Heterotic Twistor-Strings
David Skinner, Oxford & Perimeter

Based on arXiv:0807.2276 with Lionel Mason
also
Katz & Sharpe hep-th/0406226; Witten hep-th/0504078;
Adams, Distler & Ernebjerg hep-th/0506263
and standard twistor-string papers
Why reformulate twistor-string theory?

There are a number of difficulties in understanding the twistor-string models of Witten and Berkovits, including...
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  - Berkovits: vertex operators on worldsheet boundary
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  - Effective action for D-instantons themselves?
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We’d like to understand these issues better, and also see how the Witten and Berkovits pictures are related.
Outline

(0,2) Basics
   Fields & action
   Vertex operators
   Anomalies

Heterotic String Theory
   Coupling to YM
   Amplitudes

Relation to other twistor-string models
   Berkovits
   Witten

Summary
Twisted (0,2) models

A theory of smooth maps $\Phi : \Sigma \to X$ from a closed, compact Riemann surface $\Sigma$ to a complex manifold $X$.

Fields are worldsheet scalars $(\phi^i, \phi^\jmath)$ and

$$\bar{\rho}^\jmath \in \Gamma(\Sigma, \phi^* T_X) \quad \rho^i \in \Gamma(\Sigma, K_\Sigma \otimes \phi^* T_X)$$
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Susy transformations are

$$\{ \overline{Q}, \phi^i \} = 0 \quad \quad \{ \overline{Q}, \phi^\bar{j} \} = \bar{\rho}^\bar{j}$$

$$\{ \overline{Q}, \rho^i \} = \overline{\partial} \phi^i \quad \quad \{ \overline{Q}, \bar{\rho}^\bar{j} \} = 0$$

and

$$\{ \overline{Q}^\dagger, \phi^i \} = \rho^i \quad \quad \{ \overline{Q}^\dagger, \phi^\bar{j} \} = 0$$

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$\bar{Q}$ acts on functions of $\phi, \phi^\bar{j}$ as the $\overline{\partial}$ operator on $\text{Maps}(\Sigma, X)$.
Action

The basic action is

\[ S_0 = t \int_{\Sigma} g(\overline{\partial}\phi, \partial\phi) - g(\rho, \nabla\overline{\rho}) + \int_{\Sigma} \phi^* \omega \]

\[ = t \left\{ \overline{Q}, \int_{\Sigma} g(\rho, \partial\phi) \right\} + \int_{\Sigma} \phi^* \omega \]

for \( t \in \mathbb{R}^+ \) and \( g \) a Hermitian (not pseudo-Hermitian) metric on \( X \) with \( \omega(X, Y) = g(X, JY) \)
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- Action is \( \overline{Q} \)-exact \( \Rightarrow \) partition function independent of \( t, g \)
- \( S_0 = -t|\overline{\partial}\phi|^2 + \text{fermions} \Rightarrow \) localize on holomorphic maps
- Manifestly invariant under \( \overline{Q} \); also invariant under \( \overline{Q}^\dagger \) if \( X \) is Kähler
- Can generalize by coupling to \( B \)-field: \( \partial\overline{\partial}\omega = 0 \) and \( \nabla \) has torsion determined by \( B \)
**Action**

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Coupling to a bundle

We can also couple in a holomorphic bundle $\mathcal{V} \rightarrow X$ by introducing

$$
\psi^a \in \Gamma(\Sigma, \phi^*\mathcal{V}) \quad \bar{\psi}_a \in \Gamma(\Sigma, K_{\Sigma} \otimes \phi^*\mathcal{V}^\vee)
$$

$$
r^a \in \Gamma(\Sigma, \overline{K}_{\Sigma} \otimes \phi^*\mathcal{V}) \quad \bar{r}_a \in \Gamma(\Sigma, K_{\Sigma} \otimes \phi^*\mathcal{V}^\vee)
$$

