Weakly commensurable arithmetic groups and locally symmetric spaces

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Outline

1. Weak commensurability
   - Definition and motivations
   - Basic results
   - Arithmetic Groups
   - Remarks on nonarithmetic case

2. Length-commensurable locally symmetric spaces
   - Links between length-commensurability and weak commensurability
   - Main results
   - Applications to isospectral locally symmetric spaces

3. Proofs
   - “Special” elements in Zariski-dense subgroups


**Survey:**

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1 Weak commensurability
   - Definition and motivations
   - Basic results
   - Arithmetic Groups
   - Remarks on nonarithmetic case

2 Length-commensurable locally symmetric spaces
   - Links between length-commensurability and weak commensurability
   - Main results
   - Applications to isospectral locally symmetric spaces

3 Proofs
   - “Special” elements in Zariski-dense subgroups
Definition

Let $G_1$ and $G_2$ be two semi-simple groups defined over a field $F$ (of characteristic zero).

- Semi-simple $g_i \in G_i(F)$ ($i = 1, 2$) are weakly commensurable if there exist maximal $F$-tori $T_i \subset G_i$ such that $g_i \in T_i(F)$ and for some $\chi_i \in X(T_i)$ (defined over $\bar{F}$) we have
  $$\chi_1(g_1) = \chi_2(g_2) \neq 1.$$

- (Zariski-dense) subgroups $\Gamma_i \subset G_i(F)$ are weakly commensurable if every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.
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If $T \subset \text{GL}_n$ is an $F$-torus, then given $g \in T(F)$ and $\chi \in X(T)$ we have

$$\chi(g) = \lambda_1^{a_1} \cdots \lambda_n^{a_n}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $g$ and $a_1, \ldots, a_n \in \mathbb{Z}$.

- Semi-simple $g_1 \in G_1(F)$ and $g_2 \in G_2(F)$ with eigenvalues

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2}$$

are weakly commensurable if

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1$$

for some $a_1, \ldots, a_{n_1}$ and $b_1, \ldots, b_{n_2} \in \mathbb{Z}$. 
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**Main Question:** What can one say about Zariski-dense subgroups \( \Gamma_i \subset G_i(F) \) \((i = 1, 2)\) given that they are weakly commensurable?
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More specifically, *under what conditions are $\Gamma_1$ and $\Gamma_2$ necessarily commensurable*?
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**Recall:** subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of a group $G$ are commensurable if

$$[\mathcal{H}_i : \mathcal{H}_1 \cap \mathcal{H}_2] < \infty \quad \text{for} \quad i = 1, 2.$$
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$\Gamma_1$ and $\Gamma_2$ are commensurable up to an $F$-isomorphism between $G_1$ and $G_2$ if there exists an $F$-isomorphism

$$\sigma : G_1 \to G_2$$

such that $\sigma(\Gamma_1)$ and $\Gamma_2$ are commensurable in usual sense.
**General framework:** Characterization of linear groups in terms of spectra of its elements.
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Complex representations of finite groups:

Let $\Gamma$ be a finite group,

$$\rho_i : \Gamma \rightarrow GL_{n_i}(\mathbb{C}) \quad (i = 1, 2)$$

be representations. Then

$$\rho_1 \simeq \rho_2 \iff \chi_{\rho_1}(g) = \chi_{\rho_2}(g) \quad \forall g \in \Gamma,$$

where $\chi_{\rho_i}(g) = \text{tr} \rho_i(g) = \sum \lambda_j \quad (\lambda_1, \ldots, \lambda_{n_i} \text{ eigenvalues of } \rho_i(g))$
Weak commensurability

Definition and motivations

Algebraic perspective

- Data afforded by weak commensurability is much more convoluted than data afforded by character of a group representation:

  when computing

  \[ \chi(g) = \lambda_1^{a_1} \cdots \lambda_n^{a_n} \]

  one can use arbitrary integer weights \( a_1, \ldots, a_n \). So weak commensurability appears to be difficult to analyze.

- **Example.** Let \( \Gamma \subset SL_n(\mathbb{C}) \) be a neat Zariski-dense subgroup. For \( d > 0 \), let

  \[ \Gamma^{(d)} = \langle \gamma^d \mid \gamma \in \Gamma \rangle. \]

  Then any \( \Gamma^{(d)} \subset \Delta \subset \Gamma \) is weakly commensurable to \( \Gamma \).

  So, one needs to limit attention to some special subgroups in order to generate meaningful results.
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Geometric perspective

Let $M$ be a Riemannian manifold.

$L(M)$ - (weak) length spectrum (collection of lengths of closed geodesics w/o multiplicities)

- Weak commensurability (of fundamental groups) adequately reflects length-commensurability of locally symmetric space.
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$$Q \cdot L(M_1) = Q \cdot L(M_2).$$

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We will demonstrate this for **Riemann surfaces** - for now.
Let $G = SL_2$. Corresponding symmetric space:

$$SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) = \mathbb{H} \text{ (upper half-plane)}$$

- Any Riemann (compact) surface of genus $> 1$ is of the form
  $$M = \mathbb{H}/\Gamma$$

  where $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup (with torsion-free image in $PSL_2(\mathbb{R})$).

