Conformally flat hypersurfaces
with
cyclic Guichard net
(Udo Hertrich-Jeromin, 12 August 2006)

Joint work with Y. Suyama
A geometrical Problem

Classify conformally flat hypersurfaces \( f : M^{n-1} \to S^n \).

**Def.** \( f : M^{n-1} \to S^n \) is **conformally flat** if there are (local) functions so that \( e^{2u} \langle df, df \rangle \) is flat (or, equivalently, there are (local) conformal coordinates).

**Known results.**

\( n = 3 \). Every \( f : M^2 \to S^3 \) is conformally flat (Gauss’ Theorem).

\( n > 4 \). \( f \) is conformally flat \( \iff \) \( f \) is a branched channel hypersurface (Cartan 1917).

\( n = 4 \). Branched channel hypersurfaces are conformally flat;

- there are hypersurfaces that are not conformally flat (e.g., Veronese tubes);
- there are **generic** conformally flat **hypersurfaces**, i.e., with 3 distinct principal curvatures (e.g., cones, cylinders, hypersurfaces of revolution over \( K \)-surfaces).

**The problem:** Classify generic conformally flat hypersurfaces \( f : M^3 \to S^4 \).

**Observation:** There is an intimate relation

**conformally flat hypersurfaces** in \( S^4 \) \( \iff \) **curved flats** in the space of circles in \( S^4 \).
The Program

1. Conformally flat hypersurfaces
2. Curved flats
3. Isothermic surfaces
4. Conformally flat hypersurfaces revisited
Conformally flat hypersurfaces

**Cartan’s Thm.** If $f : M^{n-1} \to S^n, n \geq 5$, is conformally flat then $f$ is a branched channel hypersurface.

**Def.** Write $I = \sum_{i=1}^{3} \eta_i^2$ and $II = \sum_{i=1}^{3} k_i \eta_i^2$; then
\[
\gamma_1 := \sqrt{(k_3 - k_1)(k_1 - k_2)} \eta_1,
\gamma_2 := \sqrt{(k_2 - k_3)(k_3 - k_1)} \eta_2,
\gamma_3 := \sqrt{(k_1 - k_2)(k_2 - k_3)} \eta_3
\]
are the **conformal fundamental forms** of $f : M^3 \to S^4$.

**Lemma.** $f : M^3 \to S^4$ is conformally flat $\iff d\gamma_i = 0$.

**Cor.** If $f$ is conformally flat then there are curvature line coordinates $(x_1, x_2, x_3) : M^3 \to \mathbb{R}^3_2$ so that $dx_i = \gamma_i$.

**Observe:** $I = \sum_{i=1}^{3} l_i^2 dx_i^2$, where $\sum_{i=1}^{3} l_i^2 = 0$.

**Def.** $x : (M^3, I) \to \mathbb{R}^3_2$ is called a **Guichard net** if $I = \sum_{i=1}^{3} l_i^2 dx_i^2$ with $\sum_{i=1}^{3} l_i^2 = 0$.

**Remark.** A generic conformally flat $f : M^3 \to S^4$ gives a Guichard net $x \circ y^{-1} : \mathbb{R}^3 \to \mathbb{R}^3_2$ ($x : M^3 \to \mathbb{R}^3_2$ canonical Guichard net and $y : M^3 \to \mathbb{R}^3$ conformal coordinates).

**Thm.** A Guichard net $x : \mathbb{R}^3 \to \mathbb{R}^3_2$ gives a conformally flat $f : \mathbb{R}^3 \to S^4$ with $\gamma_i = dx_i$. 
How to prove all this?

Consider: \( f : M^{n-1} \rightarrow S^n \subset L^{n+1} \).

Observe: If \( u \in C^\infty(M^{n-1}) \) then
\[
\langle d(e^u f), d(e^u f) \rangle = e^{2u} \langle df, df \rangle.
\]

Thus: If \( f : M^{n-1} \rightarrow S^n \) is conformally flat we may choose (locally) a flat lift
\( e^u f : M^{n-1} \rightarrow L^{n+1} \).

