[
\frac{\partial \hat{Z}}{\partial E_2} = \frac{1}{24} \left[ \epsilon_1 \epsilon_2 \Omega^{ij} \partial_i \partial_j + 2(m^2 - \epsilon_+^2) \Omega^{ij} \rho_i \partial_j \right] \hat{Z}
\]

\[
\hat{Z} \equiv \frac{1}{24} \left[ \epsilon_1 \epsilon_2 \Omega^{ij} \partial_i \partial_j + 2\Omega^{ij} I_i(m, \epsilon_+) \partial_j \right] \hat{Z}
\]

\[
\frac{\partial Z_{S-dual}}{\partial E_2} = \frac{\epsilon_1 \epsilon_2}{24} \Omega^{ij} \partial_i \partial_j Z_{S-dual} : \text{heat equation}
\]

- some manipulations:

\[
Z_{S-dual} \equiv \hat{Z} \exp \left[ \frac{\Omega^{ij} I_i v_j}{\epsilon_1 \epsilon_2} + \frac{\Omega^{ij} I_i j}{\epsilon_1 \epsilon_2} E_2(\tau) \right]
\]

- dividing modular anomaly: “standard” one + anomaly of “standard” one

\[
\Omega^{ij} I_i I_j \sim \rho^2 \sim c_2 |G|
\]

\[
= N^3 - N
\]
Modular property & $N^3$

- **S-duality of $Z_{S\text{-}dual}$**: $(\delta = \frac{6}{\pi i \tau})$

$$Z_{S\text{-dual}} \left( -\frac{1}{\tau}, v, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau}; E_2(-\frac{1}{\tau}) \right) = Z_{S\text{-dual}}(\tau, v, m, \epsilon_{1,2}, E_2(\tau) + \delta)$$

$$Z_{S\text{-dual}}(\tau, v, m, \epsilon_{1,2}; E_2(\tau) + \delta) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dv'_i \, K(v, v') Z_{S\text{-dual}}(\tau, v', m, \epsilon_{1,2}; E_2(\tau))$$

$$K(v, v') = \left( \frac{i \tau}{\epsilon_1 \epsilon_2} \right)^{\frac{N}{2}} \exp \left[ -\frac{\pi i \tau}{\epsilon_1 \epsilon_2} (v - v')^2 \right]$$

- **Small $\epsilon_{1,2}$**: RHS computed by saddle point approx. $Z \sim \exp \left[ -\frac{f(\tau, v, m)}{\epsilon_1 \epsilon_2} \right], \quad Z_{S\text{-dual}} \sim \exp \left[ -\frac{f_{S\text{-dual}}(\tau, v, m)}{\epsilon_1 \epsilon_2} \right]$  

$$\tau^2 F_{S\text{-dual}} \left( \tau_D = -\frac{1}{\tau}, v_D = v + \frac{1}{2\pi i \tau} \frac{\partial f}{\partial v}, \frac{m}{\tau} \right) = F_{S\text{-dual}}(\tau, v, m) - v \frac{\partial F_{S\text{-dual}}}{\partial v}(\tau, v, m)$$

4d limit (small $T^2$) $\sim F_{S\text{-dual}} \left( -\frac{1}{\tau}, v_D \equiv \tau v + \frac{1}{2\pi i \tau} \frac{\partial f}{\partial v}, m \right)$

“Standard” S-duality in 4d Seiberg-Witten: “magnetic dual prepotential $\sim$ S-dual prepotential”

- **“S-duality anomaly”**: extra anomaly. E.g. for ADE (2,0),

$$f(\tau, v, m) = f_{S\text{-dual}}(\tau, v, m) + rf_U(1)(\tau, m) + \frac{c_2 |G|}{288} m^4 E_2(\tau)$$

$$f_U(1) = m^2 \left( \frac{1}{2} \log m - \frac{3}{4} + \frac{\pi i}{2} + \log \phi(\tau) \right) + \sum_{n=1}^{\infty} \frac{m^{2n+2} B_{2n}}{2n \cdot (2n+2)!} E_{2n}(\tau)$$

- note: The last S-duality anomaly vanishes in the 4d limit (after restoring $S^1$ radius)
Asymptotic free energy

• One can use S-dual low T ($\tau \to i\infty$) setting to study the decompactifying limit ($\tau \to i0^+$): contribution from anomalous part + 5d perturbative part

• Results:

$$-\log Z \sim \frac{f(\tau \to 0, v, m)}{\epsilon_1 \epsilon_2} \sim \frac{i}{\epsilon_1 \epsilon_2 \tau} \left[ \frac{N^3 m^4}{48\pi} - \frac{\pi N m^2}{12} \mp i \frac{N^2 m^3}{12} \right]$$

for $0 < \pm \text{Im}(m) < \frac{2\pi}{N}$

- At $m = 0$, SUSY enhances, #(boson) – #(fermion) = 0. Obstruct full cancelation by $m \neq 0$
- Mechanism, w/ light D0’s…? Interpretation?

- In a sense, one has observed “deconfined” degrees of freedom coming from light instantons/D0-branes, in an asymptotic limit
- These structures can be extended to study black holes in $AdS_7$, and more generally in $AdS_5$, $AdS_3$ (work in progress)
Challenges

- Attempts towards microscopic formulations (many of them to be discussed in this conference)

- Fully satisfactory descriptions should explain the following challenging problems.
  - 4d S-duality
  - Building in $N^3$ in a consistent manner
  - $N^2 \to N^3$ interpolation of d.o.f. by changing $S^1$ radius

- I also tried to explain that all these aspects are connected to the physics of 6d self-dual strings, especially their anomaly structures. Some connections are quantitatively established recently.

- Hopefully, these should put strong constraints on the higher structures relevant for 6d SCFTs, thus providing useful guidance.
Challenges

• 4d S-duality:
  - Highly nontrivial in 4d Yang-Mills, but geometrical in 6d. What makes 4d S-duality possible should be built-in in the 6d QFT, if one seeks for a manifestly Lorentz-covariant description.
  - There may be other ways to formulate these QFTs, without manifest covariance. [Zwanziger]
  - Perhaps equivalently, w/ auxiliary fields. [Pasti, Sorokin, Tonin] [Lambert, Papageorgakis]

• 6d self-duality:
  - Lagrangian is always unfriendly to self-dual p-form potentials in $d = 2p + 2$. One way of dealing with it is to use suitable auxiliary fields, e.g. the PST formalism.
  - Another way was recently found in the context of type IIB SUGRA [Sen] (2015)

\[
S = S'_1 + S_2
\]

\[
S'_1 = \frac{1}{2} \int dP^{(4)} \wedge *dP^{(4)} - \int dP^{(4)} \wedge Q^{(5)} - \int B^{(2)} \wedge F^{(3)} \wedge Q^{(5)} + \frac{1}{2} \int *\left( B^{(2)} \wedge F^{(3)} \right) \wedge \left( B^{(2)} \wedge F^{(3)} \right)
\]

which replaces the “conventional action” in which self-duality is imposed later by hand

\[
S_1 \equiv -\frac{1}{2} \int \hat{F}^{(5)} \wedge *\hat{F}^{(5)} + \int F^{(5)} \wedge B^{(2)} \wedge F^{(3)}
\]

  - Sort of “guaranteed” to work at quantum level, being inspired by string field theory
  - Once we find the right equation of motion, a covariant action may be available.