Higher Gauge Theory from Twistor Space

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Introduction and Motivation
One of the big challenges in M-theory is the formulation of the $\mathcal{N} = (2, 0)$ theory. This a chiral superconformal gauge theory in six dimensions with maximal $\mathcal{N} = (2, 0)$ supersymmetry. At the linearised level, we have:

- A potential 2-form $B$ with curvature 3-form $H = dB$ such that $H = \star_6 H$
- Five scalars $\phi^{IJ}$ such that $\Box \phi^{IJ} = 0$
- Four Weyl fermions $\psi^I$ such that $\mathcal{D} \psi^I = 0$

**Problem:** How can this be promoted to an interacting non-Abelian theory?
Proposal: Combine twistor theory and categorified principal bundles.
Higher Gauge Algebras
An **NQ-manifold** is a non-negatively graded manifold quipped with a nil-quadratic degree-one vector field $Q$.

An NQ-manifold concentrated in degree one is a **Lie algebra**. Indeed, let $\xi^\alpha$ be local coordinates. The most general degree-one vector field $Q$ is of the form

$$Q := \xi^\alpha \xi^\beta f_{\alpha\beta}^\gamma \frac{\partial}{\partial \xi^\gamma},$$

with $f_{\alpha\beta}^\gamma$ constant. Then $Q^2 = 0$ is equivalent to requiring $f_{\alpha\beta}^\gamma$ to satisfy Jacobi. Thus, we obtain a Lie algebra with $Q$ as its Chevalley–Eilenberg differential.
An $NQ$-manifold in degree zero and one is a Lie algebroid. Indeed, such a manifold must be of the form $E[1] \to X$. Let $(x^i, \xi^\alpha)$ be local coordinates so that

$$Q := \xi^\alpha \rho^i_\alpha \frac{\partial}{\partial x^i} + \xi^\alpha \xi^\beta f_{\alpha\beta}^\gamma \frac{\partial}{\partial \xi^\gamma}.$$ 

Now $f_{\alpha\beta}^\gamma \in \mathcal{C}^\infty(X)$ are structure functions of a Lie bracket $[-,-]$ on $\Gamma(E)$ and the $\rho^i_\alpha \in \mathcal{C}^\infty(X)$ encode a map $\rho : E \to TX$. Then $Q^2 = 0$ implies that the $f_{\alpha\beta}^\gamma$ satisfy Jacobi, $\rho$ is a Lie algebra homomorphism, and $[s_1, fs_2] = (\rho(s_1)f)s_2 + f[s_1, s_2]$ for all $f \in \mathcal{C}^\infty(M)$ and $s_{1,2} \in \Gamma(E)$. Hence, this describes a Lie algebroid with $Q$ as its Chevalley–Eilenberg differential.

A $k$-term $L_\infty$-algebroid is an $NQ$-manifold concentrated in degrees $0, 1, \ldots, k$. When concentrated in degrees $1, \ldots, k$ we call it a $k$-term $L_\infty$-algebra.
For \( k = 1, 2 \), let \((\xi^\alpha, \eta^i)\) be local coordinates. Then,

\[
Q := \xi^\alpha \xi^\beta f_{\alpha \beta}^\gamma \frac{\partial}{\partial \xi^\gamma} + f_{i}^\alpha \eta^i \frac{\partial}{\partial \xi^\alpha} + f_{i}^\alpha j^\alpha \eta^j \frac{\partial}{\partial \eta^i} + f_{\alpha \beta}^\gamma i^\alpha \xi^\beta \xi^\gamma \frac{\partial}{\partial \eta^i},
\]

where \( f_{\alpha \beta}^\gamma, f_{i}^\alpha, f_{i}^\alpha j^\alpha, \) and \( f_{\alpha \beta}^\gamma j^\alpha \) are constants. Letting \( \mathfrak{w} \) be a vector space with basis \( w_\alpha \) and \( \mathfrak{v} \) a vector space with basis \( v_i \) we may thus write