Then: The tangent bundle of a flat lift
\( f : M^3 \rightarrow L^5 \subset R^6 \) is flat.

Lemma. In this situation, the normal bundle of
\( f : M^3 \rightarrow R^6 \) is also flat.

Cor. If \( f : M^3 \rightarrow L^5 \) is a flat lift of a conformally flat hypersurface then its Gauss map
\( \gamma : M^3 \rightarrow \frac{O_1(6)}{O(3) \times O_1(3)} \), \( p \mapsto \gamma(p) = d_p f(T_p M) \) is a “curved flat”.

Note. Curved flats come with special coordinates:
\( \leadsto \) integrability of conformal fundamental forms and of Cartan’s umbilic distributons;
\( \leadsto \) conformally flat hypersurfaces come with principal Guichard nets.
Curved flats

**Setup:** Let $G/K$ be a symmetric (or reductive homogeneous) space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding symmetric decomposition of the Lie algebra, i.e.,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$ 

For a map $\gamma : M^m \rightarrow G/K$ we consider any lift $F : M^m \rightarrow G$ and decompose its connection form $F^{-1}dF = \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p}$.

**Def.** $\gamma : M^m \rightarrow G/K$ is called a **curved flat** if $[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \equiv 0$.

**Observation:** $\gamma : M^m \rightarrow G/K$ is a curved flat $\iff \Phi_\lambda := \Phi_{\mathfrak{k}} + \lambda \Phi_{\mathfrak{p}}$ is integrable for all $\lambda$, i.e., the Gauss-Ricci equations split:

$$0 = d\Phi_{\mathfrak{k}} + \frac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] \quad \iff \quad \begin{cases} 0 = d\Phi_{\mathfrak{k}} + \frac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] + \frac{1}{2}[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \\ 0 = d\Phi_{\mathfrak{p}} + [\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{p}}] \\ 0 = [\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \end{cases}$$

**Consequences:**

- curved flats come in “associated families”; and . . .
- the curved flat equations become a 0-curvature condition for the family;
- hence integrable systems methods (e.g., finite gap integration etc) can be applied.
Isothermic surfaces

Def. $f : M^2 \rightarrow S^3$ is isothermic if there are (local) conformal curvature line parameters.

Well understood:
- **Darboux pairs** of isothermic surfaces in $S^3$:
  1. envelope a Ribaucour sphere congruence
  2. induce conformally equivalent metrics

  $\leftrightarrow$ curved flats in $\frac{O_1(5)}{O(3) \times O_1(2)}$
  (in the space of point-pairs)

- **Christoffel pairs** of isothermic surfaces in $\mathbb{R}^3$
  (“limiting case of Darboux pairs”):
  1. parallel curvature directions
  2. induce conformally equivalent metrics

  $\leadsto$ curved flats in $\frac{O_1(5)}{O(3) \times O_1(2)}$

Note: Special coordinates are already “built in”. 
Curved flats come in associated families!
The associated family of curved flats yields:

- the classical **Calapso transformation** (\(T\)-transformation)
- the conformal deformation for isothermic surfaces.

**Small miracle:**
The surfaces \(f_\lambda\) of \(\gamma_\lambda = (f_\lambda, \hat{f}_\lambda)\) only depend on \(f = f_1\) and \(\lambda\).

**The limiting case:**
\((f_0, \hat{f}_0) = \lim_{\lambda \to 0}(f_\lambda, \hat{f}_\lambda)\) yields a Christoffel pair in \(\mathbb{R}^3\).
Discrete isothermic and cmc nets.