with susy transformations

$$\{Q, \psi^a\} = 0 \quad \{Q, \bar{\psi}_a\} = \bar{r}_a
$$

$$\{Q, r^a\} = \overline{D}\psi^a + F_{ij}{}^a{}_b \psi^b \rho^i \bar{\rho}^j \quad \{Q, \bar{r}_a\} = \bar{\partial}\bar{\psi}_a
$$

and action

$$
S_1 = \left\{ \overline{Q}, \int_{\Sigma} \bar{\psi}_a r^a \right\}
$$

$$
= \int_{\Sigma} \bar{\psi}_a \overline{D}\psi^a + F_{ij}{}^a{}_b \bar{\psi}_a \psi^b \rho^i \bar{\rho}^j + \bar{r}_a r^a
$$

Total action $S_0 + S_1$ is twisted version of heterotic string on general background
Twistor theory

We could choose $X = \mathbb{P}^{3|4}$, but

- Difficult to interpret bosonic worldsheet superpartners of fermionic target coordinates
- Not clear how to promote to string theory
- Can’t use D-brane to set $\bar{\psi} = 0$
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- Difficult to interpret bosonic worldsheet superpartners of fermionic target coordinates
- Not clear how to promote to string theory
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Instead, we’ll choose $X = \mathbb{P}^3$ and include the bundle $\mathcal{V} = \mathcal{O}(1) \oplus^4$

The advantages are

- $\psi$ is a worldsheet scalar, as it would be with $\mathbb{P}^{3|4}$ target, but $\bar{\psi}$ is a 1-form – naturally on different footing
- First-order action for worldsheet fermions
- Worldsheet superpartners are auxiliary
Sheaves of chiral algebras

The antiholomorphic stress tensor $T_{\bar{z}\bar{z}} = \{\bar{Q}, \bar{G}_{\bar{z}\bar{z}}\}$, so all the antiholomorphic Virasoro generators $\bar{L}_n$ are $\bar{Q}$-exact.

$[\bar{L}_0, \mathcal{O}] = \bar{h}\mathcal{O}$, but since $\bar{L}_0 = \{\bar{Q}, \bar{G}_0\}$ we find

$$\bar{h}\mathcal{O} = \{\bar{Q}, \mathcal{O}\} = \underbrace{\{\bar{Q}, [\bar{G}_0, \mathcal{O}]\}}_{\bar{Q}\text{-exact}} + \underbrace{\{[\bar{Q}, \mathcal{O}], \bar{G}_0\}}_{=0}$$

so $\bar{Q}$-cohomology is trivial except at $\bar{h} = 0$. 

In the A- or B-model, we'd similarly find $\mathcal{h} = 0$, but in a (0,2) model there is no holomorphic susy and all $\mathcal{h} \geq 0$ are allowed.

Vertex operators form "sheaf of chiral algebras" over target. (0,2) model is holomorphic (not topological) field theory.
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(0,2) moduli

Focus on operators with \((h, \bar{h}) = (1, 0)\) and ghost number \(+1\) (related to deformations of the (0,2) action via descent).
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\[
\mathcal{O}_M := g_{i\bar{k}} M^i \bar{\rho}^j \partial \phi^k \\
\mathcal{O}_b := b_{i\bar{j}} \bar{\rho}^j \partial \phi^i \\
\mathcal{O}_\mu := \mu^a_{\bar{j}} \bar{\rho}^j \bar{\psi}_a \\
\mathcal{O}_\beta := \beta_{a\bar{j}} \bar{\rho}^j \partial \psi^a
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- Non-trivial in \(\bar{Q}\)-cohomology if \([M] \in H^{0,1}(\mathbb{PT}', T_{\mathbb{PT}'})\), plus supersymmetric extensions.
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$$O_M := g_{i\bar{k}} M^i_j \bar{\rho}^j \partial \phi^\bar{k}$$

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- Non-trivial in $\bar{Q}$-cohomology if $[M] \in H^{0,1}(\mathbb{P}T', T_{\mathbb{P}T'})$, plus supersymmetric extensions.

