Virtual Hybrid Edge Detection:  
Propagation and recovery of singularities in  
Calderón’s inverse conductivity problem  

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Electrical impedance tomography (EIT)

Problem: EIT is high contrast, but low resolution:

Figure 1: EIT tank and measurements. Source: Kaipio lab, Univ. of Kuopio, Finland
Hybrid imaging

“Multi-wave” methods often combine illumination and measurement modalities, one having

- high contrast sensitivity/low resolution (EIT,..)

and the other exhibiting

- low contrast/high resolution (ultrasound...
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Use two different types of waves, linked by a **physical coupling**.
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and the other exhibiting

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Use two different types of waves, linked by a physical coupling.

Mathematically: couple an elliptic PDE with a hyperbolic / real principal type PDE
Current work: virtual ‘hybrid’ imaging

Use only one kind of wave: electrostatic.

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Good propagation of singularities is obtained not via coupling with another physics, but by mathematical analysis.

Exploit complex principal type operator geometry underlying CGO solutions \(\Rightarrow\) Singularities propagate efficiently along 2D characteristics to all of \(\partial \Omega\) (in \(\mathbb{R}^2\)).

Formally: if \(\sigma\) is pws with jumps (edges), can stably reconstruct leading singularities. Can image inclusions within inclusions.
Astala-Päivärinta CGO solutions in 2D

\[ \Omega \subset \mathbb{R}^2 = \mathbb{C}, \ (x, y) = x + iy = z, \ (\xi, \eta) = \xi + i\eta = \zeta \]

\[ \sigma \in L^{\infty}(\Omega), \ 0 < c_1 \leq \sigma(z) \leq c_2 < \infty, \]

\[ \sigma \equiv 1 \text{ near } \partial \Omega, \text{ extended to } \mathbb{R}^2. \]

\[ \implies \text{conductivity } \sigma^{-1} \text{ is similar} \]
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**Exponentially growing/decaying solutions:**

For \( k \in \mathbb{C} \) a complex frequency, \( \exists u_1, u_2 \) s.t.

\[
\nabla \cdot \sigma \nabla u_1 = 0, \quad \nabla \cdot \sigma^{-1} \nabla u_2 = 0, \quad \text{on} \quad \mathbb{R}^2,
\]

\[
u_1, u_2 \sim e^{ikz} \left(1 + O\left(\frac{1}{|z|}\right)\right), \quad |z| \to \infty.
\]
Beltrami equations

Let \( \mu = \mu_\sigma = \frac{1-\sigma}{1+\sigma} \), so \( |\mu| \leq 1 - \epsilon \), \( \mu_{\sigma-1} = -\mu_\sigma \).

Look for CGO solns \( f_\mu(z) \) of \( \overline{\partial}_z f_\mu = \mu \partial_z f_\mu \),
similarly \( f_{-\mu} \) for \( -\mu \):

\[
f_{\pm \mu}(z, k) = e^{ikz}(1 + \omega_{\pm}(z, k)), \quad \omega_{\pm} = O\left(\frac{1}{|z|}\right).
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$$f_{\pm\mu}(z, k) = e^{ikz}(1 + \omega^{\pm}(z, k)), \quad \omega^{\pm} = O\left(\frac{1}{|z|}\right).$$

$$(u_1, u_2) \leftrightarrow (f_\mu, f_{-\mu})$$

$\omega^{\pm}$ can be computed from D2N data for $\sigma$.

Focus on $\omega^{+} =: \omega$. 
Huhtanen and Perämäki solutions (2012)

Let
\[ e_k(z) = e^{i(kz + \overline{k}z)} = e^{i2\text{Re}(kz)}, \]
so that \(|e_k(z)| = 1, \quad \overline{e_k} = e_{-k} \).

Define
\[ \alpha(z, k) = -i\overline{k}e_{-k}(z)\mu(z), \quad \beta(z, k) = e_{-k}(z)\mu(z) \]
Huhtanen and Perämäki solutions (2012)

Let
\[ e_k(z) = e^{i(kz + k \bar{z})} = e^{i2\text{Re}(kz)}, \]
so that \(|e_k(z)| = 1,\) \(\overline{e_k} = e_{-k}.\)

Define
\[ \alpha(z, k) = -i\overline{k} e_{-k}(z) \mu(z), \quad \beta(z, k) = e_{-k}(z) \mu(z) \]

Then \(\omega(z, k)\) satisfies a \(\mathbb{R}\)-Beltrami equation:
\[ (1) \quad \overline{\partial} \omega - \beta \overline{\partial} \omega - \alpha \omega = \alpha. \]

H.-P. show that \(\exists! \omega \in W^{1,p}(\mathbb{C}), 2 < p < p_\epsilon.\)
Solid Cauchy and Beurling transforms

\[ P f(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(z')}{z - z'} d^2z', \quad S f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(z')}{(z' - z)^2} d^2z' \]

so that \( \overline{\partial}P = I \), \( S = \partial P \) and \( S\overline{\partial} = \partial \) on \( C_0^\infty(\mathbb{C}) \).
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so that \( \overline{\partial} P = I \), \( S = \partial P \) and \( S \overline{\partial} = \partial \) on \( C_0^\infty(\mathbb{C}) \).

