Limit Theorems in Nonextensive Statistical Mechanics

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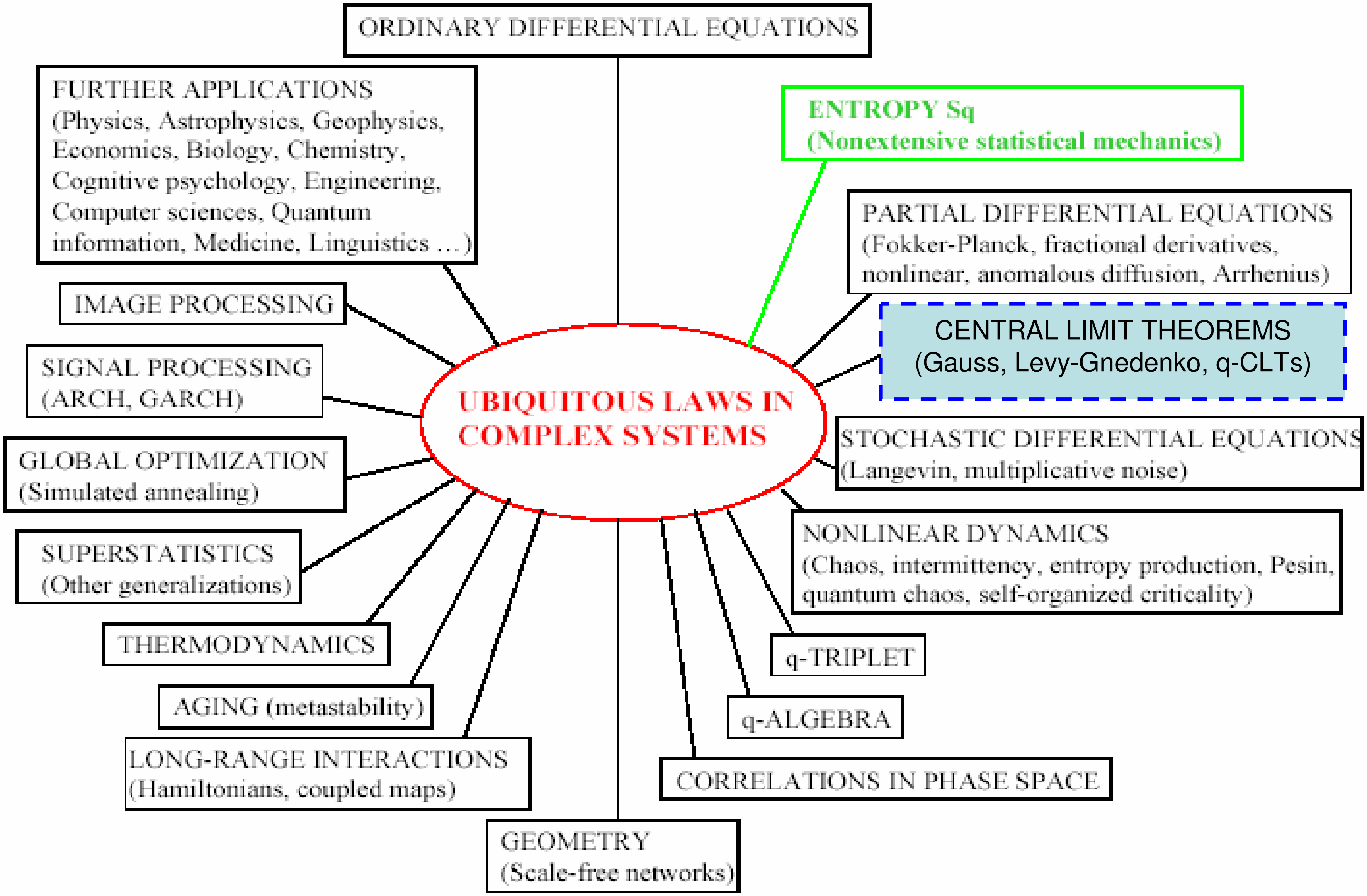

