The Linked Twist Map Approach to Fluid Mixing

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Statistical Mechanics and Dynamical Systems
Joint work with Steve Wiggins and Julio Ottino

Dynamical systems and fluids

Fluids	Dynamical systems
incompressible fluid	invertible, area-preserving dynamical system
Poincaré section	Discrete time map, $f : M \to M$
region of unmixed (stationary) fluid	invariant (periodic) set $f(A) = A$
islands forming barriers to mixing	KAM theory
“chaotic”	existence of a horseshoe

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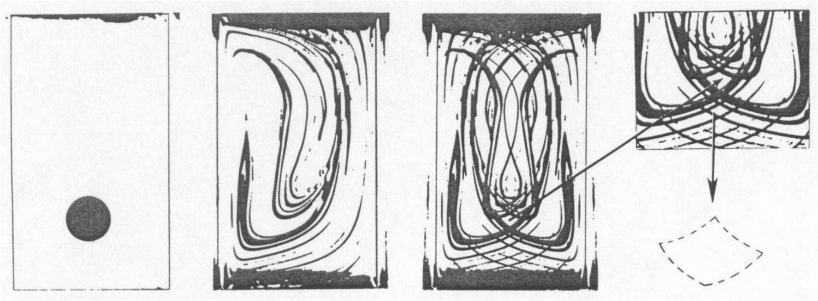
Horseshoes in fluids



from [Chien, Rising, Ottino, JFM **170** 355-77 (1986)]

Dynamical systems, ergodic theory and fluids

Topological

- topological space
- behaviour of individual trajectories
- dense orbit

Measure-theoretic

- measure space
- need an invariant measure — Lebesgue measure μ
- behaviour of sets of positive (or full) measure
- ergodicity

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f is **ergodic** if $\mu(A) = 0$ or 1 whenever $f(A) = A$.

Birkhoff ergodic thm $\implies$ “time averages = spatial averages”

Central notion is *indecomposability*

ergodicity $\implies$ “no islands of unmixed fluid"

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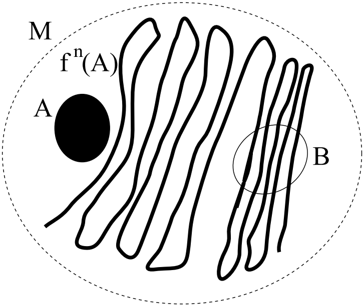
Mixing



Definition

f is **mixing** if

$$\lim_{n\to\infty} \frac{\mu(f^n(A)\cap B)}{\mu(B)} = \mu(A)$$

Intuitive definition is that upon iteration, sets become asymptotically independent of each other.

The Bernoulli property

Bernoulli means “statistically indistinguishable from coin tosses”

The Ergodic Hierarchy

Bernoulli $\Longrightarrow$ Mixing $\Longrightarrow$ Ergodicity

... plus lots more!

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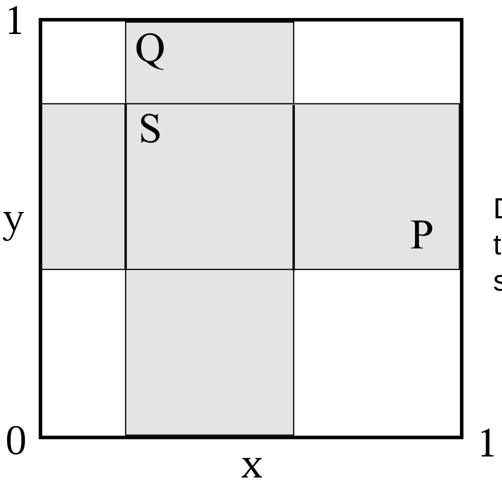
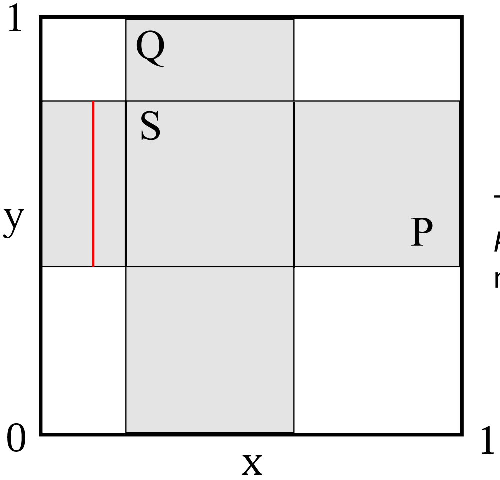
Linked Twist Maps on the torus