- Any closed geodesic $c$ in $M$ corresponds to a semi-simple $\gamma \in \Gamma$, i.e. $c = c_\gamma$, and has length
  $$\ell(c_\gamma) = (1/n_\gamma) \cdot \log t_\gamma$$

  where $t_\gamma$ is the eigenvalue of $\pm \gamma$ which is $> 1$, $n_\gamma$ is an integer $\geq 1$.

  NOTE that $\pm \gamma$ is conjugate to

$$\begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}.$$
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  NOTE that $\pm \gamma$ is conjugate to $\begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$. 
If $M_i = \mathbb{H}/\Gamma_i$ ($i = 1, 2$) are length-commensurable then:

- for any nontrivial semi-simple $\gamma_1 \in \Gamma_1$ there exists a nontrivial semi-simple $\gamma_2 \in \Gamma_2$ such that

$$n_1 \cdot \log t_{\gamma_1} = n_2 \cdot \log t_{\gamma_2}$$

for some integers $n_1, n_2 \geq 1$, and vice versa.

So,

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$$

where $\chi_i$ is the character of the maximal $\mathbb{R}$-torus $T_i \subset SL_2$

corresponding to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{n_i}$.

Thus, $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.
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**Theorem 1.** Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero. If there exist finitely generated Zariski-dense subgroups $\Gamma_i \subset G_i(F) \ (i = 1, 2)$ that are weakly commensurable then either $G_1$ and $G_2$ have the same Killing-Cartan type, or one of them is of type $B_n$ and the other is of type $C_n$ for some $n \geq 3$. 
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The first result shows that weak commensurability “almost” retains information about the type of the ambient algebraic group.

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Note that groups of types $B_n$ and $C_n$ can indeed contain Zariski-dense weakly commensurable subgroups - more later.
Let

- $G$ be a connected almost simple algebraic group defined over a field $F$ of characteristic zero,
- $\Gamma \subset G(F)$ be a Zariski-dense subgroup.

Then $K^\Gamma$ is the (minimal) field of definition of $\text{Ad} \Gamma$ (E.B. Vinberg).

Theorem 2. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero, and let $\Gamma_i \subset G_i(F)$ ($i = 1, 2$) be finitely generated Zariski-dense subgroups. If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable then $K^\Gamma_1 = K^\Gamma_2$. 
Field of definition

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Let $K_{\Gamma}$ denote the subfield of $F$ generated by $\text{Tr} \ Ad \ \gamma$ for all $\gamma \in \Gamma$. 
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Notion of arithmeticity

For a $\mathbb{Q}$-defined algebraic group $G \subset \text{GL}_n$, we set

$$G(\mathbb{Z}) = G \cap \text{GL}_n(\mathbb{Z}).$$

The subgroups of $G(F)$ (where $F/\mathbb{Q}$) commensurable with $G(\mathbb{Z})$, are called arithmetic.
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Replace $\mathbb{Z}$ with $\mathbb{Z}[1/2]$ (= ring of $S$-integers $\mathbb{Z}_S \subset \mathbb{Q}$ for $S = \{v_\infty, v_2\}$). The subgroups of $G(F)$ commensurable with

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are called $S$-arithmetic.
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are called \( S \)-arithmetic.

More generally, given a number field \( K \) and a (finite) \( S \subset V^K \) containing \( V^K_\infty \) (archimedean places), one defined the ring of \( S \)-integers

\[
\mathcal{O}_K(S) = \{a \in K^\times | v(a) \geq 0 \text{ for all } v \in V^K \setminus S\} \cup \{0\}.
\]
Given a $K$-defined algebraic group $G \subset \text{GL}_n$, we set

$$G(\mathcal{O}_K(S)) = G \cap \text{GL}_n(\mathcal{O}_K(S)).$$

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What is an arithmetic subgroup of an algebraic group which is NOT defined over a number field?
Weak commensurability

Arithmetic Groups

Notion of arithmeticity

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What is an arithmetic subgroup of an algebraic group which is NOT defined over a number field?

E.g.: What is an arithmetic subgroup of $G(\mathbb{R})$ where

$$G = SO_3(f) \quad \text{and} \quad f = x^2 + e \cdot y^2 - \pi \cdot z^2?$$
Notion of arithmeticity

We define arithmetic subgroups of $G(F)$ in terms of all possible forms of $G$ over subfields of $F$ that are number fields.
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In our example, we can consider rational quadratic forms that are $\mathbb{R}$-equivalent to $f$, e.g.:

$$f_1 = x^2 + y^2 - 3z^2 \text{ or } f_2 = x^2 + 2y^2 - 7z^2.$$
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Then $SO_3(f_i) \simeq SO_3(f)$ over $\mathbb{R}$, and

$$\Gamma_i := SO_3(f_i) \cap GL_3(\mathbb{Z})$$

are arithmetic subgroups of $G(\mathbb{R})$ for $i = 1, 2$. 