- $b \rightarrow b + \partial \chi$ changes vertex operator by total derivative (upto $\rho$ eom) $\Rightarrow \mathcal{H} = \partial b$ nontrivial in $H^{0,1}(\mathbb{P}T', \Omega^2_{c1})$, plus super extension

$(0,2)$ moduli correspond to states of $\mathcal{N} = 4$ conformal supergravity under the Penrose transform
Anomalies

Sigma model anomaly unless

\[ \text{ch}_2(T_X) - \text{ch}_2(\mathcal{V}) = 0 \quad \text{c}_1(T_\Sigma)(\text{c}_1(T_X) - \text{c}_1(\mathcal{V})) = 0 \]
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Twistor-strings: \( c(T_{\mathbb{P}^3}) = c(O(1)^{\oplus 4}) \Rightarrow \text{no sigma model anomaly} \)
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Anomalies in global symmetries

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\text{ind}(\overline{\partial}_\phi^*T_{\mathbb{P}^3}) = 4d + 3(1 - g)
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for a map of degree \(d\), genus \(g\).
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\[ \text{ind}(\overline{\partial}_{\phi^*} \mathcal{O}(1)^{\oplus 4}) = 4(d + 1 - g) \]

for a map of degree \( d \), genus \( g \).

Amplitudes with \( n_h \) external SYM states of helicity \( h \) supported on maps of degree

\[ d = g - 1 + \sum_{h=-1}^{+1} \frac{h + 1}{2} n_h \]

Coefficient of \( (\psi)^{\text{top}} \) is a section of canonical bundle of instanton moduli space
Perturbative corrections

There are also perturbative corrections to the theory. (0,2) susy ensures that $\Delta \bar{T}_{\bar{z}z}$ and $\Delta T_{z\bar{z}}$ are $\bar{Q}$-exact, but there is no such statement for $T_{zz}$.

At one loop, correction to worldsheet action is

$$\Delta S^{1{-\text{loop}}} = \left\{ \bar{Q}, \int_\Sigma R_{i\bar{j}} \rho^i \partial \phi^{\bar{j}} + g^{i\bar{j}} F_{i\bar{j}}^a b \bar{\psi}_a r^b \right\}$$

- On $\mathbb{P}^3|4$ we have $R = 0$ and no bundle
- For $\mathbb{P}^3$ and bundle $\mathcal{O}(1) \oplus 4$ we have $R_{i\bar{j}} = 4 g_{i\bar{j}}$ and $F_{i\bar{j}}^a b = \delta^a_b g_{i\bar{j}}$ so the 1-loop correction is $\propto$ classical action.

The twistor model is a holomorphic CFT provided we study correlators of $\bar{Q}$-closed operators.
Holomorphic $bc$-system

Supercurrent $\overline{G}_{\overline{z}\overline{z}}$ plays role of $\overline{b}$-antighost

No left-moving susy, so need to include holomorphic $bc$-ghost system

$$S = \int_{\Sigma} b \overline{\partial} c \quad b \in \Gamma(\Sigma, K_{\Sigma} \otimes K_{\Sigma}) \ ; \ c \in \Gamma(\Sigma, T_{\Sigma})$$

- Provides holomorphic BRST operator $Q$
- $Q + \overline{Q}$ has complete descent chain
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▶ Provides holomorphic BRST operator $Q$
▶ $Q + \bar{Q}$ has complete descent chain
▶ Fixed vertex operators $\Rightarrow$ sigma-model vertex operators of $(h, \bar{h}) = (1, 0)$, contracted with $c$

Physical string states $\Leftrightarrow (0,2)$ moduli $\Leftrightarrow \mathcal{N} = 4$ conformal supergravity
Yang-Mills current algebra

In order for $Q^2 = 0$ we need to include a holomorphic current algebra contributing central charge $c = 28 (= 26 + 2 \times (4 - 3))$, as in both Berkovits’ and Witten’s models (see later . . .)

