Integrability in $\mathcal{N} = 4$ SYM: an overview

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SYM with 16 supercharges – conformal field theory in $d = 4$

Textbook description:
- anomalous dimensions of gauge invariant operators
- 3-point functions
- sewing relations

- spectral problem of $\mathcal{N} = 4$ SYM: find all anomalous dimensions

- $\mathcal{O}_n$ “good” operators – diagonal RG flow
  \[ \langle \mathcal{O}_n(0) \mathcal{O}_m(x) \rangle = \frac{\delta_{mn}}{|x|^{2\Delta_n}} \]
  available tricks for special “good” operators

- $\mathcal{O}_n$ ”natural” operators ⇒ operator mixing – \( \langle \mathcal{O}_n(0) \mathcal{O}_m(x) \rangle \neq \delta_{mn} \)
  “good” operators = combinations of “natural” operators

- Clean formulation in terms of dilatation generator ($\mathcal{D} \in PSU(2,2|4)$)
  \[ \mathcal{D} \mathcal{O}_n = \sum_m \gamma_{nm} \mathcal{O}_m \quad \rightarrow \quad \mathcal{D} \mathcal{O}_n = \Delta_n \mathcal{O}_n \]

- Further simplification: restrict to single-trace $\mathcal{O}_n$ in ’t Hooft limit
Known explicit higher loop anomalous dimensions in $\mathcal{N} = 4$ SYM:

- BMN operators (2-loops)  \cite{Gross, Mikhailov, RR}
- single-magnon dispersion relation (all loops)  \cite{Gross, Mikhailov, RR; Santambrogio, Zanon}
- 2- and 3-loop cusp anomalous dimension  \cite{Kotikov, Lipatov, Onishchenko, Velizhanin; Bern, Dixon, Smirnov}
- 4-loop cusp anom. dim. tour de force  \cite{Bern, Czakon, Dixon, Kosower, Smirnov; Cachazo, Spradlin, Volovich}
  – further improvements

Strong coupling – from AdS/CFT

- Operators with large quantum numbers  \cite{Frolov, Tseytlin; Kruczensky, Ryzhov, Tseytlin;}
- Cusp/folded string (LO, NLO, NNLO)  \cite{Gubser, Klebanov, Polyakov; Frolov, Tseytlin; Kruczensky; Frolov, Tirziu, Tseytlin; RR, Tirziu, Tseytlin}
Outline

- What is integrability and why do we believe in it?
- Hamiltonian vs. (2-dimensional) S-matrix: 1, 2, 3, 4, ..., \infty?
- The idea of the Bethe ansatz
- BES vs direct calculations
- Is this the end?
- Outlook and open problems
What is integrability . . .

Wikipedia: “In Hamiltonian mechanics, an integrable system refers to a Hamiltonian system that has constants of motion other than the energy itself. A completely integrable system is a system that has \( n \) degrees of freedom, \( n \) constants of motion, and whose constants of motion are in involution: that is, the Poisson bracket between each pair of constants of motion vanishes.”

- spectrum – degeneracies unexplained by lowest conservation laws
- no algorithm to identify integrable Hamiltonians
Sasha Migdal: “... but why do you believe in integrability for $\mathcal{N} = 4$ SYM?”

Matthias Staudacher: “I guess unlimited optimism :)

12th Itzykson meeting, Paris 2007
From weak coupling end:

- low order evolution operators define integrable $H$ 
  Lipatov; Minahan, Zarembo; Beisert, Staudacher

- tree-level Yangian consistent with 1-loop dilatation operator 
  Dolan, Nappi, Witten

- higher-loop Yangian 
  Agarwal, Rajeev; Zwiebel

- degenerate pairs and higher multiplets 
  Beisert, Kristjansen, Staudacher

From strong coupling end:

- classical $AdS$ string – infinitely many integrals of motion 
  Bena, Polchinski, RR

- seem to survive quantization 
  Berkovits

- structures characteristic to integrable systems in quantum strings 
  – classical spectrum on cylinder 
  Kazakov, Marshakov, Minahan, Zarembo; Beisert, Kazakov, Sakai, Zarembo
Hamiltonian vs. (2-dimensional) S-matrix

\[ \mathcal{D} O_n = \sum_m \gamma_{nm} O_m \quad \rightarrow \quad \mathcal{D} O_n = \Delta_n O_n \]