One can also consider $\mathbb{K} = \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ and $f_3 = x^2 + y^2 - \sqrt{2}z^2$. Then $\Gamma_3 := SO_3(f_3) \cap GL_3(\mathbb{Z}[\sqrt{2}])$ is an arithmetic subgroup of $G(\mathbb{R})$ over $\mathbb{K}$. 

One can further replace integers by $S$-integers, etc.
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One can further replace integers by $S$-integers, etc.
Definition of arithmeticity

**Definition.** Let $G$ be an absolutely almost simple algebraic group over a field $F$, $\text{char } F = 0$, and $\pi : G \rightarrow \widetilde{G}$ be isogeny onto adjoint group.

1. a number field $K$ with a fixed embedding $K \hookrightarrow F$;
2. a finite set $S \subset V^K$ containing $V^K_\infty$;
3. an $F/K$-form $\mathcal{G}$ of $\widetilde{G}$, i.e. $F\mathcal{G} \simeq \widetilde{G}$ over $F$. 
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Convention: $S$ does not contain nonarchimedean $v$ such that $G$ is $K_v$-anisotropic.

We do NOT fix an $F$-isomorphism $F\mathcal{G} \simeq \mathcal{G}$ in $\circ \circ \circ$, and by varying it we obtain a class of groups invariant under $F$-automorphisms.
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1. a **number field** $K$ with a **fixed** embedding $K \hookrightarrow F$;
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We do **NOT** fix an $F$-isomorphism $F\mathcal{G} \simeq G$ in n° 3, and by varying it we obtain a class of groups invariant under $F$-automorphisms.
**Proposition.** Let $G_1$ and $G_2$ be connected absolutely almost simple algebraic groups defined over a field $F$, $\text{char } F = 0$, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i, K_i, S_i)$-arithmetic group ($i = 1, 2$).

Then $\Gamma_1$ and $\Gamma_2$ are commensurable up to an $F$-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if

- $K_1 = K_2 =: K$;
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In the above example, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are *pairwise* noncommensurable.

- $\Gamma_1$ and $\Gamma_2$ are *NOT* commensurable b/c the corresponding $\mathbb{Q}$-forms $G_1 = \text{SO}_3(f_1)$ and $G_2 = \text{SO}_3(f_2)$ are *NOT* isomorphic over $\mathbb{Q}$. 
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- $\Gamma_3$ is NOT commensurable with either $\Gamma_1$ or $\Gamma_2$ b/c they have different fields of definition: $\mathbb{Q}(\sqrt{2})$ for $\Gamma_3$, and $\mathbb{Q}$ for $\Gamma_1$ and $\Gamma_2$. 
Theorem 3. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero.

If Zariski-dense $(G_i, K_i, S_i)$-arithmetic $\Gamma_i \subset G_i(F)$ are weakly commensurable for $i = 1, 2$, then $K_1 = K_2$ and $S_1 = S_2$. 
**Theorem 3.** Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero.

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The forms $G_1$ and $G_2$ may NOT be $K$-isomorphic in general, but we have the following.

**Theorem 4.** Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero, of the same type different from $A_n$, $D_{2n+1}$ with $n > 1$, and $E_6$, and let $\Gamma_i \subset G_i(F)$ be a $(G_i, K, S)$-arithmetic subgroup.

If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable then $G_1 \simeq G_2$ over $K$, and hence $\Gamma_1$ and $\Gamma_2$ are commensurable up to an $F$-isomorphism between $\overline{G}_1$ and $\overline{G}_2$. 
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[1] - groups of type $\neq D_{2n}$;  [2] - groups of type $D_{2n}$ other than $D_4$;
Skip Garibaldi - type $D_4$ and alternative proof for all $D_{2n}$.
Theorem 5. (Garibaldi-R.) Let $G_1$ and $G_2$ be connected absolutely almost simple groups of types $B_n$ and $C_n$ ($n \geq 3$) respectively, defined over a field $F$ of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i, K, S)$-arithmetic subgroup.

Then $\Gamma_1$ and $\Gamma_2$ are weakly commensurable if and only if

- $\text{rk}_{K_v} G_1 = \text{rk}_{K_v} G_2 = 0$ or $n$ for all $v \in V^K$;
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Theorem 6. Let $G_1$ and $G_2$ be two connected absolutely almost simple groups defined over a field $F$ of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be a Zariski-dense $(K, S)$-arithmetic subgroup.

Then the set of Zariski-dense $(K, S)$-arithmetic subgroups $\Gamma_2 \subset G_2(F)$ which are weakly commensurable to $\Gamma_1$, is a union of finitely many commensurability classes.
Theorem 7. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i, K, S)$-arithmetic subgroup for $i = 1, 2$. If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable then $rk_K G_1 = rk_K G_2$; in particular, if $G_1$ is $K$-isotropic then so is $G_2$. 
**Theorem 7.** Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i, K, S)$-arithmetic subgroup for $i = 1, 2$.