**Bianchi permutability:**

\[ \mathcal{D}_{\lambda_1} \mathcal{D}_{\lambda_2} f = \mathcal{D}_{\lambda_2} \mathcal{D}_{\lambda_1} f \quad \text{and} \quad [f; \mathcal{D}_{\lambda_1} f; \mathcal{D}_{\lambda_1} \mathcal{D}_{\lambda_2} f; \mathcal{D}_{\lambda_2} f] = \frac{\lambda_2}{\lambda_1}. \]

**Def.** \( f : \mathbb{Z}^2 \rightarrow S^3 \) is **isothermic** if \( q_{m,n} = \frac{a(m)}{b(n)}. \)

This yields a completely analogous discrete theory:

- Christoffel transformation;
- Darboux transformation;
- Calapso transformation;
- Bianchi permutability theorems;
- \( \leadsto \) discrete minimal & cmc surfaces;
- \( \leadsto \) Weierstrass representation;
- \( \leadsto \) Bryant type representation;
- \( \leadsto \) Bonnet’s theorem;
- Polynomial conserved quantities by Burstall/Calderbank/Santos...
Conformally flat hypersurfaces with cyclic Guichard net

We saw: From a conformally flat \( f : M^3 \to S^4 \) we get
\[ \gamma : M^3 \to \frac{O_1(6)}{O(3) \times O_1(3)}, \quad p \mapsto \gamma(p) = d_pf(T_p M) \text{ curved flat (non-unique)}, \]
\[ x : (M^3, I) \to \mathbb{R}^3_2 \text{ Guichard net (unique), and} \]
\[ x \circ y^{-1} : \mathbb{R}^3 \to \mathbb{R}^3_2 \text{ Guichard net (unique up to Möbius transformation).} \]

Conversely:

- A curved flat \( \gamma : M^3 \to \frac{O_1(6)}{O(3) \times O_1(3)} \) is a “cyclic system” with
  conformally flat orthogonal hypersurfaces (analogue of the Darboux transformation);
- A Guichard net \( x : \mathbb{R}^3 \to \mathbb{R}^3_2 \) gives rise to
  a conformally flat hypersurface (unique up to Möbius transformation).

Questions:

1. How are the hypersurfaces of a curved flat related (“Darboux transformation”)?
2. What is the geometry of the associated family (“Calapso transformation”)?
3. How are the geometry of a conformally flat hypersurface and a Guichard net related?
4. How to define a suitable discrete theory?
5. . . .
4. Conformally flat hypersurfaces with cyclic Guichard net

Partial answers to the 3rd question.

**Thm.** Cones, cylinders and hypersurfaces of revolution over $K$-surfaces in $S^3$, $R^3$ and $H^3$, respectively, correspond to cyclic Guichard nets with totally umbilic orthogonal surfaces.

**Def.** A cyclic system is a smooth 2-parameter family of circles in $S^3$ with a 1-parameter family of orthogonal surfaces, i.e., a smooth map

$$\gamma : M^2 \rightarrow \frac{O_1(5)}{O(2) \times O_1(3)}$$

so that the bundle $\gamma^\perp$ of Minkowski spaces is flat.

**Example.** The normal line congruence of a surface in a space form $Q_\kappa^3$ is a cyclic system.

**Thm.** A cyclic Guichard net is a normal line congruence in some $Q_\kappa^3$ with all orthogonal surfaces linear Weingarten.

**Question:** What are the corresponding hypersurfaces?

**Classification result:** They “live” in some $Q_\kappa^4$, where the orthogonal surfaces of the cyclic system are (extrinsically) linear Weingarten surfaces in a family of (parallel) hyperspheres in $Q_\kappa^4$.

Conversely, conformally flat hypersurfaces with cyclic Guichard net can be constructed starting from suitable linear Weingarten surfaces in any space form in a unique way.
How to prove this?

Recall: If \( f : M^3 \to S^4 \) is conformally flat then there are curvature line coordinates \((x, y, z) : M^3 \to \mathbb{R}^3\) so that \( I = e^{2u} \{ \cos^2 \varphi \, dx^2 + \sin^2 \varphi \, dy^2 + dz^2 \} \).