ORDINARY DIFFERENTIAL EQUATIONS
FURTHER APPLICATIONS (Physics, Astrophysics, Geophysics, Economics, Biology, Chemistry, Cognitive psychology, Engineering, Computer sciences, Quantum information, Medicine, Linguistics ...)
ENTROPY Sq (Nonextensive statistical mechanics)
PARTIAL DIFFERENTIAL EQUATIONS (Fokker-Planck, fractional derivatives, nonlinear, anomalous diffusion, Arrhenius)
IMAGE PROCESSING
CENTRAL LIMIT THEOREMS (Gauss, Levy-Gnedenko, q-CLTs)
SIGNAL PROCESSING (ARCH, GARCH)
UBIQUITOUS LAWS IN COMPLEX SYSTEMS
STOCHASTIC DIFFERENTIAL EQUATIONS (Langevin, multiplicative noise)
GLOBAL OPTIMIZATION (Simulated annealing)
SUPERSTATISTICS (Other generalizations)
NONLINEAR DYNAMICS (Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)
THERMODYNAMICS
q-TRIPLET
AGING (metastability)
q-ALGEBRA
LONG-RANGE INTERACTIONS (Hamiltonians, coupled maps)
CORRELATIONS IN PHASE SPACE
GEOMETRY (Scale-free networks)

Contents

- q-generalization of the classic central limit theorem
- q-generalization of the classic α-stable Levy distributions

Classic CLT

Theorem. Assume a sequence of identically distributed random variables $X_1,\ldots,X_N$

1) are independent
2) have a finite variance $\sigma^2 < \infty$

Then the scaled sum

$$Z_N = \frac{X_1 + \ldots + X_N - N\mu}{\sqrt{N}\sigma} \to N(0,1)$$

where N(0,1) is the normal distribution with the mean 0 and the variance $\sigma^2 = 1$.

- ✓ density function of N(0,1) is a Gaussian G(x), i.e. the attractor is a Gaussian
- ✓ the normal distribution is stable
- ✓ The Gaussian maximizes Boltzmann-Gibbs entropy

$$S_{BG}(f) = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx$$

under natural constraints

q-generalization of S_{BG}(f)

- q-entropy $$S_q(f) = \frac{1 - \int [f(x)]^q dx}{q-1}$$
- $S_q \to S_{BG}$, if $q \to 1$.
- q-Gaussian $$G_q(\beta, x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}, \beta > 0$$

$$C_q = \begin{cases} \dfrac{2\sqrt{\pi}\,\Gamma(\frac{1}{1-q})}{(3-q)\sqrt{1-q}\,\Gamma(\frac{3-q}{2(1-q)})}, & -\infty < q < 1 \\ \sqrt{\pi}, & q = 1 \\ \dfrac{\sqrt{\pi}\,\Gamma(\frac{3-q}{2(q-1)})}{\sqrt{q-1}\,\Gamma(\frac{1}{q-1})} & 1 < q < 3 \end{cases}$$

$$e_q^x = [1 + (1-q)x]_+^{\frac{1}{1-q}},$$

$$\ln_q x = \frac{x^{1-q} - 1}{1-q}, x > 0$$

q-Gaussian maximizes q-entropy

q-mathematics

CLT does not hold for strongly correlated random variables on the basis of the classic algebra

- q-sum

$$x \oplus_q y = x + y + (1-q)xy$$

- q-product

$$x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}$$

q-sum and q-product are both

- commutative,
- associative,
- recovers usual operations, if q=1.