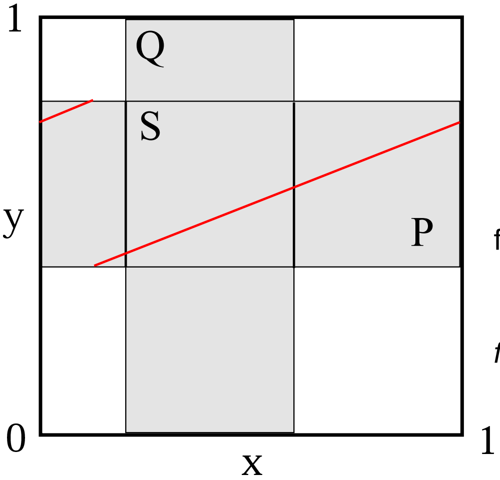
Define annuli P and Q on the torus $\mathbb{T}^2$ which intersect in region S.

Linked Twist Maps on the torus



The horizontal annulus P has a horizontal twist map....

Linked Twist Maps on the torus

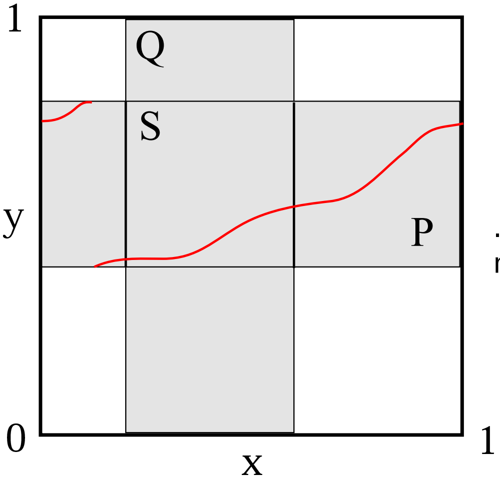


$$F(x, y) = (x + f(y), y)$$

for points in P

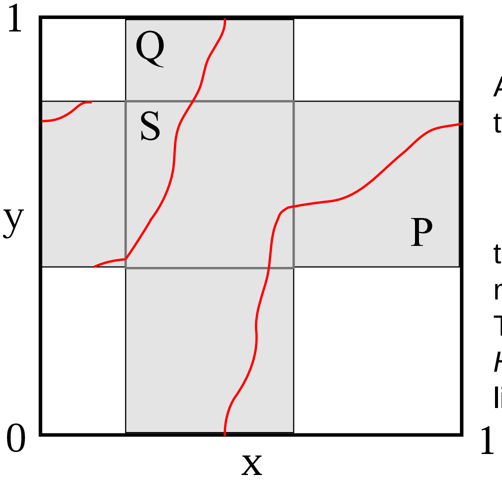
$f(y)$ could be linear...

