EQUIDISTRIBUTION AND DIRECTIONS IN HOMOLOGY FOR ANOSOV FLOWS

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Setting. $\phi_t : M \to M$ transitive Anosov flow (M a smooth compact Riemannian manifold).

The flow ϕ_t has

(1) topological entropy $h_{\rm top} > 0$;
(2) a countable infinity of prime periodic orbits γ, with period l_γ.

One may recover $h_{\rm top}$ from the periodic orbit data:

Margulis (1969): If ϕ_t is weak-mixing then

$$\#\{\gamma : l_\gamma \le T\} \sim \frac{e^{h_{\rm top}T}}{h_{\rm top}T}, \quad \text{as } T \to +\infty.$$

($\sim$ means that the ratio tends to 1).

There is a modified version if ϕ_t is not weak-mixing.

Examples.

(1) suspension of an Anosov diffeomorphism $f : N \to N$ (e.g. CAT map $\mathbb{T}^2 \to \mathbb{T}^2$): $M = N \times [0,1]/(x,1) \sim (fx,0)$. (modify the velocity to make it weak-mixing);
(2) the geodesic flow on the unit tangent bundle $M = SV$ over a negatively curved manifold V.

Homology.

$$H_1(M,\mathbb{Z})/\text{torsion} \cong \mathbb{Z}^d$$

(fixed isomorphism).

Assume $d \ge 1$.

For a periodic orbit γ define:

$[\gamma] \in \mathbb{Z}^d$ (homology class);
if $[\gamma] \neq 0$, $\theta(\gamma) := [\gamma]/\|[\gamma]\|_2 \in S^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ (direction).

Homology directions (Fried):

$$\mathcal{D}_\phi = \overline{\{\theta(\gamma) : \gamma \text{ periodic orbit, } [\gamma] \neq 0\}} \subset S^{d-1}.$$

(There is a more general definition for arbitrary flows.)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Topological view.

Theorem (Fried, 1982). *$\mathcal{D}_\phi$ contained in an open hemisphere $\implies$ ϕ_t is the suspension of an Anosov diffeomorphism (with a velocity change).*

Theorem (Sharp, 1993). *$\mathcal{D}_\phi$ not contained in a closed hemisphere $\iff$ $\mathcal{D}_\phi = S^{d-1}$ $\iff$ $\forall \alpha \in \mathbb{Z}^d\ \exists \gamma : [\gamma] = \alpha$. If the latter holds, we say that ϕ_t is homologically full.*

Furthermore, if ϕ_t is homologically full then $\exists 0 < h^* \le h_{\mathrm{top}}$:

$$\#\{\gamma : l_\gamma \le T,\ [\gamma] = \alpha\} \sim C_\alpha \frac{e^{h^* T}}{T^{1+d/2}}.$$

(For geodesic flows, $h^* = h_{\mathrm{top}}$.)

Measure theoretic view.

$$\nu_\phi^T := \frac{1}{\#\{\gamma : l_\gamma \le T, [\gamma] \neq 0\}} \sum_{\substack{l_\gamma \le T \\ [\gamma] \neq 0}} \delta_{\theta(\gamma)} \to ?, \text{ as } T \to +\infty.$$

μ_0 = measure of maximal entropy for ϕ_t.

Define $\Phi_{\mu_0} \in H_1(M, \mathbb{R})$ by its action on de Rham classes of closed 1-forms:

$$\Phi_{\mu_0}([\omega]) = \int \omega(X_\phi) d\mu_0$$

(well-defined).

Theorem (Collier & Sharp, 2006).

(1) *If $\Phi_{\mu_0} \neq 0$ then $\nu_\phi^T \to \delta_{\Phi_{\mu_0}/\|\Phi_{\mu_0}\|_2}$, as $T \to +\infty$ (weak*).*

(2) *If $\Phi_{\mu_0} = 0$ then $\nu_\phi^T \to \nu_\phi$, which is fully supported on $\mathcal{D}_\phi = S^{d-1}$. The measure ν_ϕ is given by an (inner product) norm $\|\cdot\|$ on $\mathbb{R}^d$:*

$$\nu_\phi(D) = \frac{\mathrm{Vol}_d(S(D) \cap B_{\|\cdot\|}(1))}{\mathrm{Vol}_d(B_{\|\cdot\|}(1))},$$

where $S(D)$ is the sector based on D and $B_{\|\cdot\|}(1) = \{x \in \mathbb{R}^d : \|x\| \le 1\}$.

More generally, if $\Phi_{\mu_0} = 0$ and $A \subset \mathbb{Z}^d$ has a density with respect to $\|\cdot\|$ then

$$\lim_{T \to +\infty} \frac{\#\{\gamma : l_\gamma \le T,\ [\gamma] \in A\}}{\#\{\gamma : l_\gamma \le T\}}$$

exists and is equal to the $\|\cdot\|$-density of A. (Petridis and Risager have shown this in the special case of closed geodesics on hyperbolic surfaces.)

References

1. D. Collier and R. Sharp, *Directions and equidistribution in homology for periodic orbits*, preprint (2006).
2. Y. Petridis and M. Risager, *Equidistribution of geodesics in homology classes and analogues for free groups*, preprint (2006).