- Interpret as a Schrödinger equation with $\mathcal{D}$ as Hamiltonian; $\gamma$ is its matrix representation in the basis $\{O_n\}$

- Need vacuum; choose scalar $1/2$–BPS operator $\text{Tr} [Z^J] \ J \rightarrow \infty$

Excitations:
- replace vacuum field $Z$ by one of remaining fields
- enforce gauge invariance

\[ \rightarrow \quad 4 \text{ real scalars} \]

\[ 4 \text{ fermions} \]

arbitrary covariant derivatives of vacuum fields

Berenstein, Maldacena, Nastase
Hamiltonian vs. (2-dimensional) S-matrix

\[ DO_n = \sum_m \gamma_{nm} O_m \rightarrow DO_n = \Delta_n O_n \]

- Interpret as a Schrödinger equation with \( D \) as Hamiltonian; \( \gamma \) is its matrix representation in the basis \( \{O_n\} \)

- Need vacuum; choose scalar 1/2–BPS operator

- \( \gamma \) has block-diagonal structure due to charge conservation

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\phi^1 & \phi^2 & Z & A_\mu & \psi^1 & \psi^2 & \psi^3 & \psi^4 & Q^1 & Q^2 & Q^3 & Q^4 \\
\hline
J_{12} & 1 & 0 & 0 & 0 & 1/2 & -1/2 & -1/2 & 1/2 & -1/2 & 1/2 & -1/2 \\
J_{34} & 0 & 1 & 0 & 0 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\
J_{56} & 0 & 0 & 1 & 0 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & -1/2 \\
\end{array}
\]

- \( SU(2) \) sector
- \( SU(1|1) \) sector
- \( SU(n|m) \) sector
- \( SL(2) \)

\( \text{Beisert – complete classification} \)

- vacuum \( \rightarrow \) \( \text{Tr} [Z^J], J \rightarrow \infty \)

- 't Hooft limit \( \rightarrow \) fields are ordered \( \rightarrow \) lattice
How does one find $\gamma$?

V1: from renormalization factors

$$\gamma = \lim_{\epsilon \to 0} \epsilon Z^{-1} \frac{dZ}{d \ln g_{YM}}$$

V2: from 2-point function of basis elements

V3: twist-2 operators – from scattering amplitudes/AP kernel

V4: algebraically

- Constraints on the $L$-loop Hamiltonian
  - global symmetries and structure of Feynman diagrams
  - vacuum energy and the energy of other BPS states
  - results of explicit calculations
  - allow/introduce terms not affecting the eigenvalues
    - similarity transformations: $\mathcal{H}' = U^{-1}\mathcal{H}U$ & $U^\dagger \neq U^{-1}$
- **Existing results:**
  - 5-loop $SU(2)$ assuming integrability, smooth continuum limit  
    Beisert, Dippel, Staudacher
  - 2-loop $SL(2)$ (w/o); larger sectors (w/)
  - 4-loop $SU(2)$; no assumptions  
    Beisert, McLoughlin, RR

$$H_0 = \{\}$$

$$H_1 = +2\{\} - 2\{1\}$$

$$H_2 = -8\{\} + 12\{1\} - 2\{1,2\} + \{2,1\}$$

$$H_3 = +60\{\} - 104\{1\} + 4\{1,3\} + 24\{1,2\} + \{2,1\}$$

$$- 4i\epsilon_2\{1,3,2\} + 4i\epsilon_2\{2,1,3\} - 4\{1,2,3\} + \{3,2,1\}$$

$$H_4 = +\left(-560 - 4\beta_{2,3}\right)\{\}$$

$$+ \left(+1072 + 12\beta_{2,3} + 8\epsilon_3\right)\{1\}$$

$$+ \left(-84 - 6\beta_{2,3} - 4\epsilon_3\right)\{1,3\}$$

$$- 4\{1,4\}$$

$$+ \left(-302 - 4\beta_{2,3} - 8\epsilon_3\right)\{1,2\} + \{2,1\}$$

$$+ \left(+4\beta_{2,3} + 4\epsilon_3 + 2i\epsilon_3 - 4i\epsilon_3\right)\{1,3,2\}$$