Linked Twist Maps on the torus



...or not, but must be monotonic

Linked Twist Maps on the torus

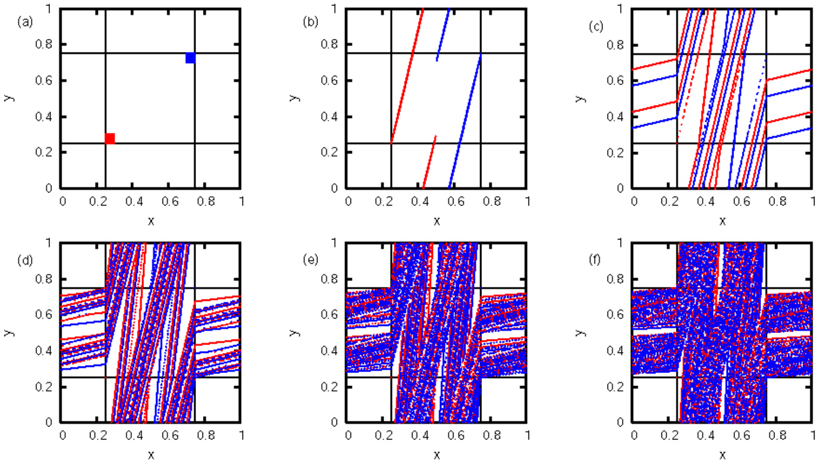


After F, apply a vertical twist

$$G(x, y) = (x, y + g(x))$$

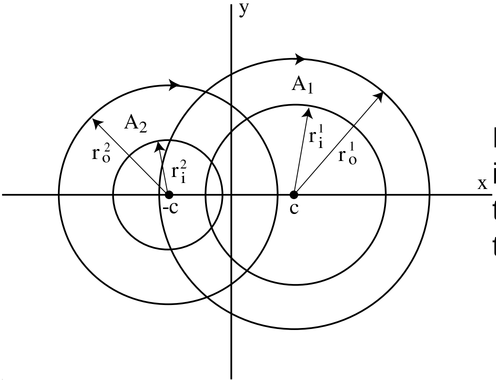
to points in Q. Again g must be monotonic.
The combined map $H(x, y) = G \circ F$ is the linked twist map.

Mixing properties of LTMs on the torus



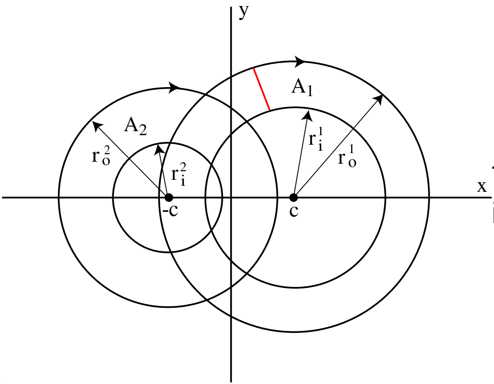
Proof of ergodic mixing due to Burton & Easton (1980), Devaney (1980), Wojtkowski (1980), Przytycki (1983)

Linked Twist Maps on the plane



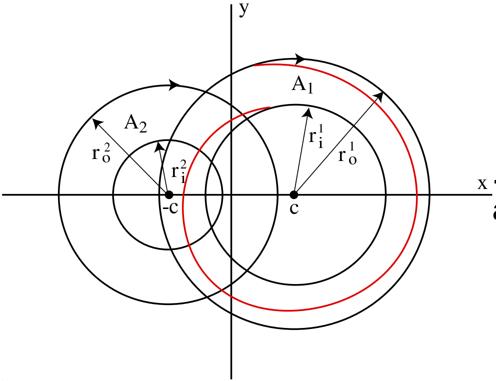
Domain is two intersecting annuli with two distinct regions of intersection

Linked Twist Maps on the plane



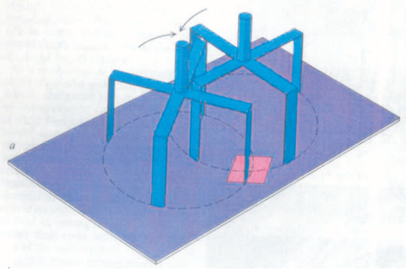
The action of a twist map is to take a line...

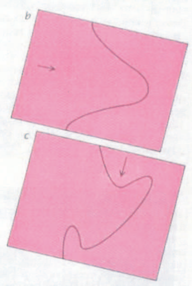
Linked Twist Maps on the plane



... and twist it around the annulus.

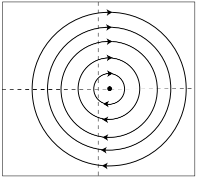
The Egg-Beater Flow

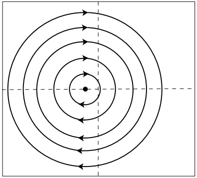




An egg-beater can be viewed as either linked twist map on the plane, or on the torus. from [Ottino, J, *Sci. Am.*, **260**, 56–67 (1989)]

The Blinking Vortex


Streamlines in the first half of the advection cycle


Streamlines in the second half of the advection cycle

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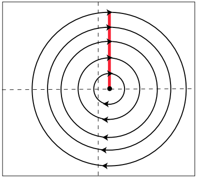

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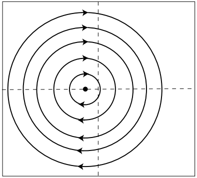