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**Theorem 8.** Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a nondiscrete locally compact field $F$ of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense lattice for $i = 1, 2$.

Assume that $\Gamma_1$ is a $(K, S)$-arithmetic subgroup of $G_1(F)$.

If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable, then $\Gamma_2$ is a $(K, S)$-arithmetic subgroup of $G_2(F)$.
Outline

1. Weak commensurability
   - Definition and motivations
   - Basic results
   - Arithmetic Groups
   - Remarks on nonarithmetic case

2. Length-commensurable locally symmetric spaces
   - Links between length-commensurability and weak commensurability
   - Main results
   - Applications to isospectral locally symmetric spaces

3. Proofs
   - “Special” elements in Zariski-dense subgroups
Two aspects:

1. Given a Zariski-dense subgroup $\Gamma_1 \subset G_1(F)$ with $K_{\Gamma_1} = K$, determine possible $K$-groups $G_2$ for which there exists a Zariski-dense subgroup $\Gamma_2 \subset G_2(K)$ which is weakly commensurable to $\Gamma_1$;

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Item 1° is closely related to the following classical question:

To what extent is an absolutely almost simple algebraic $K$-group $G$ determined by the set of isomorphism classes of its maximal $K$-tori?
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To what extent is an absolutely almost simple algebraic $K$-group $G$ is determined by the set of isomorphism classes of its maximal $K$-tori?

(Our results solve this problem for a number field $K$.)
(*) Let $D_1$ and $D_2$ be quaternion division algebras over a field $K$ (char $K \neq 2$). Assume that $D_1$ and $D_2$ have same maximal subfields. Are $D_1$ and $D_2$ necessarily isomorphic?
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**Geometric connection:**

Let

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be a (compact) **Riemann surface**, $\Gamma \subset SL_2(\mathbb{R})$ a discrete subgroup.
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(\text{.. well, one usually considers } \mathbb{Q}[\Gamma^{(2)}] \text{ where } \Gamma^{(2)} \subset \Gamma \text{ is generated by squares ...})
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Suppose that $M_1$ and $M_2$ are length-commensurable.

Then

$$Z(D_1) = Z(D_2) =: K,$$

and for any semi-simple $\gamma_1 \in \Gamma_1$ there exists a semi-simple $\gamma_2 \in \Gamma_2$ s. t.

$$\gamma_1^m \text{ and } \gamma_2^n \text{ are conjugate in } SL_2(\mathbb{R}) \text{ for some } m, n \geq 1.$$ 

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$\Rightarrow$ $K[\gamma_1^m] \subset D_1$ and $K[\gamma_2^n] \subset D_2$ are isomorphic.

Thus, length-commensurability of $M_1$ and $M_2$ implies that $D_1$ and $D_2$ have the same isomorphism classes of étale subalgebras that intersect $\Gamma_1$ and $\Gamma_2$, respectively.
On the other hand,

\[ \Gamma_1 \ & \Gamma_2 \text{ commensurable} \implies D_1 \simeq D_2. \]
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So, analysis of length-commensurability for Riemann surfaces leads to questions like (\ast) for quaternion algebras.
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So, analysis of length-commensurability for Riemann surfaces leads to questions like (\(\ast\)) for quaternion algebras.

\(\ast\) has affirmative answer over number fields \(\Rightarrow\)

\[ L(M_1) = L(M_2) \] for arithmetically defined Riemann surfaces \(M_1 \ & \ M_2\) implies that \(M_1\) and \(M_2\) are commensurable (A. Reid).
On the other hand,

\[ \Gamma_1 \ & \ \Gamma_2 \ \text{commensurable} \ \Rightarrow \ \mathcal{D}_1 \simeq \mathcal{D}_2. \]

So, analysis of length-commensurability for Riemann surfaces leads to questions like \((\ast)\) for quaternion algebras.

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\((\ast)\) can have negative answer over “large” fields (Rost, Wadsworth, Schacher ...), but remains widely open over finitely generated fields.
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**Theorem 9.** (A.R., I.R.) If \((*)\) holds over \(K\) then it also holds over the field of rational functions \(K(x)\).
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Theorem 9. (A.R., I.R.) If \((*)\) holds over \(K\) then it also holds over the field of rational functions \(K(x)\).

Definition. Let \(D\) be a finite-dimensional central division algebra /\(K\). The \textbf{genus} of \(D\) is

\[ \text{gen}(D) = \{ [D'] \in \text{Br}(K) \mid D' \text{ division algebra with same maximal subfields as } D \}. \]
Question A: When does $\text{gen}(D)$ consist of a single class? Is this the case for quaternions?
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Question B: *When is $\text{gen}(D)$ finite?*

Question A is meaningful only for algebras $D$ of exponent 2. Indeed, $D^{\text{op}}$ has the same maximal subfields as $D$. But if $D \simeq D^{\text{op}}$, then $[D] \in \text{Br}(K)$ has exponent 2.
**Question A:** *When does* \( \text{gen}(D) \) *consist of a single class? Is this the case for quaternions?*

**Question B:** *When is* \( \text{gen}(D) \) *finite?*

Question A is meaningful *only* for algebras \( D \) of exponent 2. Indeed, \( D^{\text{op}} \) has the *same* maximal subfields as \( D \). But if \( D \simeq D^{\text{op}} \) then \([D] \in \text{Br}(K)\) *has exponent* 2.