Could include further fermions $\lambda^\alpha \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* E)$ $\bar{\lambda}^\alpha \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* E^\vee)$ for some holomorphic bundle $E \to X$ (together with auxiliary superpartners).

Conformal invariance requires $c_1(E) = 0$

Freedom from sigma model anomalies requires $c_2(E) = 0$

$\Rightarrow$ $E$ corresponds to a zero-instanton spacetime bundle

Vertex operators $c^A_{\alpha \bar{\alpha}} \bar{\nu}^{\beta \bar{\lambda}} \alpha \lambda \beta$ $\Leftrightarrow$ External states in $N=4$ SYM
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Vertex operators $cA_j^{\alpha \beta} \bar{\lambda}_\alpha \lambda^\beta \leftrightarrow$ External states in $N = 4$ SYM
Yang-Mills instantons

Heterotic strings contain NS branes which couple magnetically to the NS $B$-field.

- Physical heterotic strings (10-manifold) → 5-branes
- Twisted heterotic strings (complex 3-fold) → 1-branes

\[ \text{Modified Green-Schwarz condition} \]

\[ \text{ch}^2(TX) - \text{ch}^2(V) - \text{ch}^2(E) + \sum_i [\text{NS}] i = 0 \]

⇒ instanton backgrounds allowed

\[ \text{e.g.} \ 't \text{Hooft} SU(2)^k \text{-instanton} \]

\[ A(x) = i \sum_\mu \sigma_{\mu\nu} \partial^\nu \log \Phi, \quad \Phi(x) = k \sum_i \lambda_i (x - x_i)^2 \]

wrap NS branes on the $k+1$ lines in twistor space corresponding to the $x_i$'s.
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Modified Green-Schwarz condition

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$\Rightarrow$ instanton backgrounds allowed

e.g. 't Hooft $SU(2) k$-instanton

$$A(x) = i \, dx^\mu \sigma_{\mu\nu} \partial^\nu \log \Phi, \quad \Phi(x) = \sum_{i=0}^{k} \frac{\lambda_i}{(x - x_i)^2}$$

wrap NS branes on the $k + 1$ lines in twistor space corresponding to the $x_i$s.
### A puzzle

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- Change level of current algebra?
- Include additional fields contributing to $c$?
- Promote to string theory by some other means than $bc$-system?

Clear that modular invariance is key test.
Amplitudes and contours

Choose basis of Beltrami differentials $\mu$ and compute

$$\left\langle 3g-3+n \prod_{i=1}^{3g-3+n} (\mu^{(i)}, b)(\overline{\mu}^{(i)}, \overline{G}) \prod_{j=1}^{n} O_j \right\rangle$$

where $O_j$ are fixed vertex operators.

- $bc$-ghost number anomaly absorbed by $(\mu, b)$ and vertex operators
- $U(1)_R$ anomaly is $3(1 - g) + 4d$. Remaining anomaly of $4d = \text{vdim}_{\mathbb{C}} \overline{\mathcal{M}}_{g,0}(\mathbb{P}^3, d)$
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► $U(1)_R$ anomaly is $3(1-g) + 4d$. Remaining anomaly of $4d = \text{vdim}_\mathbb{C} \overline{\mathcal{M}}_{g,0}(\mathbb{P}^3, d)$

Integrand is effectively a $(4d, 0)$ form on moduli space of stable maps ⇒ contour integral.

► Absorb anomaly by inserting Poincaré dual into path integral, soaking up remaining $\bar{\rho}$ zero-modes (Dolbeault picture).