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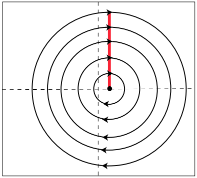

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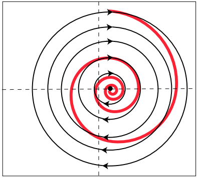

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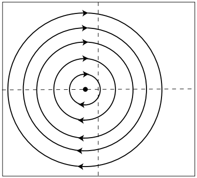

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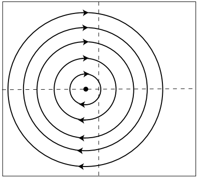

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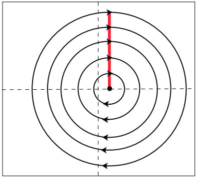

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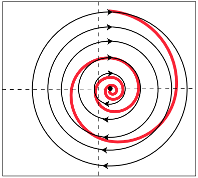

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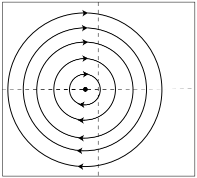

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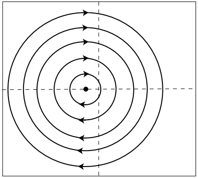

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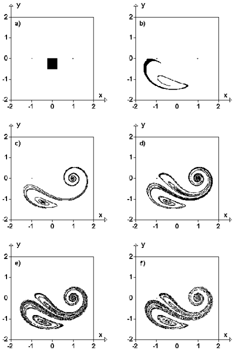
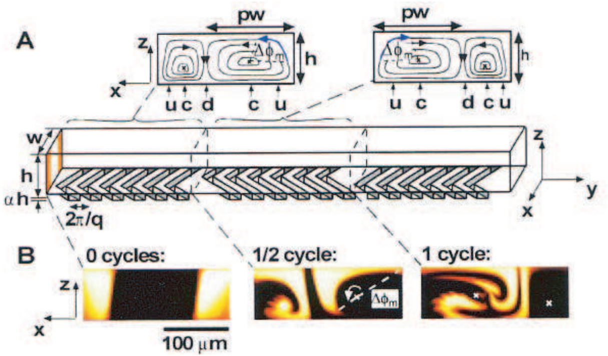
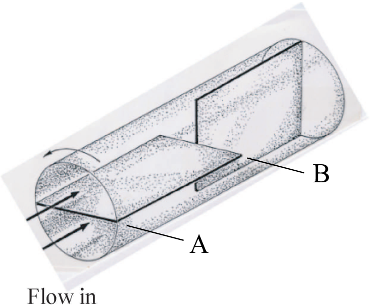


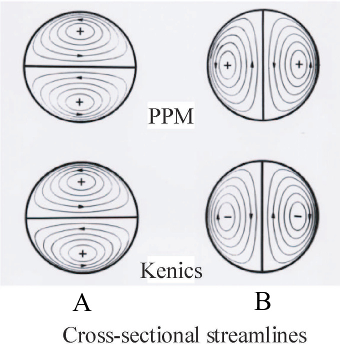
Microfluidics — patterned walls



from [Stroock, A. D. *et al.*, *Science* **295**, 647–651 (2002)]

