Cosmological Twistors

Relevance to Scattering Amplitudes:

Divergences

Infra-red: extremely long distances
Asymptotic: $P$

Ultra-violet: very short distances
Big bang: $B$

Singularity

Current standard picture of the expanding universe

- inflation?
- spatial curvature close to flat: could be positive, negative or zero.
Spectrum of the Microwave Background

Note: error bars are exaggerated by a factor of 500.
The solid curve displays the Planck black body spectrum of thermal equilibrium.
2nd Law of Thermodynamics

Entropy increases with time
\[ \uparrow \Rightarrow \text{"disorder"} \] (roughly speaking)

Gas in a box

Time increases

Entropy increases

Gravitating bodies

Maximum entropy: BLACK HOLE
Singularities

black hole

with irregularities
Fundamental Asymmetry in space-time singularity structure:

future singularity
Weyl \xrightarrow{\sim} \infty

past singularity
Weyl \xrightarrow{\sim} 0

Weyl curvature hypothesis

Implies \( \sim \) Uniformity of microwave background
2nd Law of Thermodynamics

Constraint on Big Bang: \( \frac{1}{10^{10^{123}}} \)

[from Bekenstein-Hawking black-hole entropy of \( 10^{80} \) protons/neutrons]

Quantum Gravity?
Not any conventional approach (time-asymm.)
Tod's form of the Weyl curvature hypothesis

 Requires highly constrained Weyl curvature at the Big Bang

 "mathematical trick" \( \Omega \to \infty \)

\[ \hat{g}_{\mu \nu} = s^2 g_{\mu \nu} \]

Near the big bang, energies get so great that mass becomes irrelevant and particles are effectively massless. Conformal invariance holds.
The Extremely Remote Future

Much matter collapses to black holes. Eventually, the expanding universe cools to lower than the holes’ Hawking temperatures (the larger the hole, the lower the temperature — always very low!). Then, the hole evaporates away — very slowly — until [pop!] it disappears.

\[ \sim 10^{64} \text{ yrs for } M_{\odot}, \quad \sim 10^{90} \text{ yrs for galactic} \]

Provided protons, etc., eventually decay, then matter itself disappears into radiation.

Scheme appears to require:

- decay of the mass of electrons, massive neutrinos, etc., into massless ingredients — at least asymptotically, in remote future
- mass ratios (e.g. \( m_e : m_p : m_N \)) might evolve with time — and
- other numerical constants.

Nature (e.g. fine struc. const.) might evolve.

\( G \)\( R \) metric is Milne’s “dynamical” metric (“atomic” metrics give finite aon time).
With only massless ingredients left, the universe loses track of time. All contents of the universe would be conformally invariant.

No way of constructing a clock — only the light-cone structure remains.

**Conformal geometry**

To a photon (or other massless entity) no time is experienced between beginning and end:

Eternity is no time at all, to a photon!

Conformal infinity $\mathcal{I}$

"mathematical trick" $\Omega \to 0$ $\mathcal{I}=\mathcal{I}^+\,\mathcal{I}^0$

Automatically: Weyl = 0
Singularity → Conformal rescaling → Infinity → Singularity
Conformal Cyclic Cosmology
2 Conformal "Mathematical Tricks"

Gravitational radiation

\[ \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \]

Cosmological singularities

Asymptotically flat space-time \( \Lambda = 0 \)

Trick: shrink \( \infty \) to a finite place by taking \( \Omega = 0 \) there

when \( \Lambda > 0 \), this still works (in some sense better (RP Friedrich) - easier)

but then \( S^+ \) is spacelike rather than null

\[ S = 0 \]

Trick: expand out singularity to obtain a conformally smooth initial hypersurface

Tod's form of Weyl curvature hypothesis: this works!
Anti-self-dual (complex) solutions of the Einstein vacuum equations:
2 original twistor approaches ($\Lambda = 0$).

1. Asymptotic (Newman $H$-Space)

- 3 types of geodesic on $\mathcal{C}^\phi$
- $\beta$-plane on $\mathcal{C}^\phi$
- $\alpha$-plane on $\mathcal{C}^\phi$
- $\alpha$-line
- $\beta$-line
- Dual twistor

- General definition of hypersurface twistor (projective) as $\alpha$-line.
- $\mathbb{CP}^1$ reach $\alpha$-line corr. topt.
- Asymptotic twistor space
- $\mathbb{C}$<$\phi$ complex stereographic coordinate

- $s = e^{i\phi\cot\theta/2}$
- $u = \text{retarded time (Bondi)}$

- Newman "good cut" by $\alpha$-lines defines point in ASD space
"Constructive" (non-linear graviton) produce deformed version of (a region in) $\mathbb{CP}^3$