Question B makes sense for division algebras of *any* degree.
**Question A:** When does $\text{gen}(D)$ consist of a single class? Is this the case for quaternions?

**Question B:** When is $\text{gen}(D)$ finite?

Question A is meaningful only for algebras $D$ of exponent 2. Indeed, $D^{\text{op}}$ has the same maximal subfields as $D$. But if $D \cong D^{\text{op}}$ then $[D] \in \text{Br}(K)$ has exponent 2.

Question B makes sense for division algebras of any degree.

*Both questions have the affirmative answer over number fields.*
Theorem 10. (Chernousov + R²) Let $K$ be a field of characteristic $\neq 2$. If $K$ satisfies the following property

(●) Any two finite-dimensional central division $K$-algebras $D_1$ and $D_2$ of exponent two that have the same maximal subfields are necessarily isomorphic,

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**Theorem 10.** (Chernousov + \( R^2 \)) Let \( K \) be a field of characteristic \( \neq 2 \).

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**Theorem 11.** (C + \( R^2 \)) Let \( K \) be a finitely generated field, and let \( D \) be a central division algebra \( /K \) of degree \( n \) which is prime to \( \text{char} \, K \).

Then \( \text{gen}(D) \) is finite.
**Conjecture.** Let $G_1, G_2$ be absolutely simple algebraic groups over a field $F$, $\text{char } F = 0$, let $\Gamma_1 \subset G_1(F)$ be a *finitely generated* Zariski-dense subgroup. 

Set $K = K_{\Gamma_1}$.

Then there exist a *finite collection* $G_2^{(1)}, \ldots, G_2^{(r)}$ of $F/K$-forms of $G_2$ such that if $\Gamma_2 \subset G_2(F)$ is a Zariski-dense subgroup weakly commensurable to $\Gamma_1$ then $\Gamma_2$ is contained (up to an $F$-automorphism of $G_2$) in one of the $G_2^{(i)}(K)$’s.
Conjecture. Let $G_1, G_2$ be absolutely simple algebraic groups over a field $F$, char $F = 0$, let $\Gamma_1 \subset G_1(F)$ be a finitely generated Zariski-dense subgroup. Set $K = K_{\Gamma_1}$.

Then there exist a finite collection $G_2^{(1)}, \ldots, G_2^{(r)}$ of $F/K$-forms of $G_2$ such that if $\Gamma_2 \subset G_2(F)$ is a Zariski-dense subgroup weakly commensurable to $\Gamma_1$ then $\Gamma_2$ is contained (up to an $F$-automorphism of $G_2$) in one of the $G_2^{(i)}(K)$’s.

Question: When can one take $r = 1$?
Outline

1 Weak commensurability
   - Definition and motivations
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   - Arithmetic Groups
   - Remarks on non-arithmetic case

2 Length-commensurable locally symmetric spaces
   - Links between length-commensurability and weak commensurability
   - Main results
   - Applications to isospectral locally symmetric spaces

3 Proofs
   - “Special” elements in Zariski-dense subgroups
Notations

- $G$ a connected absolutely (almost) simple algebraic group over $\mathbb{R}$; $\mathcal{G} = G(\mathbb{R})$
- $\mathcal{K}$ a maximal compact subgroup of $\mathcal{G}$; $\mathcal{X} = \mathcal{K}\backslash\mathcal{G}$ associated symmetric space, $\text{rk} \mathcal{X} = \text{rk}_{\mathbb{R}} \mathcal{G}$
- $\Gamma$ a discrete torsion-free subgroup of $\mathcal{G}$; $\mathcal{X}_\Gamma = \mathcal{X}/\mathcal{\Gamma}$
- $\mathcal{X}_\Gamma$ is \textbf{arithmetically defined} if $\Gamma$ is arithmetic (for $S = V^K_\infty$) as defined earlier
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Given $G_1, G_2, \Gamma_i \subset G_i := G_i(\mathbb{R})$ etc. as above, we will denote the corresponding **locally symmetric spaces** by $\mathcal{X}_{\Gamma_i}$. 
Two Riemannian manifolds $M_1$ and $M_2$ are:
- **commensurable** if they have a common finite-sheeted cover;
- **length-commensurable** if $Q \cdot L(M_1) = Q \cdot L(M_2)$, where $L(M_i)$ is the set of lengths of all closed geodesics in $M_i$. 

Question: When does length-commensurability imply commensurability?