$$+ \left(+4\beta_{2,3} + 4\epsilon_3 - 2i\epsilon_3 + 4i\epsilon_3\right)\{2,1,3\}$$

$$+ \left(4 - 2i\epsilon_3\right)\{1,2,4\} + \{1,4,3\}$$

$$+ \left(4 + 2i\epsilon_3\right)\{1,3,4\} + \{2,1,4\}$$

$$+ \left(+96 + 4\epsilon_3\right)\{1,2,3\} + \{3,2,1\}$$

$$+ \left(-12 - 2\beta_{2,3} - 4\epsilon_3\right)\{2,1,3,2\}$$

$$+ \left(+18 + 4\epsilon_3\right)\{1,3,2,4\} + \{2,1,4,3\}$$

$$+ \left(-8 - 2\epsilon_3 - 2i\epsilon_3\right)\{1,2,4,3\} + \{1,4,3,2\}$$

$$+ \left(-8 - 2\epsilon_3 + 2i\epsilon_3\right)\{2,1,3,4\} + \{3,2,1,4\}$$

$$- 10\{1,2,3,4\} + \{4,3,2,1\}$$

- $\{...,bac\} = \ldots P_{b,b+1}P_{a,a+1}P_{c,c+1}$
- $\beta =$undetermined; directly computable
- $\epsilon =$similarity parameters

$$H \rightarrow U(\epsilon)^{-1}HU(\epsilon)$$

Higher-loop $H$ is computable in principle; Direct calculation of the all-loop $H$ appears impractical; partial calculations contain nontrivial information
Eigenvectors and eigenvalues: illustrate on $SU(2)$ sector

- **2-particle states:** $\mathcal{H}|\Psi\rangle = E|\Psi\rangle$

$$|\Psi\rangle = \sum_{1 \leq l_1 < l_2 \leq J} \psi(l_1, l_2) |\ldots Z\phi Z \ldots Z\phi Z \ldots\rangle$$

- If $l_1 - l_2 \geq L + 1$ then $\psi(l_1, l_2) = e^{i(l_1p_1 + l_2p_2)}$

- If $l_1 - l_2 \leq L$ details of $\mathcal{H}$ matter;

- **Bethe ansatz – asymptotic plane waves**

$$\psi(l_1, l_2) = e^{i(l_1p_1 + l_2p_2)} + S(p_1, p_2, \lambda) e^{i(l_1p_2 + l_2p_1)}$$

- We may think of $S(p_1, p_2, \lambda)$ as a $(1 + 1)$-dimensional S-matrix

- **Spectrum from periodicity of wave function**

$$e^{ip_1J} = S(p_1, p_2, \lambda) \quad e^{ip_2J} = S(p_2, p_1, \lambda)$$
Eigenvectors and eigenvalues: illustrate on $SU(2)$ sector

- **2-particle states:** $\mathcal{H}\ket{\psi} = E\ket{\psi}$

\[
\ket{\psi} = \sum_{1 \leq l_2 < l_2 \leq J} \psi(l_1, l_2)\ket{\ldots Z\phi Z \ldots Z\phi Z \ldots}
\]

- If $l_1 - l_2 \geq L + 1$ then $\psi(l_1, l_2) = e^{i(l_1 p_1 + l_2 p_2)}$
- If $l_1 - l_2 \leq L$ details of $\mathcal{H}$ matter;

For example, 1-loop $SU(2)$ sector $\mathcal{H}_1 = 1 - \mathcal{P}$

\[
\psi(l_1, l_2) = e^{i(l_1 p_1 + l_2 p_2)} + S(p_1, p_2, \lambda) e^{i(l_1 p_2 + l_2 p_1)}
\]

\[
E\psi(l_1, l_2) = 2\psi(l_1, l_2) - \psi(l_1 - 1, l_2) - \psi(l_1, l_2 + 1)
\]

\[
S(p_1, p_2) = \frac{e^{i(p_1 + p_2)} - 2e^{ip_1} + 1}{e^{i(p_1 + p_2)} - 2e^{ip_2} + 1}
\]

\[
E = E(p_1) + E(p_2)
\]

\[
E(p) = 4\sin^2\frac{p}{2}
\]
Integrability = factorization $\iff$ Yang-Baxter equation

$$S(p_1, p_2)S(p_1, p_3)S(p_2, p_3) = S(p_2, p_3)S(p_1, p_3)S(p_1, p_2)$$

- multi-particle S-matrix is a sequence of 2-particle S-matrices
- particle momenta are separately conserved ($\exists$ higher IoM)