q-Fourier transform

Definition: $$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \otimes_q e_q^{ix\xi} dx = \int_{-\infty}^{\infty} f(x) e_q^{ix\xi[f(x)]^{q-1}} dx$$

q=1: $F_q[f]$ recovers the Fourier transform

If q≥1, then $F_q[f]$ exists for all $f \in L_1(-\infty, \infty)$

If q<1, then $F_q[f]$ exists for $f \in L_1(-\infty, \infty)$ satisfying the condition

$$f \propto |x|^{-\gamma}, \gamma > \frac{2-q}{1-q}$$

Properties of q-FT

$$F_q[G_q(\beta;x)](\xi) = e_{q_1}^{-\beta_*(q)\xi^2}$$

$$\beta_*(q) = \frac{3-q}{8\beta^{2-q}C_q^{2(q-1)}} \qquad q_1 = z(q) = \frac{1+q}{3-q} \qquad q_{-1} = z^{-1}(q) = \frac{3q-1}{q+1}$$

$$q_{-1} + \frac{1}{q_1} = 2$$

$$F_q : G_q \to G_{q_1} \qquad F_{\frac{1}{q_1}} : G_{\frac{1}{q_1}} \to G_{\frac{1}{q}}$$

$$F_{q_{-1}} : G_{q_{-1}} \to G_q \qquad F_{\frac{1}{q}} : G_{\frac{1}{q}} \to G_{\frac{1}{q_{-1}}}$$

Properties of q-FT

sequence $q_n = z(q_{n-1})$:

$$q_n = \frac{2q + n(1-q)}{2 + n(1-q)}$$

1. If q=1, then $q_n = 1, n = 0, \pm 1, \pm 2, \ldots$

2. $q_{n-1} + \dfrac{1}{q_{n+1}} = 2, n = 0, \pm 1, \pm 2, \ldots$

3. $\ldots \xrightarrow{F_{q_{-2}}} G_{q_{-1}} \xrightarrow{F_{q_{-1}}} G_q \xrightarrow{F_q} G_{q_1} \xrightarrow{F_{q1}} G_{q_2} \xrightarrow{F_{q2}} \ldots$

4. $\ldots \xleftarrow{F_{q_{-2}}^{-1}} G_{q_{-1}} \xleftarrow{F_{q_{-1}}^{-1}} G_q \xleftarrow{F_q^{-1}} G_{q_1} \xleftarrow{F_{q1}^{-1}} G_{q_2} \xleftarrow{F_{q2}^{-1}} \ldots$

q-correlation

Two random variables X and Y are called to be q-correlated, if

$$F_q[X+Y]=F_q[X]\otimes_q F_q[Y]$$

or equivalently

$$\int_{-\infty}^{\infty} f_{X+Y}(x)\otimes_q e_q^{ix\xi}dx = F_q[f_X]\otimes_q F_q[f_Y]$$

Note: if q=1 then $F[f*g]=F[f]\cdot F[g]$

q-central limit theorem (k=0)

Theorem. Assume a sequence of identically distributed random variables

$$X_1,\ldots,X_N$$

1) are q-correlated
2) have a finite (2q-1)-variance $\sigma_{2q-1} < \infty$

Then the sum $$Z_N = \frac{X_1 + \ldots + X_N - N\mu_q}{(\sqrt{N\nu_{2q-1}}\sigma_{2q-1})^{\frac{1}{2-q}}} \rightarrow N_{z^{-1}(q)}$$

The corresponding attractor is q_{-1}-Gaussian, with $\beta = (\frac{3-q_{-1}}{4qC_{q_{-1}}^{2q_{-1}-2}})^{\frac{1}{2-q_{-1}}}$

q-central limit theorem (∀k)

Theorem. Let {…, q_{-1}, q, q_1, …} is given as above.

Assume a sequence of identically distributed random variables

$$X_1,\ldots,X_N$$

1) are q_k-correlated for some integer k;
2) have a finite ($2q_k$ -1)-variance $\sigma_{2q-1} < \infty$

Then the sum

$$Z_N = \frac{X_1 + \ldots + X_N - N\mu_{q_k}}{(\sqrt{N\nu_{2q_k-1}}\sigma_{2q_k-1})^{\frac{1}{2-q_k}}} \to N_{q_{k-1}}$$

The corresponding attractor is q_{k-1}-Gaussian, with

$$\beta_k = \left(\frac{3 - q_{k-1}}{4qC_{q_{k-1}}^{2q_{k-1}-2}}\right)^{\frac{1}{2-q_{k-1}}}$$

q-central limit theorem: schematic illustration

$$\{f : \sigma_{2q-1} < \infty\} \xrightarrow{F_{q_k}} G_{q_k}(2) \xleftarrow{F_{q_{k-1}}} G_{q_{k-1}}(2)$$