$X_{\Gamma_1}$ and $X_{\Gamma_2}$ are commensurable $\iff \Gamma_1$ and $\Gamma_2$ are commensurable up to an isomorphism between $G_1$ and $G_2$.

**Fact.** Assume that $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are of finite volume. If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are length-commensurable then (under minor technical assumptions) $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.
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If $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ are length-commensurable then (under minor technical assumptions) $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.
The proof relies:

- **in rank one case** - on the result of Gel’fond and Schneider (1934): 
  \[ \text{if } \alpha \text{ and } \beta \text{ are algebraic numbers } \neq 0, 1 \text{ then } \frac{\log \alpha}{\log \beta} \text{ is either rational or transcendental.} \]

- **in higher rank case** - on the following **Conjecture** (Shanuel) 
  \[ \text{If } z_1, \ldots, z_n \in \mathbb{C} \text{ are linearly independent over } \mathbb{Q}, \text{ then the transcendence degree of the field generated by } \]
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So, our results for higher rank spaces are *conditional.*
Outline

1. Weak commensurability
   - Definition and motivations
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2. Length-commensurable locally symmetric spaces
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3. Proofs
   - “Special” elements in Zariski-dense subgroups
Theorem 12. Let $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ be locally symmetric spaces of finite volume. If they are length-commensurable then

1. either $G_1$ and $G_2$ are of the same Killing-Cartan type, or one of them is of type $B_n$ and the other is of type $C_n$;
2. $K_{\Gamma_1} = K_{\Gamma_2}$.
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- $K_{\Gamma_1} = K_{\Gamma_2}$.

Theorem 13. Let $\mathcal{X}_{\Gamma_1}$ be an arithmetically defined locally symmetric space. The set of arithmetically defined locally symmetric spaces $\mathcal{X}_{\Gamma_2}$ which are length-commensurable to $\mathcal{X}_{\Gamma_1}$, is a union of finitely many commensurability classes. It consists of a single commensurability class if $G_1$ and $G_2$ have the same type different from $A_n, D_{2n+1}$ with $n > 1$ and $E_6$. 
Corollary.

1. Let $d$ be even or $\equiv 3 \pmod{4}$, and let $M_1$ and $M_2$ be arithmetic quotients of the $d$-dimensional real hyperbolic space. If $M_1$ and $M_2$ are not commensurable, then (after a possible interchange of $M_1$ and $M_2$) there exists $\lambda_1 \in L(M_1)$ such that for any $\lambda_2 \in L(M_2)$, the ratio $\lambda_1 / \lambda_2$ is transcendental over $\mathbb{Q}$ (in particular, $M_1$ and $M_2$ are not length-commensurable.)

2. For any $d \equiv 1 \pmod{4}$ there exist length-commensurable, but not commensurable, arithmetic quotients of the real hyperbolic $d$-space.
Corollary.

1. Let $d$ be even or $\equiv 3 \pmod{4}$, and let $M_1$ and $M_2$ be arithmetic quotients of the $d$-dimensional real hyperbolic space.

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Theorem 14. Let $X_{\Gamma_1}$ and $X_{\Gamma_2}$ be locally symmetric spaces of finite volume which are length-commensurable. Assume that one of the spaces is arithmetically defined. Then

1. the other space is also arithmetically defined;
2. compactness of one of the spaces implies compactness of the other.
**Theorem 14.** Let $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ be locally symmetric spaces of finite volume which are length-commensurable. Assume that one of the spaces is **arithmetically defined**. Then

1. the other space is also **arithmetically defined**;
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- It would be interesting to find a **geometric** explanation of item 2°.
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- Is 2° remains valid without any assumptions on arithmeticaly?
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• Is 2° remains valid without any assumptions on arithmeticity?

RECALL that for any lattice $\Gamma$, compactness of $\mathcal{X}_\Gamma$ is equivalent to the existence of nontrivial unipotents in $\Gamma$. So, one can ask: Suppose two lattices are weakly commensurable. Does the existence of nontrivial unipotents in one of them implies their existence in the other? This question makes sense for arbitrary Zariski-dense subgroups.
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   - “Special” elements in Zariski-dense subgroups
Two compact Riemannian manifolds are **isospectral** if they have the **same spectra** of the Laplace-Beltrami operator (same **eigenvalues** and **same multiplicities**).
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**Fact.** Let $M_1$ and $M_2$ be two compact locally symmetric spaces. If $M_1$ and $M_2$ are isospectral then $L(M_1) = L(M_2)$. 
Two compact Riemannian manifolds are \textit{isospectral} if they have the same spectra of the Laplace-Beltrami operator (same \textit{eigenvalues} and same \textit{multiplicities}).

\textbf{Fact.} Let $M_1$ and $M_2$ be two compact locally symmetric spaces. If $M_1$ and $M_2$ are isospectral then $L(M_1) = L(M_2)$.

$\implies$ if $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_1}$ are compact and isospectral then $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.
Two compact Riemannian manifolds are **isospectral** if they have the **same spectra** of the Laplace-Beltrami operator (same *eigenvalues* and same *multiplicities*).