\[
\mu_1(v_i) := f_{i}^\alpha w_\alpha, \quad \mu_2(w_\alpha, w_\beta) := f_{\alpha \beta}^\gamma w_\gamma, \\
\mu_2(v_i, w_\alpha) := f_{i}^\alpha j^\alpha v_j, \quad \mu_3(w_\alpha, w_\beta, w_\gamma) := f_{\alpha \beta}^\gamma i^\alpha v_i,
\]

i.e. we obtain a 2-term complex \( \mathfrak{v} \xrightarrow{\mu_1} \mathfrak{w} \) with \( \mu_2 : \mathfrak{w} \wedge \mathfrak{w} \to \mathfrak{w} \), \( \mu_2 : \mathfrak{v} \wedge \mathfrak{w} \to \mathfrak{v} \), \( \mu_3 : \mathfrak{w} \wedge \mathfrak{w} \wedge \mathfrak{w} \to \mathfrak{v} \)

and \( Q^2 \) yields higher homotopy Jacobi identities e.g.

\[
\mu_1(\mu_3(W_1, W_2, W_3)) = \mu_2(\mu_2(W_1, W_2), W_3) \\
+ \mu_2(\mu_2(W_3, W_1), W_2) + \mu_2(\mu_2(W_2, W_3), W_1)
\]
Higher Gauge Group(oid)s
The simplex category $\Delta$ is the category that has finite totally ordered sets $[p] := \{0, 1, \ldots, p\}$ as objects and order-preserving maps as morphisms.

The objects of $\Delta$ have a geometric realisation as standard topological simplices.

The morphisms of $\Delta$ are generated by the coface maps, $\phi^p_i$, and codegeneracy maps, $\delta^p_i$, defined by

\[
\phi^p_i : [p - 1] \to [p]
\]

\[
\delta^p_i : [p + 1] \to [p]
\]
A simplicial set (manifold) is a functor $\mathcal{X} : \Delta^{op} \to \text{Set (Mfd)}$.

Hence, $\mathcal{X} = \bigcup_p X_p$ and $X_p := \mathcal{X}([p])$ is the set of simplicial $p$-simplices; the elements of $X_0$ are the vertices of $\mathcal{X}$. We obtain the face maps, $f_p^i := \mathcal{X}(\phi^p_i) : X_p \to X_{p-1}$, and the degeneracy maps, $d_p^i := \mathcal{X}(\delta^p_i) : X_p \to X_{p+1}$ subject to the simplicial identities

\[
\begin{align*}
  f_i \circ f_j &= f_{j-1} \circ f_i \quad \text{for } i < j, \\
  d_i \circ d_j &= d_{j+1} \circ d_i \quad \text{for } i \leq j, \\
  f_i \circ d_j &= d_{j-1} \circ f_i \quad \text{for } i < j, \\
  f_i \circ d_j &= d_j \circ f_{i-1} \quad \text{for } i > j + 1, \\
  f_i \circ d_i &= \text{id} = f_{i+1} \circ d_i.
\end{align*}
\]

Depict simplicial sets by writing arrows for the face maps,

\[
\bigl\{ \cdots \implies X_2 \implies X_1 \implies X_0 \bigr\}.
\]

For a ordinary set $X$ write $\bigl\{ \cdots \implies X \implies X \implies X \bigr\}$ with all face and degeneracy maps identities. Such a set is called a simplicially constant simplicial set.
A simplicial map between simplicial sets is a natural transformation between the defining functors.

The standard simplicial $p$-simplex, $\Delta^p$, is the simplicial set $\text{hom}_\Delta(-, [p]) : \Delta^{op} \to \text{Set}$.

For any simplicial set $X = \bigcup_p X_p$, one can show that $X_p \cong \text{hom}_{sSet}(\Delta^p, X)$.