Define \( u = -\partial \omega = -\overline{(\partial \omega)} \in L^p \). Then 
\[
\omega = -P \overline{u} \quad \text{and} \quad \partial \omega = -S \overline{u}
\]
and (1) becomes

\[
(1') \quad (I + A \rho) u = -\overline{\alpha},
\]

where \( \rho = \) complex conjugation and 
\[
A = -(\overline{\alpha} P + \overline{\beta} S)
\]
Neumann series

Expand \( u \sim \sum_{n=0}^{\infty} u_n, \ u_0 = -\bar{\alpha}, \ u_{n+1} = -A\bar{u}_n \)

\[ \implies \omega = -P\bar{u} \sim \sum_{n=0}^{\infty} \omega_n, \ \omega_n = -P\bar{u}_n. \]
Neumann series

Expand $u \sim \sum_{n=0}^{\infty} u_n$, $u_0 = -\bar{\alpha}$, $u_{n+1} = -Au_n$

$\implies \omega = -Pu \sim \sum_{n=0}^{\infty} \omega_n$, $\omega_n = -Pu_n$.

Focus on

$u_0 = -\bar{\alpha}$, $\omega_0 = P\alpha$,

$u_1 = A\alpha = - (\bar{\alpha}P + \bar{\beta}S')(\alpha)$, $\omega_1 = P(\alpha\bar{P}\alpha + \beta\bar{S}\alpha)$. 
Neumann series

Expand $u \sim \sum_{n=0}^{\infty} u_n$, $u_0 = -\bar{\alpha}$, $u_{n+1} = -A\bar{u}_n$

$\implies \omega = -P\bar{u} \sim \sum_{n=0}^{\infty} \omega_n$, $\omega_n = -P\bar{u}_n$.

Focus on

$u_0 = -\bar{\alpha}$, $\omega_0 = P\alpha$,

$u_1 = A\alpha = -(\bar{\alpha}P + \bar{\beta}S)(\alpha)$, $\omega_1 = P(\alpha\bar{P}\alpha + \beta\bar{S}\alpha)$.

$\omega_0|_{z \in \partial \Omega}$: Stably determines singularities of $\mu$.

$\omega_n|_{z \in \partial \Omega}, n \geq 1$: Contribute scattering, which explains artifacts in numerics.

Note: $\omega_n$ is an $(n+1)$-linear operator of $\mu$. 
We can currently carry this out on level of

- WF set analysis: all \( \tilde{\omega}_n \) for general \( \sigma \in L^\infty \).
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- **WF set analysis:** all \( \tilde{\omega}_n \) for general \( \sigma \in L^\infty \).

- **Operator theory:** \( \sigma \) \text{ pws with jumps} across \text{curved interfaces} \implies \( \tilde{\omega}_1, \tilde{\omega}_2 \) are in \( I^{p,l} \) spaces.
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- **WF set analysis**: all $\tilde{\omega}_n$ for general $\sigma \in L^\infty$.

- **Operator theory**: $\sigma$ pws with jumps across curved interfaces $\Rightarrow \tilde{\omega}_1, \tilde{\omega}_2$ are in $L^{p,l}$ spaces.

- Higher order terms in Neumann series create a strong artifact at $t = 0$ and weaker ones via multiple scattering of points in $WF(\mu)$. 
\[ \omega_0(z, k) = \frac{ik}{\pi} \int_{\mathbb{C}} e^{i 2 \text{Re}(kz')} \frac{\mu(z')}{z - z'} \, d^2 z' \]
\[ \omega_0(z, k) = \frac{ik}{\pi} \int_{C} \frac{e^{i2 \text{Re}(kz')} \mu(z')}{z - z'} d^2z' \]

1. Polar coordinates in \( k \): write \( k = \tau e^{i\varphi} \)
\[
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\]