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If $M_1$ and $M_2$ are isospectral then $L(M_1) = L(M_2)$.

$\Rightarrow$ if $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_1}$ are **compact** and **isospectral** then $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.

**Theorem 15.** Let $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ be isospectral compact locally symmetric spaces. If $\Gamma_1$ is **arithmetic** then $\Gamma_2$ is also **arithmetic**.
Theorem 16. Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral compact locally symmetric spaces, and at least one of the subgroups $\Gamma_1$ or $\Gamma_2$ is arithmetic. Then $G_1 = G_2 =: G$. Moreover, unless $G$ is type $A_n$, $D_{2n+1}$ ($n > 1$) or $E_6$, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.
Theorem 16. Assume that $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are isospectral compact locally symmetric spaces, and at least one of the subgroups $\Gamma_1$ or $\Gamma_2$ is arithmetic. Then $G_1 = G_2 = G$. Moreover, unless $G$ is type $A_n$, $D_{2n+1}$ ($n > 1$) or $E_6$, the spaces $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are commensurable.

It would be interesting to determine if Theorem 16 remains valid without any assumptions of arithmeticity.
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Proofs rely on the existence of “special” elements in Zariski-dense subgroups.
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**Question 1:** Let $G$ be a compact Lie group, and let $\Gamma \subset G$ be a dense subgroup. Does there exist $\gamma \in \Gamma$ such that $\langle \gamma \rangle$ is a maximal torus of $G$?

**Question 2:** Let $G$ be a reductive algebraic group over a field $K$ (of characteristic zero), and let $\Gamma \subset G(K)$ be a Zariski-dense subgroup. Does there exist a semi-simple $\gamma \in \Gamma$ such that the Zariski closure $\langle \gamma \rangle$ is a maximal torus of $G$? Elements of this kind will be called generic (this notion will be specialized further later on).
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Elements of this kind will be called **generic** (this notion will be specialized further later on).
The answer is **No** to both questions if $G$ (resp., $G$) is a **torus**.
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**Example 1:** Let $G = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, and let

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\Gamma = (\sqrt{2}\mathbb{Z} + \mathbb{Z})/\mathbb{Z} \times (\sqrt{2}\mathbb{Z} + \mathbb{Z})/\mathbb{Z}.
$$

Then $\Gamma$ is dense in $G$, but for any

$$
\gamma = \left( \sqrt{2}m(\text{mod } \mathbb{Z}), \sqrt{2}n(\text{mod } \mathbb{Z}) \right) \in \Gamma
$$

we have $\langle \gamma \rangle \subset \{ (a(\text{mod } \mathbb{Z}), b(\text{mod } \mathbb{Z})) \mid na - mb \equiv 0(\text{mod } \mathbb{Z}) \}$, so $\langle \gamma \rangle \neq G$. 

**Example 2:** Let $G = C \times C \times C \times C$, and let $\varepsilon \in C \times C$ be **not** a root of unity.

Then $\Gamma = \langle \varepsilon \rangle \times \langle \varepsilon \rangle$ is Zariski-dense in $G$, but for any $\gamma = (\varepsilon^m, \varepsilon^n) \in \Gamma$, we have $\langle \gamma \rangle \subset \{ (x, y) \in G \mid x^n = y^m \}$, so $\langle \gamma \rangle \neq G$. 

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**Question 1** reducing to **Question 2** (b/c in compact groups, Zariski-dense subgroups are also dense in the usual topology), so we will focus on **Question 2**.
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**Example 3:** Let $G$ be a simple $\mathbb{Q}$-group with $\text{rk}_\mathbb{R} G = 1$. Then $\Gamma = G(\mathbb{Z})$ is Zariski-dense. Let $T \subset G$ be a maximal $\mathbb{Q}$-torus.
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T = T' \cdot T''
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(almost direct product), so \( T(\mathbb{Z}) \) is commensurable with \( T'(\mathbb{Z}) \cdot T''(\mathbb{Z}) \).
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(almost direct product), so $T(\mathbb{Z})$ is commensurable with $T'(\mathbb{Z}) \cdot T''(\mathbb{Z})$.

Thus, for any $\gamma \in T \cap \Gamma$, we have $\gamma^n \in T'$ or $T''$, and therefore $T \neq \langle \gamma \rangle$. 
In this example, $T$ can only be generated by a single element $\gamma \in T \cap \Gamma$ if it contains NO proper $\mathbb{Q}$-subtori.
In this example, $T$ can only be generated by a single element $\gamma \in T \cap \Gamma$ if it contains NO proper $\mathbb{Q}$-subtori.

Conversely, if $T$ is a $\mathbb{Q}$-torus without proper $\mathbb{Q}$-subtori then any $\gamma \in T(\mathbb{Q})$ of infinite order generates a Zariski-dense subgroup of $T$. 

\textbf{Definition.} Let $T$ be an algebraic torus defined over a field $K$. Then $T$ is $\mathbb{K}$-irreducible if it does not any proper $K$-defined subtori.