For two simplicial sets $X$ and $Y$, a simplicial homotopy between two simplicial maps $g, \tilde{g} : X \to Y$ is a simplicial map $h : X \times \Delta^1 \to Y$ that renders

\[
\begin{array}{ccc}
X \times \Delta^0 & \cong & X \\
\downarrow \text{id} \times \phi_1^1 & & \downarrow g \\
X \times \Delta^1 & \cong & Y \\
\downarrow \text{id} \times \phi_0^1 & & \downarrow \tilde{g} \\
X \times \Delta^0 & \cong & X
\end{array}
\]

commutative. Here, $\phi_0^1$ and $\phi_1^1$ are the coface maps.
For each $i$, the $(p, i)$-horn $\Lambda_i^p$ of $\Delta^p$ is the simplicial subset of $\Delta^p$ given by all faces of $\Delta^p$ except for the $i$-th one. The $(p, i)$-horns of a simplicial set $Y$ is the set $\text{hom}_{sSet}(\Lambda_i^p, Y)$.

The horns $\Lambda_i^p$ of $\Delta^p$ can always be filled (i.e. completed) to $\Delta^p$. For a simplicial set $Y$ this is, in general, not the case.

A Kan simplicial set is a simplicial set such that any horn $\lambda : \Lambda_i^p \rightarrow Y$ can be filled, that is,

$$\Lambda_i^p \xrightarrow{\lambda} Y \xrightarrow{\delta} \Delta^p$$

is commutative. Put differently, the natural restrictions

$$Y_p \cong \text{hom}_{sSet}(\Delta^p, Y) \rightarrow \text{hom}_{sSet}(\Lambda_i^p, Y)$$

are surjective. For a simplicial manifold, these are surjective submersions.
Let $X$ and $Y$ be two simplicial sets. Consider the relation $g \sim \tilde{g}$ on the set of all simplicial maps between $X$ and $Y$ defined by saying that $g$ is related to $\tilde{g}$ whenever there exists a simplicial homotopy from $g$ to $\tilde{g}$. If $Y$ is a Kan simplicial set then this is an equivalence relation.

A **quasi-groupoid** is a Kan simplicial set. A **Lie quasi-groupoid** is a Kan simplicial manifold. A **Lie $k$-quasi-groupoid** is a Lie quasi-groupoid for which the $(p, i)$-horns can be filled uniquely for $p > k$, $i \in \{0, \ldots, p\}$.

Every Lie $k$-quasi-groupoid differentiates to a $k$-term $L_{\infty}$-algebroid following a method due to Ševera in which the algebroid is given as the 1-jet of the quasi-groupoid.
Examples $k = 1$

Let $f : Y \to X$ be a surjective submersion between two manifolds $Y$ and $X$. Consider

$$Y \times_X Y := \{ (y_1, y_2) \in Y \times Y \mid f(y_1) = f(y_2) \}.$$

- The Čech groupoid $\check{C}(Y \to X)$ is the Lie groupoid $Y \times_X Y \rightrightarrows Y$ with pairs $(y_1, y_2) \in Y \times_X Y$ as its morphisms and

  $$s(y_1, y_2) := y_2, \quad t(y_1, y_2) := y_1, \quad \text{id}_Y := (y, y),$$

  $$\quad (y_1, y_2) \circ (y_2, y_3) := (y_1, y_3).$$

- The Čech nerve of the Čech groupoid $\check{C}(Y \to X)$ is the Lie 1-quasi-groupoid

$$N(\check{C}(Y \to X)) := \{ \cdots \rightrightarrows Y \times_X Y \times_X Y \rightrightarrows Y \times_X Y \rightrightarrows Y \} ,$$

with face and degeneracy maps given by

$$f^p_i (y_0, \ldots, y_p) := (y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_p),$$

$$d^p_i (y_0, \ldots, y_p) := (y_0, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_p).$$
Let $G$ be a Lie group.

- The **delooping** $BG$ is the Lie groupoid $G \xrightarrow{\sim} \ast$, where the source and target maps are trivial, $\text{id}_\ast = 1_G$, and the composition is group multiplication in $G$.