Some parallels and applications

1) Triplet (q_{k-1}, q_k, q_{k+1}) with:

- q_{k-1} is the index of attractor
- q_k is the index of correlation
- q_{k+1} is the index of scaling rate

q-triplet in nonextensive sttistical mechanics:

$(q_{sens}, q_{relax}, q_{st.state})$

Some parallels and applications

2) Moyano,Tsallis,Gell-Mann. Europhys. Lett (2006)

q-correlated variables with $(2-\frac{1}{q})$ -Gaussian (attractor) was obtained numerically

This follows from q-CLT: k=-1$\Rightarrow$ $q_{-2} = 2 - \frac{1}{q}$

3) Tsallis,Bukman. Phys. Rev. 1996 (Nonlinear Fokker-Planck)

Scaling rate $\delta = \frac{2}{3-q}$ obtained subject to finite q-variance

This follows from q-CLT: k=1$\Rightarrow$ $\delta = q_2 = \frac{1}{2-q} = \frac{2}{3-Q}, Q = 2q-1$

(q,α)-stable distributions

Definition. $$L(q,\alpha) = \{ f : F_q f = a e_q^{-b|\xi|^\alpha} \}$$

Theorem. Let $X_1, X_2, \ldots, X_N$ be a sequence of q-correlated random variables with a symmetric density

$$f(x) \approx C\left|x\right|^{-\frac{1+\alpha}{1+\alpha(q-1)}}$$

Then the sum

$$Z_N = \frac{X_1 + \ldots + X_N}{(\mu_{q,\alpha} N)^{\frac{1}{\alpha(2-q)}}} \to Z_\infty \in L(q,\alpha)$$

F_q-description

$$f \in L(q,\alpha) \Rightarrow \exists G_{q^L}(\beta;x) : f \approx G_{q^L}(\beta;x)$$

$$q^L = \frac{3+Q\alpha}{1+\alpha}, Q = 2q-1.$$

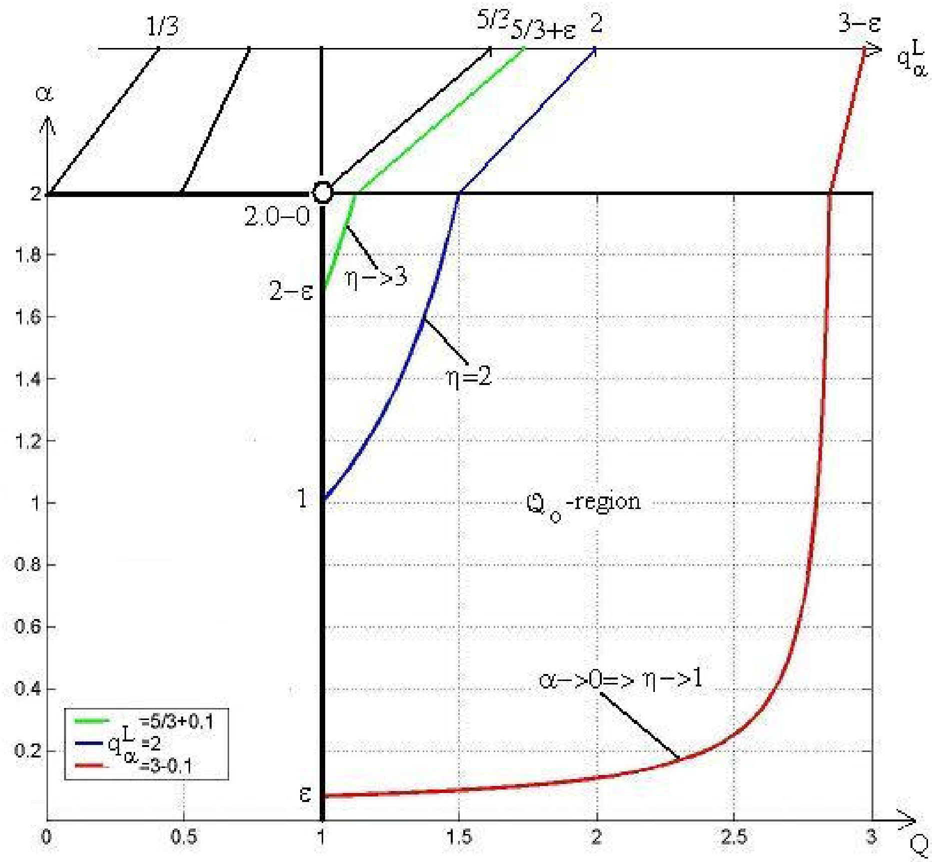
$$\eta = \frac{2(\alpha+1)}{2+\alpha(Q-1)}, Q = 2q-1.$$