1. Polar coordinates in \( k \): write \( k = \tau e^{i\varphi} \)

2. Partial Fourier transform \( \tau \rightarrow t \):

\[
\tilde{\omega}_0(z, t, e^{i\varphi}) := \int_{\mathbb{R}} e^{-it\tau} \omega_0(z, \tau e^{i\varphi}) \, d\tau
\]

\[
= \frac{e^{i\varphi}}{\pi} \int_{\mathbb{R}} \int_{\mathbb{C}} (i\tau) \frac{e^{-i\tau(t-2\Re(e^{i\varphi}z'))}}{z - z'} \mu(z') \, d^2z' \, d\tau
\]

\[
= -2e^{i\varphi} \int_{\mathbb{C}} \delta'(t - 2\Re(e^{i\varphi}z')) \frac{\mu(z')}{z - z'} \, d^2z'
\]
Recall: $\sigma \in L^\infty$, $\sigma \equiv 1$ near $\partial \Omega$, $\mu \equiv 0$ near $\partial \Omega$.
Assume $\text{supp}(\mu) \subset \Omega_0 \subset \subset \Omega$.

Define $T_0 : \mathcal{E}'(\Omega_0) \to \mathcal{D}'(\mathbb{C} \times \mathbb{R} \times S^1)$,
\[ \mu'(z) \mapsto (T_0 \mu)(z, t, e^{i\varphi}) := \tilde{\omega}_0(z, t, e^{i\varphi}). \]
Recall: $\sigma \in L^\infty$, $\sigma \equiv 1$ near $\partial \Omega$, $\mu \equiv 0$ near $\partial \Omega$. Assume $\text{supp}(\mu) \subset \Omega_0 \subset \subset \Omega$.

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\[
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\]

Schwartz kernel of $T_0$:
\[
K_0(z, t, e^{i\varphi}, z') = \left(\frac{-2e^{i\varphi}}{z - z'}\right)\delta'(t - 2\text{Re}(e^{i\varphi}z')).
\]

First factor is smooth for $z \notin \Omega_0$, $z' \in \Omega_0$ $\implies$

$T_0$ is a generalized Radon transform and thus a Fourier integral operator (FIO).
Define $T_0^{z_0} : \mathcal{E}'(\Omega_0) \rightarrow \mathcal{D}'(\mathbb{R} \times S^1)$ by

$$\mu(z') \mapsto (T_0^{z_0} \mu)(t, e^{i\varphi}) := \tilde{\omega}_0(z_0, t, e^{i\varphi}).$$
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$$\mu(z') \mapsto (T_0^{z_0} \mu)(t, e^{i\varphi}) := \tilde{\omega}_0(z_0, t, e^{i\varphi}),$$

- $T_0^{z_0}$ is a weighted and differentiated version of the Radon transform on $\mathbb{C} \simeq \mathbb{R}^2$.
- $T_0^{z_0}$ is an FIO of order $\frac{1}{2}$, $T_0^{z_0} \in I^{\frac{1}{2}}(C)$, with same canonical relation as std. Radon transf.

$$C = N^* \{ t = 2 \text{Re}(e^{i\varphi}z') \}' \subset T^*(\mathbb{R} \times S^1) \times T^*\Omega_0.$$

- $C$ is a canonical graph.
• $C$ is independent of $z_0$, but symbol

$$\sigma_{prin}(T_0^{z_0}) = \frac{(-ie^{i\varphi}) \text{sgn}(\tau)|\tau|^{\frac{1}{2}}}{z_0 - z'},$$

is not.

• The factor $(z_0 - z')^{-1}$ is smooth and $\neq 0$, but causes

  (i) A fall-off in detectability of jumps, at rate $\sim d(z', z_0)^{-1}$.

  (ii) Artifacts, esp. when $\mu$ has singularities at $z'$ close to $z_0$, due to the large magnitude and phase gradient of $(z_0 - z')^{-1}$. 
Figure 2: Conductivity phantom: a small circular inclusion.
Figure 3: $Re \tilde{\omega}(z_0, t, e^{i\phi})$ (axes: $\phi = \text{horiz.}, t = \text{vert.}$)
Figure 4: Backprojected reconstruction from $\omega(z_0, \cdot, \cdot)$. 
Weighted ‘averages’ in \( z_0 \)

Let \( a(z_0) \) be a \( \mathbb{C} \)-valued weight on \( \partial \Omega \). Form

\[
\tilde{\omega}_0^a(t, \varphi) := \frac{1}{2\pi i} \int_{\partial \Omega} \tilde{\omega}_0(z_0, t, \varphi) a(z_0) \, dz_0,
\]

Let \( T_0^a \) be the operator \( \mu \to \tilde{\omega}_0^a \).
Weighted ‘averages’ in $z_0$

Let $a(z_0)$ be a $\mathbb{C}$-valued weight on $\partial \Omega$. Form

\[(2) \quad \tilde{\omega}_0^a(t, \varphi) := \frac{1}{2\pi i} \int_{\partial \Omega} \tilde{\omega}_0(z_0, t, \varphi) a(z_0) \, dz_0,\]

Let $T_0^a$ be the operator $\mu \mapsto \tilde{\omega}_0^a$.