- The **nerve** $N(BG)$ of the delooping $BG$ is the Lie $1$-quasi-groupoid

$$N(BG) := \{ \cdots \xrightarrow{\sim} G \times G \xrightarrow{\sim} G \xrightarrow{\sim} \ast \}$$

with the obvious face and degeneracy maps.
Higher Principal Bundles
For $G$ a Lie group, a **principal $G$-bundle** over a manifold $X$ subordinate to a surjective submersion $Y \to X$ is a simplicial map $g: N(\check{C}(Y \to X)) \to N(BG)$.

Take an ordinary cover $\bigcup_a \{(x, a) \mid x \in U_a\} \to X$ so that the set of morphisms of the corresponding Čech groupoid is $\bigcup_{a,b} \{(x, a, b) \mid x \in U_a \cap U_b\}$ with the composition $(x, a, b) \circ (x, b, c) = (x, a, c)$.

Hence, a simplicial map $g: N(\check{C}(Y \to X)) \to N(BG)$ consists of

$$g_a(x) := g^0(x, a) = \ast, \quad g_{ab}(x) := g^1(x, a, b) \in G,$$

$$g_{abc}(x) := g^2(x, a, b, c) = (g^1_{abc}(x), g^2_{abc}(x)) \in G \times G,$$

and as it commutes with the face and degeneracy maps,

$$g^1_{abc}(x) = g_{ab}(x), \quad g^1_{abc}(x)g^2_{abc}(x) = g_{ac}(x), \quad g^2_{abc}(x) = g_{bc}(x).$$
Since, in addition, homotopies yield equivalent bundles, we give the following definition ...

For $\mathcal{G}$ a Lie quasi-groupoid, a Lie quasi-groupoid bundle or principal $\mathcal{G}$-bundle over $X$ subordinate to a surjective submersion $Y \rightarrow X$ is a simplicial map $g : N(\mathcal{C}(Y \rightarrow X)) \rightarrow \mathcal{G}$. Two such principal $\mathcal{G}$-bundles $g, \tilde{g} : N(\mathcal{C}(Y \rightarrow X)) \rightarrow \mathcal{G}$ are called equivalent if and only if there is a simplicial homotopy between $g$ and $\tilde{g}$.

This can be generalised to higher bases spaces i.e. base spaces which are Kan simplicial manifolds.
Generalising the above construction, we can also infer the connective structure on such principal $G$-bundles as well its patching transformations. The latter follow from computing the 1-jet of the simplicial manifold $\text{hom}(\Delta^1, G)$ appearing in 

$$\text{hom}_{sSMfd}(\mathcal{X} \times \Delta^1, G) \cong \text{hom}_{sSMfd}(\mathcal{X}, \text{hom}(\Delta^1, G))$$

Let $G$ be a Lie 2-quasi group with the induced 2-term $L_\infty$ algebra $\mathfrak{v} \xrightarrow{\mu_1} \mathfrak{w}$. Let $\bigcup_a \{(x, a) | x \in U_a\} \to X$. A Deligne cocycle describing a principal $G$-bundle with connective structure consists of the transition functions $\{g_{ab}, g_{abc}, \Lambda_{ab}\}$ with $\Lambda_{ab} \in \Omega^1(U_a \cap U_b) \otimes \mathfrak{w}$ and the connective structure $\{A_a, B_a\} \in \Omega^1(U_a) \otimes \mathfrak{w} \oplus \Omega^2(U_a) \otimes \mathfrak{v}$ with curvatures

$$\mathcal{F}_a := dA_a + \frac{1}{2} \mu_2(A_a, A_a) - \mu_1(B_a),$$

$$H_a := dB_a + \mu_2(A_a, B_a) + \frac{1}{3!} \mu_3(A_a, A_a, A_a).$$
6D Self-Dual Higher Gauge Theory
Consider $\mathcal{N} = (2, 0)$ superspace $M := \mathbb{C}^{6|16}$ with coordinates $(x^{AB}, \eta^A_I)$ with $A, B, \ldots = 1, \ldots, 4$ and $I, J, \ldots = 1, \ldots, 4$. Then,

$$P_{AB} := \partial_{AB}, \quad D^I_A := \partial_A - 2\Omega^{IJ}\eta^B_J\partial_{AB}$$

have the non-vanishing (anti-)commutation relations

$$\{D^I_A, D^J_B\} = -4\Omega^{IJ}P_{AB}.$$

Define the correspondence space $F$ to be $F := \mathbb{C}^{4|16} \times \mathbb{P}^3$ with coordinates $(x^{AB}, \eta^A_I, \lambda_A)$.