F_q-description

$$L_{q,\alpha} \xrightarrow{F_q} G_q(\alpha) \xleftarrow{F_q} G_{q^L}(2),$$

$$0 < \alpha < 2$$

F_q-description



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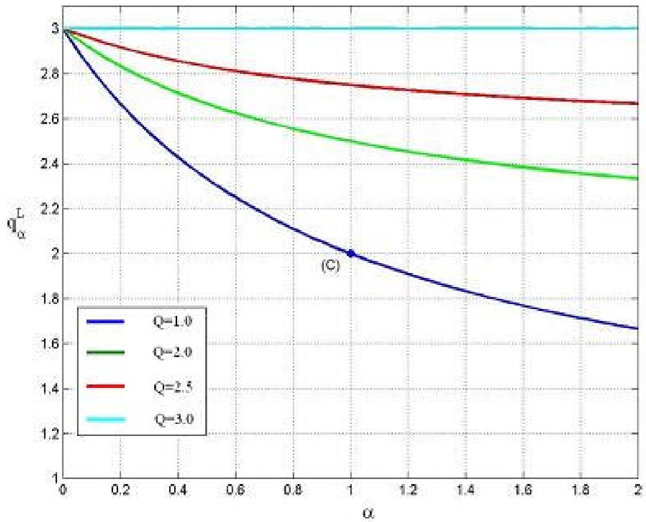
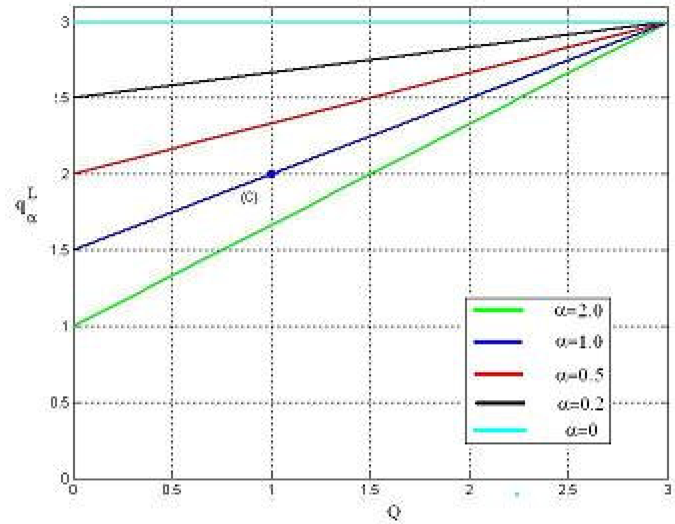
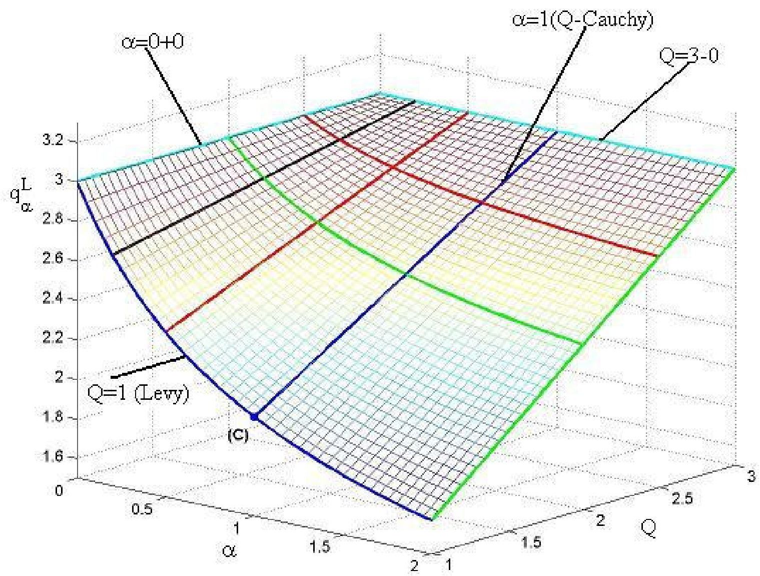
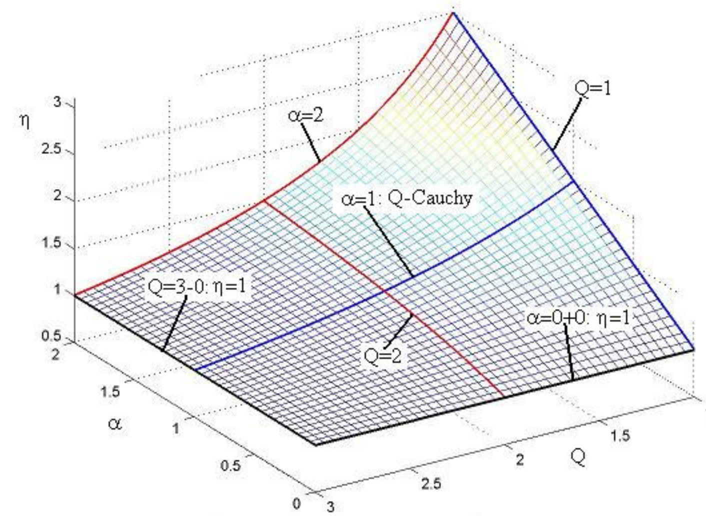