\textbf{Lemma 1.} If $T$ is irreducible over $K$ then for any $\gamma \in T(\mathbb{K})$ of infinite order, $\langle \gamma \rangle = T$. Thus, a regular semi-simple $\gamma \in \Gamma \subset G(\mathbb{K})$ is "generic" if $T = C_G(\gamma)$ is $\mathbb{K}$-irreducible.
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Thus, an element of infinite order $\gamma \in T(K)$, where $T$ is generic over $K$, is generic (as previously defined).
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Explicit construction can be implemented for other classical types.

Additional problem: embed resulting generic tori into a given group.
**General Case:**

**Fact (Voskresenskii)** There exists a purely transcendental extension $\mathcal{K} = K(x_1, \ldots, x_r)$ and a $\mathcal{K}$-defined maximal torus $T \subset G$ such that

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If $K$ is a number field (or, more generally, a finitely generated field) then one can use Hilbert’s Irreducibility Theorem to specialize parameters and get “many” maximal $K$-tori $T \subset G$ such that

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For \( K \) a number field, one can construct such generic tori with *prescribed local behavior* at finitely many places.

Then, if \( \Gamma \) is \( S \)-arithmetic, one can find generic tori containing \( \gamma \in \Gamma \) of infinite order.
Generic tori **constructed by this method** may not contain elements \( \gamma \in \Gamma \) of infinite order if \( \Gamma \) is not \( S \)-arithmetic.

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**Definition.** Let \( G \) be a semi-simple real algebraic group. An element \( \gamma \in G(\mathbb{R}) \) is **\( \mathbb{R} \)-regular** if the number of eigenvalues of \( \text{Ad} \gamma \), counted with multiplicities, of modulus 1, is minimal possible.
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**Theorem 17.** Let \( G \) be a connected semi-simple real algebraic group. Then any Zariski-dense subsemigroup \( \Gamma \subset G(\mathbb{R}) \) contain a regular \( \mathbb{R} \)-regular \( \gamma \) such that \( \langle \gamma \rangle \) is Zariski-dense in \( T = C_G(\gamma) \).
Theorem 18. Let $G$ be a semi-simple algebraic group over a field $K$ of characteristic zero, and let $\Gamma \subset G(K)$ be a Zariski-dense subgroup. Then there exists a regular semi-simple $\gamma \in \Gamma$ such that $\langle \gamma \rangle$ is Zariski-dense in $T = C_G(\gamma)^\circ$. 
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1. $\Gamma$ is finitely generated;
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**SKETCH OF PROOF** for $G$ almost absolutely simple simply connected. Can assume

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We want to construct a regular semi-simple $\gamma \in \Gamma$ of infinite order such that $T = C_G(\gamma)^\circ$ is *generic* over $K$. 
Proposition. Let $K$ be a finitely generated field, and $R \subset K$ be a finitely generated ring. There exists an infinite set of primes $\Pi$ such that for each $p \in \Pi$ there exists an embedding $\varepsilon: K \hookrightarrow \mathbb{Q}_p$ such that $\varepsilon_p(R) \subset \mathbb{Z}_p$.

- Pick a maximal $K$-torus $T_0 \subset G$ and fix a conjugacy class $C$ in $W(G, T_0)$.
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Observe that given maximal tori $T_1, T_2$ of $G$, the Weyl groups $W(G, T_1)$ and $W(G, T_2)$ are identified canonically, up to an inner automorphism; in particular, the conjugacy classes are identified canonically.

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Using Galois cohomology, we find an open $\Omega_p(C) \subset G(\mathbb{Q}_p)$ satisfying

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Pick $r$ primes $p_1, \ldots, p_r \in \Pi$, and consider $\Omega_{p_i}(C_i) \subset G(Q_{p_i})$.

One shows that
\[ \Omega := \bigcap_{i=1}^{r} (\Gamma \cap \Omega_{p_i}(C_i)) \neq \emptyset, \]
and any $\gamma \in \Omega$ is generic.
Some other applications of $p$-adic embeddings:

- **(Platonov)** Let $\pi : \tilde{G} \to G$ be a nontrivial isogeny of semi-simple groups over a finitely generated field $K$. Then $\pi(\tilde{G}(K)) \neq G(K)$.

- **(R.)** Let $\Gamma$ be a group with bounded generation, i.e.

  \[ \Gamma = \langle \gamma_1 \rangle \cdots \langle \gamma_d \rangle \quad \text{for some} \quad \gamma_1, \ldots, \gamma_d \in \Gamma. \]

  Assume that any subgroup of finite index $\Gamma_1 \subset \Gamma$ has finite abelianization $\Gamma_1^{ab} = \Gamma_1 / [\Gamma_1, \Gamma_1]$. Then there are only finitely many inequivalent irreducible representations $\rho : \Gamma \to \text{GL}_n(\mathbb{C})$.

- **(Prasad-R.)** Let $G$ be an absolutely almost simple algebraic group over a field $K$ of characteristic zero.

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