Lemma. \( \varphi \) satisfies
\[
d\alpha = 0, \quad \text{where} \quad \alpha := -\varphi_{xz} \cot \varphi \, dx + \varphi_{yz} \tan \varphi \, dy + \frac{\varphi_{xx} - \varphi_{yy} - \varphi_{zz} \cos 2\varphi}{\sin 2\varphi} \, dz, \quad \text{and}
\]
\[
0 = \frac{\varphi_{xx} + \varphi_{yy} + \varphi_{zz}}{2} + \frac{\varphi_z (\varphi_{xx} - \varphi_{yy} - \varphi_{zz} \cos 2\varphi)}{\sin 2\varphi} - \varphi_x \varphi_{xz} \cot \varphi + \varphi_y \varphi_{yz} \tan \varphi.
\]

Conversely, \( f \) can be reconstructed from \( \varphi \).

Lemma. The \( z \)-lines are circular arcs if and only if
\( \varphi_{xz} = \varphi_{yz} \equiv 0 \).

Cor. Conformally flat hypersurfaces with cyclic principal Guichard net correspond to \( \varphi \)'s satisfying:
\[
\varphi(x, y, z) = u(x, y) + g(z) \quad \text{with} \quad u_{xx} - u_{yy} = A \sin 2u \quad \text{and} \quad g'^2 = C + A \cos 2g;
\]
or: similar formulas with \( \cosh \varphi \) and \( \sinh \varphi \) (then, more cases occur).

Observation: Separation of variables considerably simplifies the PDE's for \( \varphi \).
Symmetry breaking.

From the structure equations, define $T = T(z) \in S^5_1$ and $Q = Q(z) \in \mathbb{R}^6_1 \setminus \{0\}$ with

$$T' = \frac{1}{1+g''} Q \quad \text{and} \quad Q' = \frac{-\kappa}{1+g''} T,$$

where $\kappa := -|Q|^2 \equiv (1+C)^2 - A^2$.

In particular, with $\zeta(z) = \int_0^z \frac{dz}{1+g''(z)}$,

$$T = \cosh \sqrt{\kappa} \zeta \quad T_z = 0 + \frac{1}{\sqrt{\kappa}} \sinh \sqrt{\kappa} \zeta \quad \text{and} \quad Q = \kappa \frac{1}{\sqrt{\kappa}} \sinh \sqrt{\kappa} \zeta \quad Q_z = 0 + \cosh \sqrt{\kappa} \zeta \quad Q_z = 0.$$

Consequences:

- $\text{span}\{T, Q\}$ is a fixed sphere pencil;
- $Q(0)$ defines a space form $Q^4_\kappa$;
- $T(z)$ are parallel hyperspheres in $Q^4_\kappa$;
- each surface
  $$(x, y) \mapsto \frac{f(x, y, z)}{\langle T(z), T(0) \rangle \sqrt{1+g''(z)}} \in T(z) \cap Q^4_\kappa$$
  is a linear Weingarten surface.

Explicitely:

$$f = \frac{\sqrt{1+A+C} \cos g}{\sqrt{1+g''} \cosh \sqrt{\kappa} \zeta} \{f_0 + \frac{\tan g}{1+A+C} \cdot n + \frac{\sqrt{1+g''}}{\sqrt{1+A+C} \cos g} \cdot t\},$$

where $f_0 = f(., ., 0)$, with Gauss map $n$ in $T(0) \subset Q^4_\kappa$, and $t$ the unit normal of $T(0) \subset Q^4_\kappa$. 

Thank you!

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References

1. É. Cartan: *La déformation des hypersurfaces dans l’espace conforme réel à \( n \geq 5 \) dimensions*; Bull. Soc. Math. France 56, 57–121 (1917)


4. C. Guichard: *Sur les systèmes triplement indéterminé et sur les systèmes triple orthogonaux*; Scientia 25, Gauthier-Villars (1905)