► Choice of contour $\Leftrightarrow$ choice of spacetime signature

► Leading-trace SYM amplitudes agree with Witten’s & Berkovits’ models. Sub-leading trace $= \text{cSUGRA}$ (by unitarity)
Instanton corrections and twistor actions

At degree $d$, the heterotic generating function for amplitudes in $\mathcal{N} = 4$ csugra + SYM is

$$
\int_{\mathcal{M}_{g,d}} d\mu \ exp \left( \frac{-A(C)}{2\pi} + i \int_C B \right) \frac{\det \partial E \otimes S_-}{\det' \partial N_{C|\text{PT}_s}}
$$

(\star)

- $\mathcal{M}_{g,d}$ is contour in space of genus $g$, degree $d$ curves, measure $d\mu$ ($= d^{4|8}$ at $g = 0, d = 1$)
- $A(C) =$ area of curve $C$ (from the restriction of the Kähler form)
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In compactifications on $\mathcal{CY} \times \mathbb{R}^4$, $(\star)$ describes instanton corrections to $4d$ superpotential.

Here, the $d = 1$ contribution can be used together with the Chern-Simons ($d = 0$ term) as a twistor action.
Berkovits’ model I

On contractible open patch $U \subset P{T}$

- Action becomes free
- $H^p(U, S) = 0$ for $p > 0 \Rightarrow$ Vertex ops independent of $\bar{\rho}$
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Cover target with patches, each supporting free $\beta\gamma$ system

- Anomaly conditions arise from consistency in gluing
- Higher vertex operators described by Čech cohomology
Berkovits’ model II

Equivalently, work on non-projective space

\[ S = \int_{\Sigma} Y_I \overline{D} Z^I \quad I = (\alpha|a) = (1, \ldots, 4|1, \ldots, 4) \]

with \( \overline{D} = \overline{\partial} + A \) a \( GL(1, \mathbb{C}) \) connection. To recover previous description: integrate out \( A \Rightarrow Y_I Z^I = 0 \) and solve on patches \( Z^\alpha \neq 0 \).
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- Introduce holomorphic \( bc \)-system and current algebra as before
- Path integral only involves holomorphic \( Z \)s \( \Rightarrow \) contour still needed

Given antiholomorphic involutions on \( \Sigma \) and \( \mathbb{P}^3 \), perform orientifold projection. \(+ + --\) orientifolded theory \( \Leftrightarrow \) “open string theory” on \( \Sigma' \) with action

\[ S = \int_{\Sigma'} Y_\alpha \overline{D} Z^\alpha + \overline{Y}_\tilde{\alpha} D Z^{\tilde{\alpha}} + b \overline{\partial} c + \overline{b} \partial \overline{c} + S_{YM} \]

where \( Z(\partial \Sigma') \subset \mathbb{RP}^3 \), and \( Z^\alpha|_{\partial \Sigma'} = \overline{Z}^{\tilde{\alpha}}|_{\partial \Sigma'} \) etc.
Witten’s model

The D1-D5 strings in Witten’s B-model give a factor
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There are also D1-D1 strings. On the worldvolume of a single D1-brane wrapping $C$, their effective action is

$$\int_C \Phi_1 \overline{\partial} \Phi_0 ; \quad \Phi_0 \in \Gamma(C, N_C|_{\text{PT}_s}) , \quad \Phi_1 \in \Gamma(C, K_C \otimes N_C^\vee|_{\text{PT}_s}) .$$

(from dimen. reduc. of Chern-Simons)

$\Rightarrow \ 1/ \det \overline{\partial}_{N_C|_{\text{PT}_s}}$
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Finally, WB & NOV propose D1-branes themselves \( \Rightarrow \exp(A(C)/2\pi + i \int_C B) \) (electric source for \( B + i \omega \))
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Combining these ingredients gives exactly the same contribution as the heterotic worldsheet instantons.
Conclusions & Outlook

We’ve given a construction of twistor-string theory as a heterotic string.

- Entire D1/D5 system in B-model equivalent to heterotic string
  - Should generalize to non-pert. top. str. on standard CY
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  - Derivation of RSV? Connected/disconnected equivalence?
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- Penrose transform complete action
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  - Derivation of RSV? Connected/disconnected equivalence?
- Replace $\mathcal{O}(1)^{\oplus 4}$ by another bundle?
- Poincaré supergravity? Pure SYM? Phenomenology?