Wave function periodicity yields spectrum (Bethe equations)

$$e^{ip_k J} = \prod_{j \neq k=1}^M S(p_k, p_j, \lambda) \quad E = \sum_{k=1}^M E(p_k, \lambda)$$

Larger sectors

- Nested Bethe ansatz:
  - spectrum from periodicity
  - kind of row reduction
  - repeat 1d Bethe ansatz with a different choice of vacuum at each stage

$\diamond$ The message: it suffices to know $S(p_1, p_2, \lambda)^{kl}_{ij}$ with $i, j, k, l \in (4|4)$
Constraints on the S-matrix from \( \mathcal{N} = 4 \) SYM

- symmetries preserved by the choice of vacuum (\( PSU(2|2)^2 \))

  \(!\) symmetry algebra: extended and w/ quantum corrections

\[
[R^a_b, J^c] = \delta^c_b J^a - \frac{1}{2} \delta^a_b J^c, \quad Q^\alpha_a |\phi^b\rangle = a(p, \lambda) \delta^b_a |\psi^\alpha\rangle,
\]
\[
[L^\alpha_\beta, J^\gamma] = \delta^\gamma_\beta J^\alpha - \frac{1}{2} \delta^\alpha_\beta J^\gamma, \quad Q^\alpha_a |\psi^\beta\rangle = b(p, \lambda) \varepsilon^{\alpha\beta} \varepsilon_{ab} |\phi^b Z^+\rangle,
\]
\[
\{Q^\alpha_a, S^b_{\beta}\} = \delta^b_a L^\alpha_\beta + \delta^\alpha_\beta R^b_a + \delta^b_\alpha \delta^\alpha_\beta C, \quad S^a_\alpha |\phi^b\rangle = c(p, \lambda) \varepsilon^{ab} \varepsilon_{\alpha\beta} |\psi^\beta Z^-\rangle,
\]
\[
\{Q^\alpha_a, Q^\beta_b\} = \varepsilon^{\alpha\beta} \varepsilon_{ab} \mathcal{P}, \quad S^a_\alpha |\psi^\beta\rangle = d(p, \lambda) \delta^\beta_a |\phi^a\rangle
\]

\(!\) symmetry may change the vacuum charge

- unusual feature: scattering may change the vacuum charge

- factorization (à la Yang-Baxter)

- “algebraic crossing”

  \(!\) only indirect string theory justification

  \(!\) enhancement of \( psu(2|2) \) to a Hopf algebra
\[ S_{12} | \phi_1^a \phi_2^b \rangle = A_{12} | \phi_2^{(a} \phi_1^{b)} \rangle + B_{12} | \phi_2^{[a} \phi_1^{b]} \rangle + \frac{1}{2} C_{12} \varepsilon^{ab} \varepsilon_{\alpha \beta} | \psi_2^a \psi_1^b Z^- \rangle, \]
\[ S_{12} | \psi_1^a \psi_2^b \rangle = D_{12} | \psi_2^{(a} \psi_1^{b)} \rangle + E_{12} | \psi_2^{[a} \psi_1^{b]} \rangle + \frac{1}{2} F_{12} \varepsilon^{\alpha \beta} \varepsilon_{ab} | \phi_2^a \phi_1^b Z^+ \rangle, \]
\[ S_{12} | \phi_1^a \psi_2^b \rangle = G_{12} | \psi_2^a \phi_1^b \rangle + H_{12} | \phi_2^a \psi_1^b \rangle, \]
\[ S_{12} | \psi_1^a \phi_2^b \rangle = K_{12} | \psi_2^a \phi_1^b \rangle + L_{12} | \phi_2^a \psi_1^b \rangle. \]

\[ S_{0}(x_1, x_2) = \frac{1 - g^2/2x_k^+ x_l^-}{1 - g^2/2x_k^- x_l^+} \sigma_{kl}^2 \]

\[ \sigma_{kl} = e^{i \theta_{kl}} - \text{universal} \]

\[ \neq 0 \text{ at } \lambda \rightarrow \infty \]

\[ = \mathcal{O}(\lambda^4) \text{ at } \lambda \rightarrow 0 \]