$\mathbb{CP}^3$

$\mathbb{CP}'$ of projective $\pi^A$, spinors

Cover $\mathbb{CP}'$ by 2 overlapping hemispheres
deform

ASD sol'n of Einstein

Structure of the deformed twistor space: Let's phrase things in terms of non-projective space

$\gamma = (z^\alpha \partial_{\bar{z}^\alpha})$

$\lambda = (\pi^A \partial \pi^A)$

$\tau = (\frac{1}{2} \partial \pi^A \partial \pi^A)$

$\theta = (\frac{1}{8} \epsilon_{\alpha \beta \gamma \delta} z^\alpha dz^\beta dz^\gamma dz^\delta)$

$\phi = (dz^\alpha dz^\beta dz^\gamma dz^\delta)$

Euler curves (tangents $z^\alpha / \partial z^\alpha$)
In the case $\Lambda = 0$ (RP 1976), we have
\[ d\lambda = 2\tau \quad d\theta = 4\phi \]
$\tau \perp \tau = 1 \quad \phi \perp \tau = \theta$
and $\tau = \theta \div \phi$ in the sense $d\alpha \lor \theta = \tau \lor \phi$
where $\tau$ is "simple":
\[ \tau \lor \tau = 0 \quad \text{i.e.} \quad \tau \lor \tau = 0 \]
We also have the homogeneity relations
\[ d\phi \tau = 2\tau, \quad d\theta \tau = 2\tau, \quad d\theta = 4\phi, \quad d\phi = 4\phi \]

In 1980, R.S. Ward generalized this to $\Lambda \neq 0$, in effect just by relaxing the "simplicity" conditions above to:
\[ \tau \lor \tau = \frac{1}{3} \Lambda \theta \quad \text{i.e.} \quad \tau \lor \tau = \frac{1}{3} \Lambda \theta \]

Question: what is the analogue of $\mathbf{0}$ (Newman's "$\mathbf{R}$-space" method) in the case $\Lambda > 0$, which appears to be the cosmologically appropriate case?
When $\Lambda > 0$, we have a spacelike $J^+$ (and we may also assume that the conformally "stretched out" big bang hypersurface $B$ is spacelike; and according to $\mathcal{C}$, $B$'s conformal structure has the same character as that of $J^+$). 

Hypersurface twistors for a spacelike hypersurface $\Sigma$:

- An $\alpha$-surface would have tangent vectors $\xi^A \pi^A$ for arbitrary $\xi^A$ where $\pi^A \nabla_{AA'} \pi_{B'} = 0$.
- Virtual $\alpha$-surface.
- $N^A \alpha$-curve.
- Tangent to $\Sigma$.

The solutions of $\Sigma$ give us the $\alpha$-curves and therefore the points of hyp. twistor space for $\Sigma$. 

This is too strong, so we restrict to $\infty$.
Now choose $f$ to be $\phi^+$ (or $\phi$). We find (rather surprisingly) that with the natural choice

$$\hat{N}_a = -\nabla_a \Omega$$

where

$$\hat{g}_{ab} = \Omega^2 g_{ab}$$

is the (rescaled) metric to make $\phi^+$ finite, the conformal factor

physical metric

$$(\hat{N}_a \hat{N}^a = \frac{1}{3} \Lambda \text{ on } \phi^+)$$

that

$$\left( N^{AC} \pi_c \right) \pi^{A'} \nabla_{A'} \left\{ (N^{BD} \pi_d) \pi^{B'} \right\} = 0$$

as a consequence of the "asymptotic Einstein condition"

$$\hat{\nabla}_{A'} (A \hat{N}_B) B' = 0 \text{ on } \phi^+$$

(with only massless fields present on $\phi^+$).

It follows that the $\alpha$-lines on $\phi^+$ are actually geodesics on $\phi^+$ (as they are in the $\Lambda=0$ case), whence by symmetry (when $\Lambda > 0$) they must also be $\beta$-lines!
This has various striking implications (apparently):

- The asymptotic twistor space is a complex-symplectic manifold [so: L, T exist as in Ward’s const.]
  - We expect to find 3 distinct families of holomorphic CP’s giving 3 different analogues of Newman’s $\mathcal{N}$-space:
    - an SD Ward space
    - an ASD Ward space
    - a conformally flat Ward space

- We can take various kinds of $\Lambda \to 0$ limits

![Schematic diagram](image-url)

null $J^+$

spacelike $J^+$

($\Lambda = \epsilon > 0$)

Schematic only