Introduce a rank-3|12 distribution $\langle V^A, V^I_{AB} \rangle \hookrightarrow TF$ by $V^A := \lambda_B \partial^{AB}$ and $V^I_{AB} := \frac{1}{2}\varepsilon^{ABCD}\lambda_C D^I_D$ which is integrable. Hence, we have foliation $P := F/\langle V^A, V^I_{AB} \rangle$. 

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On $P$, we may use coordinates $(z^A, \eta_I, \lambda_A)$ with $z^A \lambda_A = \Omega^{IJ} \eta_I \eta_J$ and thus

$$F$$

$$\pi_1 \quad \pi_2$$

$$P \quad M$$

with $\pi_2$ being the trivial projection and

$$\pi_1 : (x^{AB}, \eta^A_I, \lambda_A) \mapsto (z^A, \eta_I, \lambda_A) = ((x^{AB} + \Omega^{IJ} \eta^A_I \eta^B_J) \lambda_B, \eta^A_I \lambda_A, \lambda_A)$$

A point $x \in M$ corresponds to a $\mathbb{P}^3$ in $P$, while a point $p \in P$ corresponds to a $3|12$-superplane

$$x^{AB} = x^{AB}_0 + \varepsilon^{ABCD} \mu_C \lambda_D + 2\Omega^{IJ} \varepsilon^{CDE}[A \lambda_C \theta_{IDE} \eta^B_{0J}] ,$$

$$\eta^A_I = \eta^A_{0I} + \varepsilon^{ABCD} \lambda_B \theta_{ICD}.$$
Let $G$ be a Lie 2-quasi-group. There is a bijection between equivalence classes

(i) of holomorphic $M$-trivial principal $G$-bundles on $P$ and

(ii) of solutions to the constraint system on the chiral superspace $M$

\[
F_A^B = \mu_1(B_A^B), \quad F_{AB}^I = \mu_1(B_{AB}^I), \quad F^{IJ}_{AB} = \mu(B^{IJ}_{AB}), \\
H^{AB} = 0, \\
H_A^{BI} = \delta^B_C \psi_A^I - \frac{1}{4} \delta^B_A \psi_C^I, \\
H_{ABCD}^{IJ} = \varepsilon_{ABCD} \phi^{IJ}, \quad \text{with} \quad \phi^{IJ} \Omega_{IJ} = 0 \\
H_{ABC}^{IJ} = 0.
\]

This is a quasi-isomorphism of $L_\infty$-algebras.
4D Super Yang–Mills Theory
Consider $M := \mathbb{C}^{4|12}$ with coordinates $(x^{\alpha \dot{\alpha}}, \theta^{i \alpha}, \eta^{i \dot{\alpha}})$ where $\alpha, \dot{\alpha}, \ldots = 1, 2$ and $i, j, \ldots = 1, \ldots, 3$. Then,

$$P_{\alpha \dot{\alpha}} := \partial_{\alpha \dot{\alpha}} , \quad D_{i \alpha} := \partial_{i \alpha} + \eta^{i \dot{\alpha}} \partial_{\alpha \dot{\alpha}} , \quad D^{i}_{\dot{\alpha}} := \partial^{i}_{\dot{\alpha}} + \theta^{i \alpha} \partial_{\alpha \dot{\alpha}}$$

have the non-vanishing (anti-)commutation relations

$$\{ D_{i \alpha}, D^{j}_{\dot{\alpha}} \} = 2\delta^{j}_{i} P_{\alpha \dot{\alpha}} .$$

Define $F := \mathbb{C}^{4|12} \times \mathbb{P}^{1} \times \mathbb{P}^{1}$ with coordinates $(x^{\alpha \dot{\alpha}}, \theta^{i \alpha}, \eta^{i \dot{\alpha}}, \mu_{\alpha}, \lambda_{\dot{\alpha}})$.