\[ x^+ + \frac{g^2}{2x^+} - x^- - \frac{g^2}{2x^-} = i \]
Asymptotic all-loop Bethe equations

\[
1 = \frac{K_2}{\prod_{j=1}^{u_2,k - u_2,j + i} K_3}{\prod_{j=1}^{u_2,k - u_2,j - i}} x_3,k - x_4,j, \quad \prod_{j=1}^{u_3,k - u_2,j - i} x_3,k - x_4,j
\]

\[
1 = \frac{K_2}{\prod_{j=1}^{u_3,k - u_2,j + i}} x_3,k - x_4,j, \quad \prod_{j=1}^{u_3,k - u_2,j - i} x_3,k - x_4,j
\]

\[
1 = \left( \frac{x_4,k}{x_4,k} \right)^L \prod_{j=1}^{u_4,k - u_4,j + i} x_4,k - x_4,j - i \quad \prod_{j=1}^{u_4,k - u_4,j - i} x_4,k - x_4,j
\]

\[
1 = \frac{K_6}{\prod_{j=1}^{u_5,k - u_6,j + i}} x_5,k - x_4,j, \quad \prod_{j=1}^{u_5,k - u_6,j - i} x_5,k - x_4,j
\]

\[
1 = \frac{K_6}{\prod_{j=1}^{u_6,k - u_6,j + i}} x_6,k - x_4,j, \quad \prod_{j=1}^{u_6,k - u_6,j - i} x_6,k - x_4,j
\]

\[
E(g) = 2 \sum_{j=1}^{K_4} \left( \frac{i}{x_4,j} - \frac{i}{x_4,j} \right), \quad \Delta = \Delta_0 + g^2 E(g).
\]

\[
1 = \prod_{j=1}^{K_4} \left( \frac{x_4,j}{x_4,j} \right), \quad u_k = x_k + \frac{g^2}{x_k}, \quad u_k \pm \frac{i}{2} = x_k \pm \frac{g^2}{x_k}.
\]
The phase:

- Universal to all sectors; general structure: \[ \theta(u_1, u_2) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g) \left[ q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1) \right] \]

\[ q_{r,k} = \frac{1}{r-1} \left( \frac{i}{(x_+^k)^{r-1}} - \frac{i}{(x_-^k)^{r-1}} \right) \quad g = \frac{\sqrt{\lambda}}{\pi} \]

- various suggestions for the origin of the phase:
  - relativistic 2d sigma model
  - fill to physical vacuum
  - constrained by crossing at strong coupling
  - morally similar to \( S^{p,q}(i\pi - \theta) = C^p S^{\bar{p},q}(\theta) C^p \)
  - for AdS: \( \sigma_{12}\sigma_{\bar{1}\bar{2}} = f_{12} \) but \( f_{12} \neq f_{\bar{1}\bar{2}} \)

  \[ \text{solution: compatibility of algebra and } C: 1 \mapsto \bar{1} \mapsto \bar{\bar{1}} \neq 1 \]
The phase:

- Universal to all sectors; general structure: Arutyunov, Frolov, Staudacher; Beisert, Klose; Beisert, Tseytlin

\( \theta(u_1,u_2) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g) \left[ q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1) \right] \)

- strong coupling: \( c_{r,s}(g) = g^{2-r-s} \beta_{r,s}(g) = \sum_n c_{r,s}^{(n)} g^{1-n} \)

\[
c_{r,s}^{(n)} = \frac{(1 - (-)^{r+s})(r - 1)(s - 1) \Gamma(\frac{1}{2}(s + r + n - 3)) \Gamma(\frac{1}{2}(s - r + n - 1))}{2(-2\pi)^n \Gamma(n - 1) \Gamma(\frac{1}{2}(s + r - n + 1)) \Gamma(\frac{1}{2}(s - r - n + 3))} \]

! not unique Beisert, Hernandez, Lopez

- weak coupling: \( \beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu} \beta_{r,r+1+2\nu}^{(r+\nu+\mu)} \)

- analytic continuation from strong coupling Beisert, Eden, Staudacher
An inspired guess?

\[ f(g) = -\frac{1/g}{1 - 1/g} = \frac{1}{1 - g} \]