F_q-description





$F_{z(q)}$-description

Sequence $q_{\alpha,n}$, $q^*_{\alpha,n}$:

$$q_{\alpha,n} = \frac{\alpha q + n(1-q)}{\alpha + n(1-q)},$$

$$q^*_{\alpha,n} = \frac{\alpha + (n-2)(1-q)}{\alpha + n(1-q)}$$

Properties:

$$q^*_{2,n} = q_{2,n}, n = 0, \pm 1, \pm 2, \ldots$$

$$q^*_{\alpha,n-1} + \frac{1}{q^*_{\alpha,n+1}} = 2, n = 0, \pm 1, \pm 2, \ldots$$

$F_{z(q)}$-description

$$\ldots \xrightarrow{F_{q^*_{\alpha,-2}}} G_{q_{\alpha,-1}}(\alpha) \xrightarrow{F_{q^*_{\alpha,-1}}} G_{q_{\alpha,0}}(\alpha) \xrightarrow{F_{q^*_{\alpha,0}}} G_{q_{\alpha,1}}(\alpha) \xrightarrow{F_{q^*_{\alpha,1}}} G_{q_{\alpha,2}}(\alpha)\ldots$$

$$\ldots \xleftarrow{F^{-1}_{q^*_{\alpha,-2}}} G_{q_{\alpha,-1}}(\alpha) \xleftarrow{F^{-1}_{q^*_{\alpha,-1}}} G_{q_{\alpha,0}}(\alpha) \xleftarrow{F^{-1}_{q^*_{\alpha,0}}} G_{q_{\alpha,1}}(\alpha) \xleftarrow{F^{-1}_{q^*_{\alpha,1}}} G_{q_{\alpha,2}}(\alpha)\ldots$$