Then $T_0^a \in L^\frac{1}{2}(C')$ and $(T_0^a)^* T_0^a \in \Psi^1(\Omega_0)$, with

$\sigma_{\text{prin}}((T_0^a)^* T_0^a)(z', \zeta') = 2\pi^2 |\alpha(z')|^2 |\zeta'|$, $z' \in \Omega_0$,

where

$$\alpha(z') = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{a(z_0) \, dz_0}{z_0 - z'}, \quad z' \in \Omega$$

is the Cauchy (line) integral of $a(\cdot)$
Pick $a \equiv 1/\sqrt{2}$ on $\partial \Omega$ in (2). ($\int_{\partial \Omega} a \, dz_0 = 0$ !)

Then $\alpha(z') \equiv 1/\sqrt{2}$ on $\Omega_0$ and

$$(T^a_0)^* T^a_0 = (-\Delta)^{1/2} \mod \Psi^0(\Omega_0),$$
Pick \( a \equiv 1/\sqrt{2} \) on \( \partial \Omega \) in (3). (\( \int_{\partial \Omega} a \, dz_0 = 0! \))

Then \( \alpha(z') \equiv 1/\sqrt{2} \) on \( \Omega_0 \) and

\[
(T_0^a)^* T_0^a = (-\Delta)^{\frac{1}{2}} \bmod \Psi^0(\Omega_0),
\]

Gives local-tomography type imaging of \( \mu \), good for detection of singularities of \( \sigma \) from the singularities of \( \tilde{\omega} \) (which correspond to high frequency behavior of \( \omega \)).
Figure 5: $\text{Re} \tilde{\omega}^a(z_0, t, e^{i\varphi})$ for $a \equiv 1/(2\sqrt{2})$
Figure 6: Reconstruction from $\tilde{\omega}^a(\cdot, \cdot)$ for $a \equiv 1/\sqrt{2}$. 
So far, microlocal analysis does not seem to be needed: can express $\omega_0^a$ in terms of the Radon transform.

However: the figures above were created after filtering out certain artifacts.
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Singularities of $\tilde{\omega}^z_0$, $\tilde{\omega}^a$ occur at

(i) $t = 0$ for any $\mu$ with singularities, and

(ii) at other values of $t, \varphi$, depending on $\mu$.

Explained by wave-front set analysis of the higher order terms in the Neumann series, which are multilinear FIOs.
\[ \tilde{\omega}_1^{z_0}(t, \varphi) = \int e^{-it\tau} \omega_1(z_0, \tau e^{i\varphi}) \, d\tau \]

\[ = \int_{\Omega} \int_{\Omega} K_1^{z_0}(t, e^{i\varphi}; z', z'') \cdot \mu(z') \cdot \mu(z'') \, d^2z' \, d^2z'' \]

**Bilinear operator acting on** \( \mu \otimes \mu \), **w/ kernel**

\[ K_1^{z_0}(t, e^{i\varphi}; z', z'') = \frac{1}{\pi^2} \left( \frac{e^{2i\varphi} \delta''(t + 2 \text{Re}(e^{i\varphi}(z' - z'')))}{(z' - z_0)(\bar{z''} - \bar{z'})} \right) \]

\[ + \frac{e^{i\varphi} \delta'(t + 2 \text{Re}(e^{i\varphi}(z' - z'')))}{(z' - z_0)(\bar{z''} - \bar{z'})^2} \]
• $WF(\tilde{\omega}_n)$ can be described in terms of $(n + 1)$-fold scatterings of $WF(\mu)$.

• Still to do: estimates to control smoothness of $\tilde{\omega}_n$ for $n \geq 3$. 
• $WF(\tilde{\omega}_n)$ can be described in terms of $(n + 1)$-fold scatterings of $WF(\mu)$.

• Still to do: estimates to control smoothness of $\tilde{\omega}_n$ for $n \geq 3$.

• Can currently do this for $n = 1, 2$ under prior on $\sigma$ which includes pws with jumps across curved interfaces.

• Rigorous justification of the Neumann series will require a mixture of multilinear microlocal and harmonic analysis.
Figure 7: Stroke phantoms within low conductivity skull: Clot (left), haemorrhage (right).
Figure 8: Stroke phantoms reconstructions: Clot (left), haemorrhage (right).