- Series expansion:
  
  \[
  \begin{array}{c|c}
  g \to \infty & g \to 0 \\
  \hline
  f(g) = \sum b_n g^{-n} \quad (b_n = -1) & f(g) = \sum a_n g^{+n} \quad (a_n = +1)
  \end{array}
  \]

◊ Analytic continuation “rule”: \( a_n = -b_{-n} \) 

\[
c_{rs}(g) = \sum_n c_{r,s}^{(n)} g^{1-n} \mapsto -\sum_n c_{r,s}^{(-n)} g^{1+n}
\]

Various appealing features:

- only integer powers of \( \lambda \)
- first nonzero contribution at 4 loops
- Lipatov’s transcendentality
- expected radius of convergence

\[
c_{23}/(c_{23} + 1) \text{ vs. } \sqrt{\lambda}/(\sqrt{\lambda} + \pi)
\]
The phase:

- **Universal to all sectors; general structure:**

\[
\theta(u_1,u_2) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g) \left[ q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1) \right]
\]

- **weak coupling:**

\[
\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu} \beta_{r,r+1+2\nu}(r+\nu+\mu)
\]

- **analytic continuation from strong coupling — several possibilities:**

<table>
<thead>
<tr>
<th>(L)</th>
<th>no (\zeta(2n+1))</th>
<th>with (\zeta(2n+1))</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>(\beta_{2,3}^{(3)} = 2\zeta(3))</td>
<td>(\beta_{2,3}^{(3)} = 4\zeta(3))</td>
</tr>
<tr>
<td>5</td>
<td>(\beta_{2,3}^{(4)} = -20\zeta(5))</td>
<td>(\beta_{2,3}^{(4)} = -40\zeta(5))</td>
</tr>
<tr>
<td>6</td>
<td>(\beta_{2,3}^{(5)} = 210\zeta(7),) (\beta_{3,4}^{(5)} = 12\zeta(5),) (\beta_{2,5}^{(5)} = -4\zeta(5))</td>
<td>(\beta_{2,3}^{(5)} = 420\zeta(7),) (\beta_{3,4}^{(5)} = 24\zeta(5),) (\beta_{2,5}^{(5)} = -8\zeta(5))</td>
</tr>
</tbody>
</table>
phase coefficients may be identified at the level of the Hamiltonian

\[ \mathcal{H}_4 = + \left( -560 - 4 \beta_{2,3} \right) \{ \} \\
+ \left( +1072 + 12 \beta_{2,3} + 8 \epsilon_{3a} \right) \{ 1 \} \\
+ \left( -84 - 6 \beta_{2,3} - 4 \epsilon_{3a} \right) \{ 1, 3 \} \\
- 4 \{ 1, 4 \} \\
+ \left( -302 - 4 \beta_{2,3} - 8 \epsilon_{3a} \right) \{ 1, 2 \} + \{ 2, 1 \} \\
+ \left( +4 \beta_{2,3} + 4 \epsilon_{3a} + 2 i \epsilon_{3c} - 4 i \epsilon_{3d} \right) \{ 1, 3, 2 \} \\
+ \left( +4 \beta_{2,3} + 4 \epsilon_{3a} - 2 i \epsilon_{3c} + 4 i \epsilon_{3d} \right) \{ 2, 1, 3 \} \\
+ \left( 4 - 2 i \epsilon_{3c} \right) \{ 1, 2, 4 \} + \{ 1, 4, 3 \} \\
+ \left( 4 + 2 i \epsilon_{3c} \right) \{ 1, 3, 4 \} + \{ 2, 1, 4 \} \\
+ \left( +96 + 4 \epsilon_{3a} \right) \{ 1, 2, 3 \} + \{ 3, 2, 1 \} \\
+ \left( -12 - 2 \beta_{2,3} - 4 \epsilon_{3a} \right) \{ 2, 1, 3, 2 \} \\
+ \left( +18 + 4 \epsilon_{3a} \right) \{ 1, 3, 2, 4 \} + \{ 2, 1, 4, 3 \} \\
+ \left( -8 - 2 \epsilon_{3a} - 2 i \epsilon_{3b} \right) \{ 1, 2, 4, 3 \} + \{ 1, 4, 3, 2 \} \\
+ \left( -8 - 2 \epsilon_{3a} + 2 i \epsilon_{3b} \right) \{ 2, 1, 3, 4 \} + \{ 3, 2, 1, 4 \} \\
- 10 \{ 1, 2, 3, 4 \} + \{ 4, 3, 2, 1 \} \]