$$L_{q_{\alpha,k},\alpha} \xrightarrow{F_{q_{\alpha,k}}} G_{q_{\alpha,k}}(\alpha) \xleftarrow{F_{q^*_{\alpha,k}}} G_{q^*_{\alpha,k}}(2),$$

$$0 < \alpha \leq 2, q \neq 1$$

Full description

$$\begin{array}{ccc} L_{q_{\alpha,k},\alpha} \xrightarrow{F_{q_{\alpha,k}}} & G_{q_{\alpha,k}}(\alpha) & \xleftarrow{F_{q^*_{\alpha,k}}} G_{q^*_{\alpha,k}}(2) \\ & \updownarrow F_{q_{\alpha,k}} & \\ & G_{q^L_{\alpha,k}}(2) & \end{array}$$

Again some parallels and applications

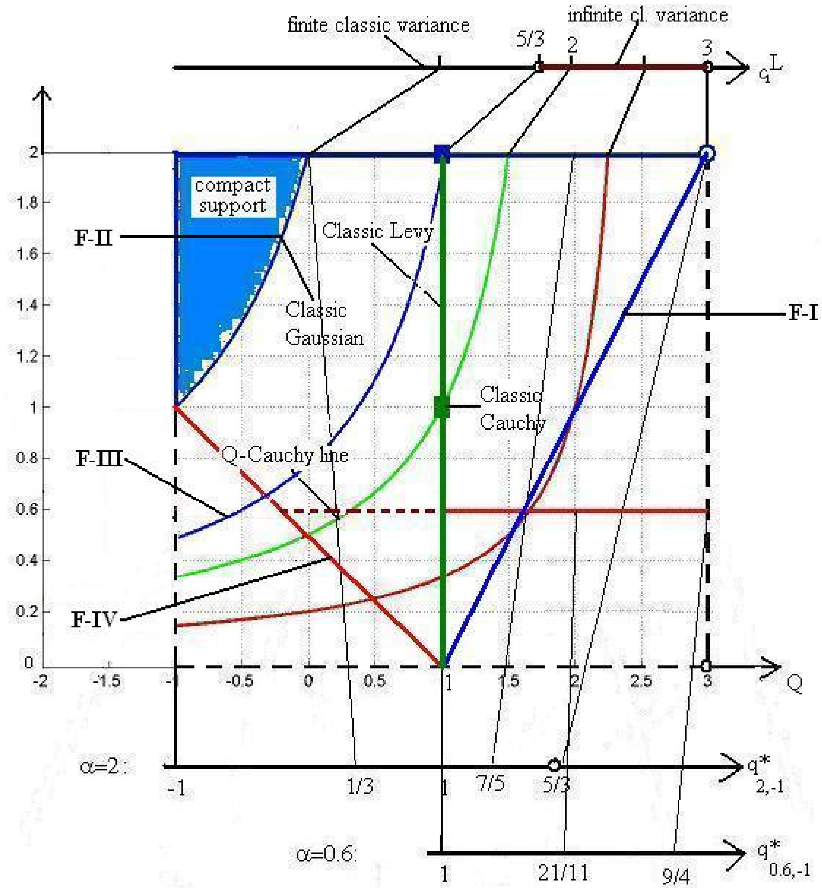
Triplet $(q^{*}_{\alpha,k-1}, q_{\alpha,k}, q^{*}_{\alpha,k+1})$

$$\delta = q^{*}_{\alpha,k+1} = \frac{2}{\alpha}\frac{2\alpha+(k-1)(1-Q)}{2\alpha+(k+1)(1-Q)}$$

Fractional F-P (conjecture) k=1:

$$\delta = q^{*}_{\alpha,2} = \frac{2}{\alpha+1-Q}, Q = 2q-1$$

Summary



finite classic variance
infinite cl. variance
5/3
2
3
q^L
compact support
F-II
Classic Levy
Classic Gaussian
F-I
Classic Cauchy
F-III
Q-Cauchy line
F-IV
Q
α=2:
-1
1/3
1
7/5
5/3
$q^*_{2,-1}$
α=0.6:
1
21/11
9/4
$q^*_{0.6,-1}$