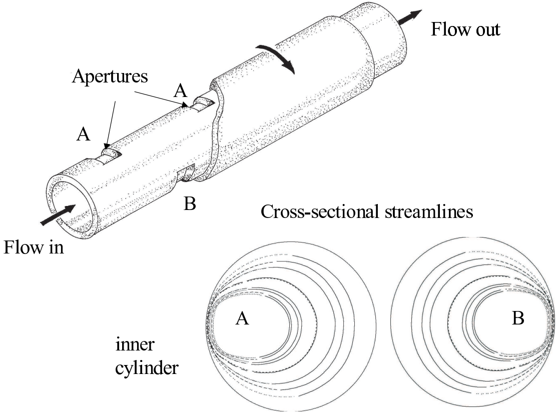
The Partitioned Pipe Mixer





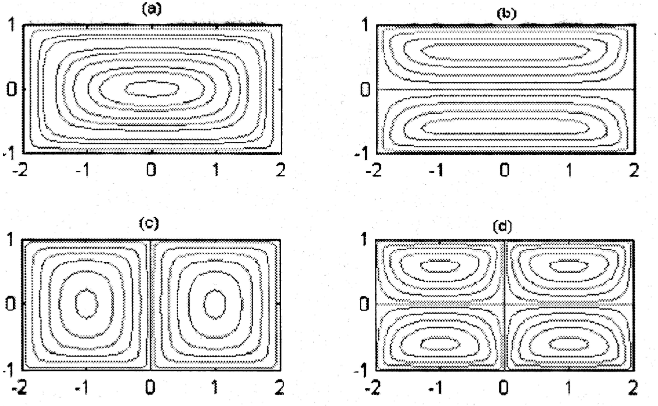
from [Khakar, D.V. *et al.*, *Chem. Eng. Sci.*, **42**, 2909–2926 (1987)]

The Rotated Arc Mixer



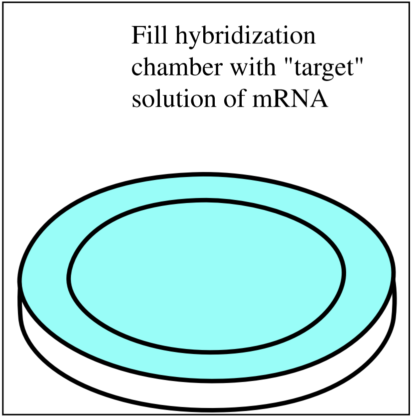
from [Metcalfe, G. *et al.*, *Am. Inst. Chem. Eng. Journal*, **52**(1), 9–38 (2006)]

Microfluidics — electroosmotic flow



from [Qian, S. & Bau, H. H., *Anal. Chem.*, **74**, 3616–3625 (2002)]

DNA Hybridization



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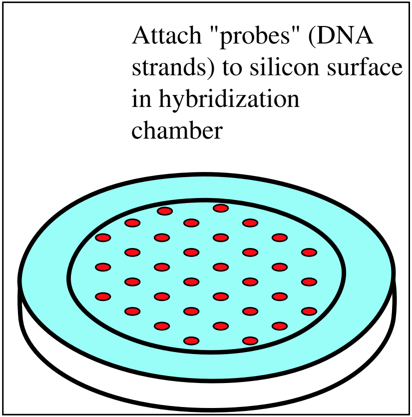

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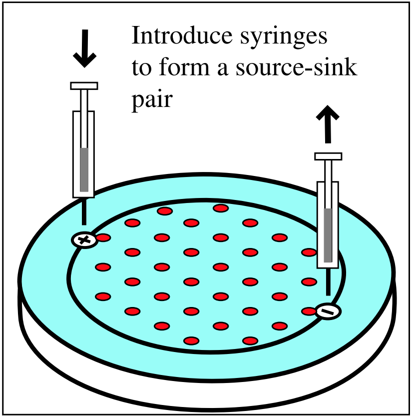

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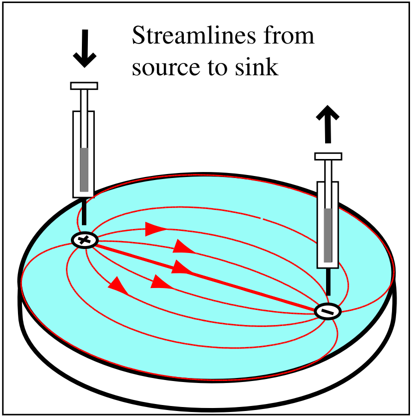

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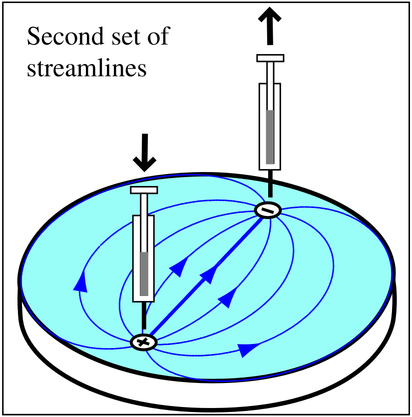

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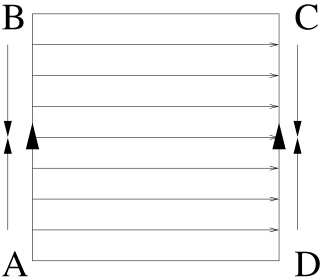

