Non-commutative Field Theory with Twistor-like Coordinates

Tomasz Taylor

Northeastern University
LMS Durham 2007
Problem

Quantum field theories are singular at short distances ? presence of ultra-violet divergences
These are handled by renormalization (whenever it is possible), often leaving some unpleasant “naturalness” problems
Non-renormalizable theories like quantum gravity are even worse:
  infinite number of input parameters (UV counterterms) ? no predictive power
So it is a good thing to construct UV softer, or even finite theories - c.f. superstrings, N=8 SUGRA (???) etc.
One way to change the short-distance behavior is to change the “particle” content. SUSY does it by pairing scalars with fermions, superstring theory by upgrading point-like particles to extended objects like strings. What happens in N=8 SUGRA is still unclear…
A more radical and profound idea is to change spacetime, replacing space-time continuum by some discrete or fuzzy “medium”
Very important (and in some way generic) examples of fuzzy spacetimes are those with non-commuting position coordinates
Questions

• *Does spacetime non-commutativity improve short-distance behavior of QFT?*
• *Twistor spacetime is fuzzy – can one think of fuzzy twistors in terms of some non-commutative geometry?*
• *If yes, how does it affect QFT at short (and long) distances?*
Outline

I. Non-commuting coordinates
II. Non-commutative Field Theory on Moyal Plane
III. Twistor Theory Revisited
IV. Quantum Fields with Twistor–like Coordinates

arXiv:0704.2071 [hep-th]
Non-commuting Coordinates

Simplest and tractable example is the Groenewold-Moyal plane $R$ with

$$[x^1, x^0] = i \hbar^{10}$$

A radical step – the algebra of functions (fields) on $R$ is modified – the product is deformed to a star (Moyal) product:

$$(\hat{A}_1 \ast \hat{A}_2)(x) = e^{i \int \hbar^{10} \hat{A}_1(y) \hat{A}_2(z) j_{y=z=x}}$$

Interesting mathematics. Appears in open string theory in the presence of a constant $B$-field $B=T$ (Seiberg-Witten, ‘99). But is it physically sensible?
Non-commutative Field Theory

Is it physically sensible?

Non-commutative Lagrangians involve non-local interactions with star products

\[ R \int d^4x \left[ \frac{1}{2} (\partial \hat{A})^2 + \frac{1}{2} m^2 \hat{A}^2 + \frac{1}{4!} \hat{A}^4 \right] \]

Free Feynman propagators are not affected, but the perturbative interaction vertices are modified by the factors

\[ e^{i \sum_{i < j} C_{i j} k_i \cdot k_j} \quad (Minwalla et al., '99) \]

They affect UV and IR behavior of Feynman diagrams
Non-commutative Field Theory

Is it physically sensible and useful?

\[ \frac{\lambda}{12} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i p \cdot k} \Theta_{\mu \nu}}{k^2 + m^2} \]

from Rivasseau, ‘07

\[ \frac{\lambda}{48\pi^2} \sqrt{\frac{m^2}{(\Theta p)^2}} K_1(\sqrt{m^2(\Theta p)^2}) \approx p^{-2}. \]

UV-IR mixing: ?

8, p 0 limits’ order matters

Renormalizable $F^{**4}$, without UV/IR mixing, can be constructed by modifying quadratic terms (Grosse, Wulkenhaar ‘04)

In general, no significant improvement in UV – Feynman diagrams still have the same degree of divergences as commutative QFT

Moyal Plane non-commutativity isn’t too useful for improving short-distance behavior
Twistor Theory (Penrose ’67)

Twistors: Spinors representing null geodesics (light rays, world lines) in $M$?

Intersections Points

Notation (Penrose, Rindler ’86)

$$V^{AA^0} = \begin{pmatrix} V^{00^0} & V^{01^0} \\ V^{10^0} & V^{11^0} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 i & iV^2 V^0 i & V^3 \end{pmatrix}$$

momentum $p^{AA^0} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}$

$$L^{AA^0B^0} = i! (A_{1/4} 2A^{0B^0} i^2 AB \frac{1}{4} (A^{0_{1/4}} B^0_{1/4}))$$

? angular momentum
Twistor Theory

**Twistors**

\[ Z^\circ = (!^A; 1/\Lambda_0) \quad \circ = 1; 2; 3; 4 \]

\[ !^A = !^\pm A \quad i \times A A_{\Lambda_0}^{0 \pm} \quad 1/\Lambda_0 = i/\Lambda_0 \]

? reference point ( ? line) of angular momentum \( L \)

**Dual Twistors**

\[ \tilde{Z}^\circ = (1/\Lambda; !^A^0) \]

\[ p^0 = p^{1/2}_0 (Z^3 \tilde{Z}_1 + Z^2 \tilde{Z}_0); \ldots \]

\[ L^{01} = i \quad L^{10} = i/2 (Z^0 \tilde{Z}_0 i \quad Z^1 \tilde{Z}_1 i \quad Z^2 \tilde{Z}_2 + Z^3 \tilde{Z}_3); \ldots \]
Interpretation

“Angular” twistor

$$ A^\dagger = i^\dagger A \frac{i}{A^0} $$

“incidence relation”

Usually, one considers a fixed reference point, often setting $$ A^\dagger = 0 $$

Then one thinks about x as running along the light ray

$$ x^{A^0} = x^{A^0} + k^{\frac{1}{4}} \frac{1}{A^0} $$

Another viewpoint: consider a fixed light ray:

light ray

observer
By using 2 reference light rays, and measuring

\[ Z_a^\circ = (\frac{A}{a}; \frac{1}{4}A^0) \quad a = 1; 2 \]

the observer can determine his/her position:

\[ X^{AA^0} = \frac{i}{\frac{1}{4}B 0\frac{1}{2}B} \left[ (\frac{A}{1} i \frac{i}{1} i) \frac{1}{2}A^0 i (\frac{A}{2} i \frac{i}{2}) \frac{1}{4}A^0 \right] \]
Twistor Quantization

\[ [L; P^{1/2}] = \overset{\frac{1}{2}}{1} P_0 \quad i \quad \overset{0}{1} P_1 \]

canonical quantization of Poincaré algebra

\[ [Z^\circ; \dot{Z}^-] = \sim^{\circ} \]

\[ [\frac{1}{A}; ! B^0] = \sim^{B^0} \]

\[ [\frac{1}{A_0}; ! B^0] = \sim^{B^0} \]

Fuzzy space-time needs no quantum gravity?