Introduce a rank-1|6 distribution $\langle V, V_{i}, V^{i} \rangle \hookrightarrow TF$ by $V := \mu^{\alpha} \lambda^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}}$, $V_{i} := \mu^{\alpha} D_{i \alpha}$, and $V^{i} := \lambda^{\dot{\alpha}} D^{i}_{\dot{\alpha}}$ which is integrable. Hence, we have foliation $L := F/\langle V, V_{i}, V^{i} \rangle$. 
On $L$, we may use coordinates $(z^\alpha, w^{\dot{\alpha}}, \theta^i, \eta_i, \mu_\alpha, \lambda_{\dot{\alpha}})$ with $z^\alpha \mu_\alpha - w^{\dot{\alpha}} \lambda_{\dot{\alpha}} = 2\theta^i \eta_i$ and thus

$$F$$

$\xrightarrow{\pi_1}$ $L$

$\xrightarrow{\pi_2}$ $M$

with $\pi_2$ being the trivial projection and

$$\pi_1 : (x^{\alpha\dot{\alpha}}, \theta^{i\alpha}, \eta^{\dot{i}}_i, \mu_\alpha, \lambda_{\dot{\alpha}}) \mapsto (z^\alpha, w^{\dot{\alpha}}, \theta^i, \eta_i, \mu_\alpha, \lambda_{\dot{\alpha}}) =$$

$$= ((x^{\alpha\dot{\alpha}} - \theta^{i\alpha} \eta^{\dot{i}}_i) \lambda_{\dot{\alpha}}, (x^{\alpha\dot{\alpha}} + \theta^{i\alpha} \eta^{\dot{i}}_i) \mu_\alpha, \theta^{i\alpha} \mu_\alpha, \eta^{\dot{i}}_i \lambda_{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}})$$

A point $x \in M$ corresponds to a $\mathbb{P}^1 \times \mathbb{P}^1$ in $L$, while a point $p \in L$ corresponds to a 1|6-superline

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + t\mu^\alpha \lambda^{\dot{\alpha}} + t^i \mu^\alpha \eta^{\dot{i}}_i - t_i \theta^{i\alpha} \lambda^{\dot{\alpha}},$$

$$\theta^{i\alpha} = \theta_0^{i\alpha} + t^i \mu^\alpha, \quad \eta^{\dot{i}}_i = \eta_0^{\dot{i}}_i + t_i \lambda^{\dot{\alpha}}.$$
Due to Witten and Isenberg–Yasskin–Green we have the following result. Let $G$ be a Lie group. There is a bijection between equivalence classes

(i) of holomorphic $M$-trivial principal $G$-bundles on $L$ and

(ii) of solutions to the constraint system of maximally supersymmetric Yang–Mills theory on $M$

\[
F_{i\alpha j\beta} = \epsilon_{\alpha\beta} \epsilon_{ijk} \phi^k, \quad F_{i\dot{\alpha}j\dot{\beta}} = \epsilon_{i\dot{\alpha}\dot{\beta}} \epsilon_{ijk} \phi^k, \quad F_{i\alpha} = 0.
\]

To prove this theorem, one makes use of the Čech description of holomorphic principal bundles. This is an intrinsically on-shell approach as the holomorphicity of the bundles encodes the equations of motion. How do we go off-shell?
To go off-shell, we make use of the Dolbeault approach. In particular, a holomorphic principal $G$-bundle can be described by a smooth principal $G$-bundle equipped with a $(0, 1)$-connection locally given by a Lie($G$)-valued $(0, 1)$-form $A^{0,1}$ subject to

$$F^{0,2} = \bar{\partial}A^{0,1} + \frac{1}{2}[A^{0,1}, A^{0,1}] = 0.$$ 

For a three-dimensional Calabi–Yau manifold, this equation is variational as it follows from the holomorphic Chern–Simons action functional

$$S := \int \Omega^{3,0} \wedge \text{tr} \left\{ A^{0,1} \wedge \bar{\partial}A^{0,1} + \frac{2}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1} \right\}.$$

Ambitwistor space is a Calabi–Yau supermanifold, however, its bosonic part is five-dimensional, and so we cannot use this action functional.
We propose to consider higher holomorphic Chern–Simons theory which we can motivate from string field theory of the B type topological sigma model on higher-dimensional Calabi–Yau spaces.