- Direct calculation of \( \mathcal{H}_4 \) \( \implies \ \beta^{(3)}_{23} = 4 \zeta(3) \) Beisert, McLoughlin, RR

- expected \( \zeta \)-constants at 5, 6, 7, 8-loops McLoughlin, RR (unpublished)
Twist operators; $SL(2)$ sector

$$O_{n_1...n_L} = \text{Tr} \left[ D^{n_1}Z \cdots D^{n_L}Z \right] \sum_{i=1}^{L} n_i = S \quad L = \Delta_0 - S$$

- Hamiltonian is complicated; unknown beyond 2-loops

Various scaling limits:

1) $S \ll L$
2) $L \to \infty$, $S \to \infty$, $(\ln S)/L < 1$, $\frac{\lambda S}{L^2} \ln \frac{S}{L}$
3) $L \to \infty$, $S \to \infty$, $(\ln S)/L \gg 1$, $\sqrt{\lambda} \ln S/\sqrt{\lambda}$

- Contact with perturbative gauge theory data – need small $L$

- AP kernel – close relation to $L = 2$;

$$\gamma(g, S) \xrightarrow{S \to \infty} f(g) \ln S \quad f(g) = \Gamma_{\text{cusp}}$$

However...

- BA defined with $L \to \infty$ and fixed $S$; order of limits issue?

- $S \gg L$ really removes twist dependence?

- small twist is outside asymptotic regime; does it matter?
Twist operators; $SL(2)$ sector; Bethe equations for $(\ln S')/\mathcal{L} \gg 1$

\[
\left( \frac{x^+_k}{x^-_k} \right)^\mathcal{L} = \prod_{\substack{j=1 \\ j \neq k}}^{S} \frac{x^-_k - x^+_j}{x^+_k - x^-_j} \frac{1 - g^2/x^+_k x^-_j}{1 - g^2/x^-_k x^+_j} e^{2i\theta(u_k, u_j)} \quad u_{\pm} = x_{\pm} + \frac{g^2}{2x_{\pm}}
\]

- take logarithm; sums $\rightarrow$ integrals as $S \gg 1$; $u_i \rightarrow \rho(u)$
- take $d/du$; perturb around $\rho_0 = \rho\big|_{g=0} : \rho(u) = \rho_0(u) - g^2 E_0 S \sigma(u)$;

Fourier-transform $\sigma(u) \mapsto \hat{\sigma}(t)$

\[
\hat{\sigma}(t) = \frac{t}{e^t - 1} \left[ K(2gt, 0) - 4g^2 \int_0^\infty dt' \hat{K}(2gt, 2gt') \hat{\sigma}(t') \right]
\]

\[
\hat{K}(t, t') = \frac{J_1(t)J_0(t') - J_1(t')J_0(t)}{t - t'} + \hat{K}_d(t, t')
\]

\[
\hat{K}_d(t, t') = \frac{4}{tt'} \sum_{\mu > \nu} (-)^\nu g^{2\mu + 1} (\beta^{(2\mu+1+\nu)}_{2\mu+1+2\nu}) J_{2\mu+2\nu}(t) J_{2\mu+1}(t') + \beta^{(2\mu+1+\nu)}_{2\mu+1+2\nu} J_{2\mu}(t) J_{2\mu+1+2\nu}(t')
\]

\[
f(g) = E(g)/\ln S = 16g^2 \hat{\sigma}(0)
\]
• Weak coupling:

\[ \pi^2 f(g) = \frac{\lambda}{2} - \frac{\lambda^2}{96} + \frac{11 \lambda^3}{23040} - \left( \frac{73}{2580480} + \frac{\zeta(3)^2}{1024 \pi^6} \right) \lambda^4 \ldots \]

- agrees within numerical accuracy with direct calculation of \( \Gamma_{\text{cusp}} \)
  Bern, Czakon, Dixon, Kosower, Smirnov; Cachazo, Spradlin, Volovich

- \( \lambda^5 \) agrees with Padé extrapolation of 4-loop \( \Gamma_{\text{cusp}} \)
  Bern, Czakon, Dixon, Kosower, Smirnov

◊ Bethe ansatz works unexpectedly well despite potential issues

• Strong coupling limit:

- Direct extrapolation of BES
  Benna, Benvenutti, Klebanov, Scardicchio

  ◊ correct LO and NLO; uncertainties w/ procedure at NNLO

  ◊ NNLO: apparent inconsistency with ws Feynman diagramatics
    RR, Tirziu, Tseytlin

- Use strong coupling phase

  ◊ correct LO and NLO; NNLO N/A

Casteill, Kristjansen
Does this imply that asymptotic BA is complete?