What are the consequences of such non-commutativity?

n.b. units: \( ! = x^p \bar{p}; \frac{1}{4} = p \bar{p} \)

A commonly expressed view: Quantized metric  “fuzzy light cone”
More on (obvious) interpretation

\[
[!^A ; \frac{1}{^B}] = \sim \frac{A}{B} \quad [\frac{1}{^A_0} ; !^B_0] = \sim \frac{B_0}{A_0}
\]

\[\& \times ? \quad \& \, p? \quad , \quad \sim \frac{1}{2}\]

What about angular momenta w.r.t. different points?

\[
[!^A (x_1); !^A (x_2)] = i \sim (x_{2^A A_0} i x_{1^A A_0})
\]
Spacetime parameterized by non-commuting twistor-like position coordinates

\[ [!_{a}^{A}(x_{1}); !_{b}^{A^0}(x_{2})] = i \sim (x_{2}^{AA} \ i \ x_{1}^{AA^0}) \pm_{ab} \]

**Locally Commuting But**

**Non-locally Non-commuting**

Uncertainty grows with separation (like in a fog…)

Reference light rays

\[ \frac{1}{4} \quad \frac{1}{2} \]

Lorentz symmetry broken by time-like

\[ l_{A^0} = \frac{1}{4A} \frac{1}{4A^0} + \frac{1}{2A} \frac{1}{2A^0} \]

**SO(1,3) ? SO(3)**

\[ l = (\mu \ ; \ 0) \]

So it’s a…
Free Fields propagating in *FOGGY ÆTHER*,

\[
\tilde{A}(! ; ! ) = \int \frac{d^3p}{(2^{1/4})^3 2j\rho j} \quad a_p e^{i \frac{A}{a} \tilde{A}_A} \approx + a_p e^{i \frac{1}{a} \tilde{a}_A 0} ! A^0 a \approx
\]

SECOND QUANTIZATION:

\[
p_{AA^0} = \tilde{a}_{A}^{1/4} A^0 \quad [a_p; a_p^0] = \pm 3 (p_i \quad p^0)
\]

FEYNMAN PROPAGATOR:

\[
i D(x^0_i \quad x) = \hbar O j T (\tilde{A}(! 0 ; ! 0) \tilde{A}(! ; ! )) j 0 i
\]

NON-COMMUTATIVE (BAKER-HAUSDORFF)

\[
e^A e^B = e^{(A+B + \frac{1}{2} [A;B])}
\]

\[
e^{i \frac{A}{a} (x^0)! \tilde{a}_A (p^0) = \approx e^{i \frac{1}{A} \tilde{b}_0 (p) ! A^0 (x) = \approx
\]

\[
= e^{[i \frac{A}{a} (x^0)! \tilde{a}_A (p^0) + \frac{1}{A} \tilde{b}_0 (p) ! A^0 (x) + \frac{i}{2} (x_i \quad x^0) A^A A^0 ! \tilde{a}_A (p^0) \tilde{a}_A (p)] = \approx}
\]
Feynman Propagator

\[ iD(x^0, x) = R \frac{d^3 p}{(2\pi)^3} e^{i p \phi(x^0, x)} \left( (p+2l)^2 = (4^1 2 \sim) \mu(t^0, t) \right) \]

\[ + e^{i p \phi(x^0, x)} (p+2l)^2 = (4^1 2 \sim) \mu(t, t^0) \]

\[ = i R \frac{d^4 k}{(2\pi)^4} e^{i k (x^0, x)} \mathcal{D}(k) \]

\[ \mathcal{D}(k) = \frac{1}{k^2} \int p \frac{p^2}{j k j = 1 + 1 (j k j = 1 + 1 + 1)} \]

UV: \[ j k j \leq 1 \quad \mathcal{D}(k) \gg \frac{1}{k^3} \]

IR: \[ j k j \geq 1 \quad \mathcal{D}(k) \gg \frac{1}{k^2} \]

Above \( \mu \) non-commutativity scale

Standard propagator
Interacting Fields (very preliminary)

Coordinates are locally commuting? local interactions unchanged

UV behavior of Feynman diagrams determined by propagators

\[ \mathcal{Z} \int \frac{d^4k}{k^3} \]

Linear (instead of quadratic)
UV divergence

No UV/IR mixing

gauge theories in FOGGY ÆTHER: perturbatively finite?
Conclusions

• Non-local (foggy) non-commutativity in twistor space
• Determined by 2 constants: $h, \mu$
• Lorentz symmetry broken: preferred time direction, fundamental time unit $\zeta = \sim (\frac{1}{2} c^2)$ and rotational invariance in the Æther frame
• Assuming Æther = CMB ? $\mu > 10$ TeV
• Foggy spacetime tames UV divergences of QFT, no UV/IR mixing
• Many open questions: interacting QFT formalism, divergences, gravity,…

Research supported by the National Science Foundation grant PHY-0600304.
Opinions expressed are those of the author and do not necessarily reflect the views of NSF.