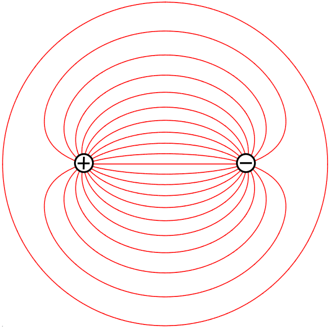

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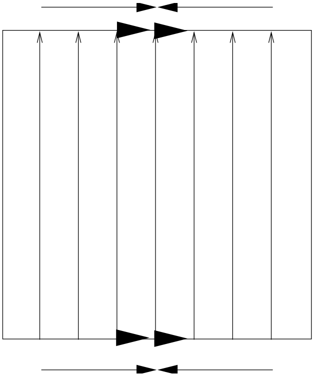

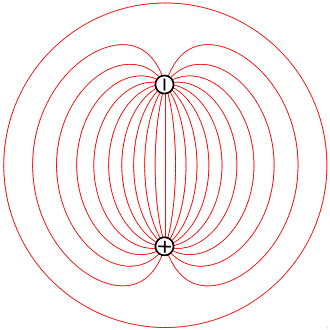

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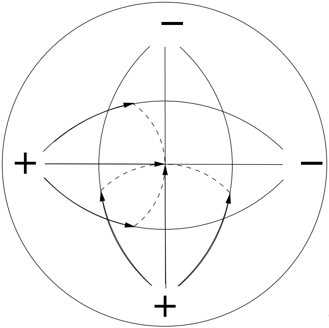
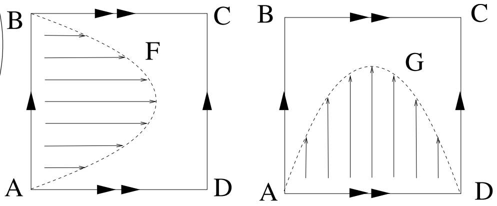


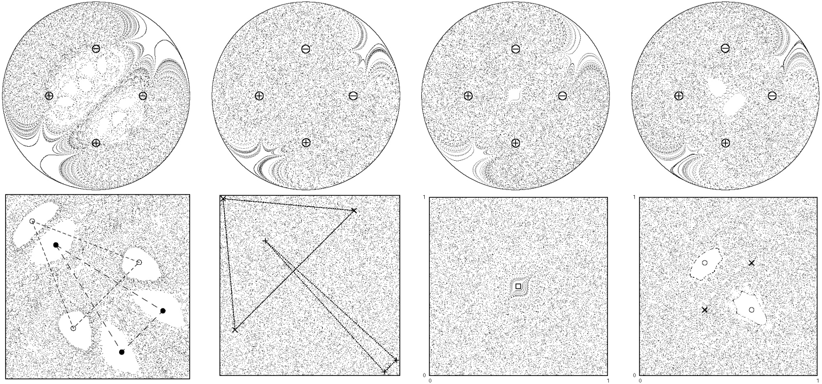
$$f(y) = ry(1-y)$$

$$g(x) = rx(1-x)$$



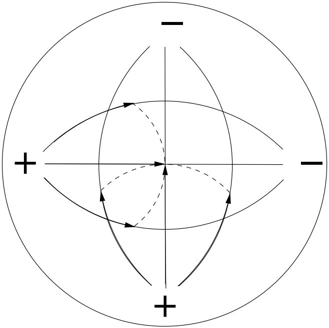
DNA Hybridization

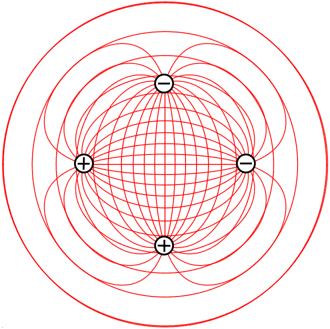
Short pumping time $\longrightarrow$ Long pumping time



from [J.M. Hertzsch, R. Sturman & S. Wiggins, 2006]