- Further tests at finite spin and finite length
  - use BFKL

\[
\frac{\omega^2}{-g^2} = \psi(-g^2 E(g)) + \psi(1 + g^2 E(g)) - 2\psi(1)
\]

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x)
\]

\[
\omega = S + 1
\]

Kotikov, Lipatov, Rej, Staudacher, Velizhanin

- Encodes the \( t \)-channel exchange of pomeron resonance

\[
\text{Tr } [ZD^{-1} + \omega Z]
\]

- Sensitive to complete dependence on \( \lambda \) and \( S \)
  - different organization of Feynman diagrams

- valid for negative spin around \( \omega = 0 \)

Balitsky, Fadin, Kuraev, Lipatov

- Comparison with asymptotic BA predictions require
  - calculation at finite spin
  - continuation to \( S < 0 \) and expansion around \( S = -1 \)
The comparison:

- **BFKL**

\[ E(g)^{\text{BFKL}} = \frac{-4g^2}{\omega} - 0 \left( \frac{-4g^2}{\omega} \right)^2 + 0 \left( \frac{-4g^2}{\omega} \right)^3 - 2\zeta(3) \left( \frac{-4g^2}{\omega} \right)^4 + \ldots \]

- **ABA**

\[ E(g)^{\text{ABA}} = \frac{-4g^2}{\omega} - 0 \left( \frac{-4g^2}{\omega} \right)^2 + 0 \left( \frac{-4g^2}{\omega} \right)^3 - \frac{(-4g^2)^4}{\omega^7} + \ldots \]

- ABA breaks down at finite spin and finite length at 4-loops

Kotikov, Lipatov, Rej, Staudacher, Velizhanin
The comparison:

- ABA breaks down at finite spin and finite length at 4-loops
  Kotikov, Lipatov, Rej, Staudacher, Velizhanin

- Proposed fix: change dressing phase coefficients \(\beta_{23}^{(3)}\)

\[
\zeta(3) \mapsto \frac{47}{24} \zeta(3) - \frac{1}{4} S_{-4} + \frac{3}{4} S_{-2} S_1 + \frac{3}{8} S_1 S_2 + \frac{3}{4} S_3 + \frac{1}{6} S_{-2,1} - \frac{17}{24} S_{2,1}
\]

- makes S-matrix state-dependent (depends on \(S\)) – Universality?

- Assuming all previous assumptions hold → conjecture for \(\gamma_{\text{Konishi}}\)

\[
O = \sum_i \text{Tr} [\phi^i \bar{\phi}_i] \quad \gamma(g) = 12g^2 - 48g^4 + 336g^6 - \left(\frac{5307}{2} + 564\zeta(3)\right) g^8 \ldots
\]

Kotikov, Lipatov, Rej, Staudacher, Velizhanin
Outlook . . .

• Great progress toward finding spectrum of $\mathcal{N} = 4$ SYM

• Bethe Ansatz works better than expected but incomplete

• Not immediately clear what it is diagonalizing
  – $\lambda \to 0$ complete 1-loop; higher loops in some sectors
  – $\lambda \to \infty$ a continuum leading order Hamiltonian
...and some open problems

★ Prove integrability

★ What integrable model describes the $\mathcal{N} = 4$ spectral problem? What about the world sheet sigma model?

★ Is the dressing phase truly correct and who ordered it?
  – Other solutions? Other analytic continuation?
  – more reliable computations are needed at strong coupling

★ Use of integrability for finite quantum numbers?
  – no direct calculation of a wrapping effect
  – why some quantities are insensitive to it?

★ New computational techniques?

★ are there extra symmetries waiting to be discovered?

★ Is integrability restricted to the spectral problem?
  – consequences/extra structure in scattering amplitudes?