Let $\mathcal{G}$ be a Lie 3-quasi-group. Consider a smooth principal $\mathcal{G}$-bundle equipped with Lie($\mathcal{G}$)-valued $(0, p|0)$-forms $A^{0,1|0}$, $B^{0,2|0}$, and $C^{0,3|0}$ with

$$S := \int \Omega^{5|6,0} \wedge \left\{ \langle A^{0,1|0}, \bar{\partial}C^{0,3|0} \rangle + \langle B^{0,2|0}, \mu_1(C^{0,3|0}) \rangle + \right.$$ 

$$+ \frac{1}{2} \langle B^{0,2|0}, \bar{\partial}B^{0,2|0} \rangle + \frac{1}{2} \langle A^{0,1|0}, \mu_2(A^{0,1|0}, C^{0,3|0}) \rangle +$$

$$+ \frac{1}{2} \langle A^{0,1|0}, \mu_2(B^{0,2|0}, B^{0,2|0}) \rangle +$$

$$+ \frac{1}{3!} \langle A^{0,1|0}, \mu_3(A^{0,1|0}, A^{0,1|0}, B^{0,2|0}) \rangle +$$

$$+ \frac{1}{5!} \langle A^{0,1|0}, \mu_4(A^{0,1|0}, A^{0,1|0}, A^{0,1|0}, A^{0,1|0}) \rangle \right\},$$

where the fermionic integration is in the sense of Berezin.
The corresponding equations of motion are

\[ \bar{\partial} A^{0,1|0} + \frac{1}{2} \mu_2 (A^{0,1|0}, A^{0,1|0}) + \mu_1 (B^{0,2|0}) = 0, \]

\[ \bar{\partial} B^{0,2|0} + \mu_2 (A^{0,1|0}, B^{0,2|0}) + \]

\[ + \frac{1}{3!} \mu_3 (A^{0,1|0}, A^{0,1|0}, A^{0,1|0}) + \mu_1 (C^{0,3|0}) = 0, \]

\[ \bar{\partial} C^{0,3|0} + \mu_2 (A^{0,1|0}, C^{0,3|0}) + \frac{1}{2} \mu_2 (B^{0,2|0}, B^{0,2|0}) + \]

\[ + \frac{1}{2} \mu_3 (A^{0,1|0}, A^{0,1|0}, B^{0,2|0}) + \frac{1}{4!} \mu_4 (A^{0,1|0}, A^{0,1|0}, A^{0,1|0}, A^{0,1|0}) = 0. \]

Every \( L_\infty \)-algebra is quasi-isomorphic to an \( L_\infty \)-algebra which has \( \mu_1 = 0 \) (Minimal Model Theorem). For this algebra, the first equation turns into

\[ \bar{\partial} A^{0,1|0} + \frac{1}{2} \mu_2 (A^{0,1|0}, A^{0,1|0}) = 0, \]

and by means of the Penrose–Ward transform this will correspond to maximally supersymmetric Yang–Mills theory in four dimensions.
Conclusions and Outlook
In general, we have seen that the area of twistor geometry and categorified principal bundles can be fruitfully combined to formulate self-dual higher gauge theory in six dimensions.

The advantage of twistor geometry is that the e.o.m. and the gauge transformations follow directly from complex algebraic data on twistor space.

Furthermore, we have seen that higher gauge theory enables us to write down a twistor action principle for maximally supersymmetric Yang–Mills theory in four dimensions.

Many open questions remain, such as the choice of higher gauge group, the explicit constructions of higher bundles, including the dimensional reductions.
Thank You!