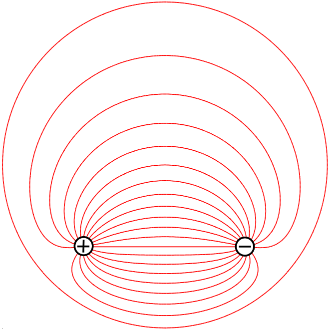
DNA Hybridization





DNA Hybridization

Off-centre sources and sinks



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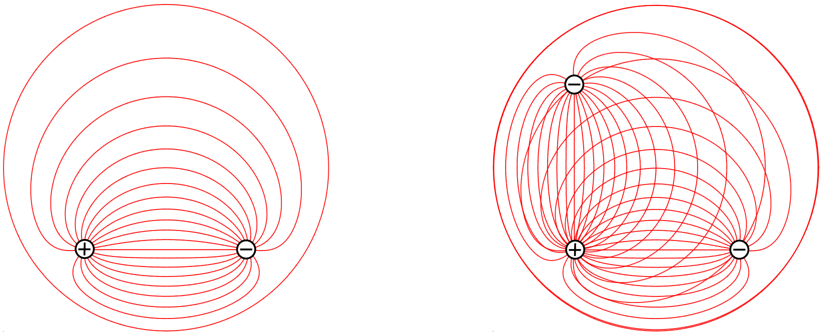

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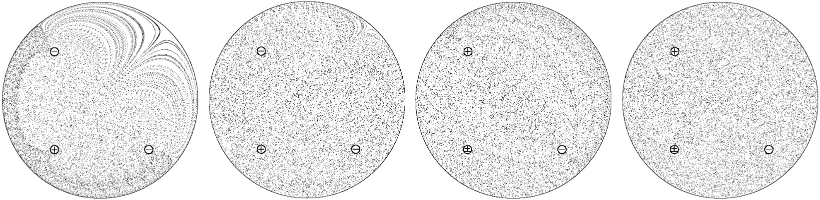
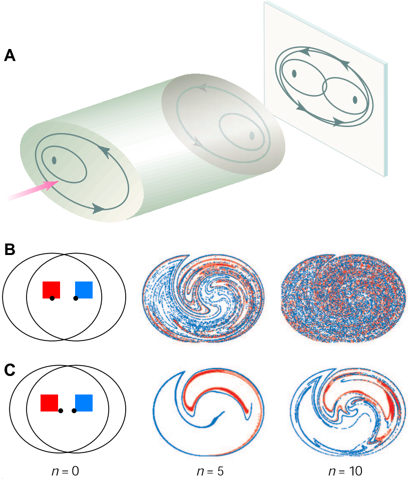


from [J.M. Hertzsch, R. Sturman & S. Wiggins, 2006]

Future Directions

- Monotonicity
- Transversality
- Speed of mixing
- Diffusion

Duct flows



- Schematic view of a duct flow with concatenated mixing elements
- Red and blue blobs of fluid mix well under a small number of applications
- Changing only the position of the centres of rotation can have a marked effect on the quality of mixing

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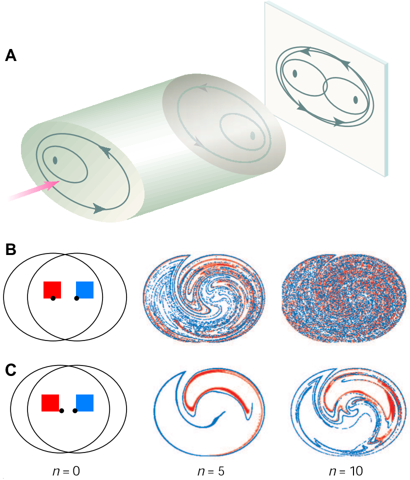
Duct flows



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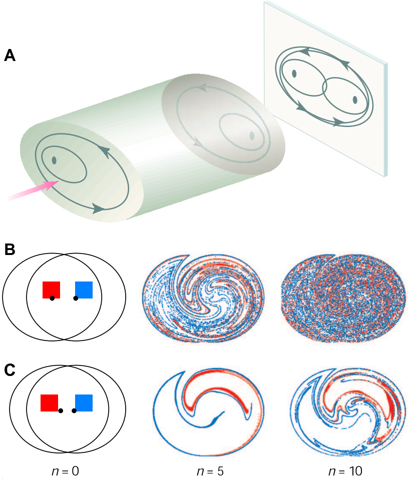

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