On critical behaviour of coupled map lattices

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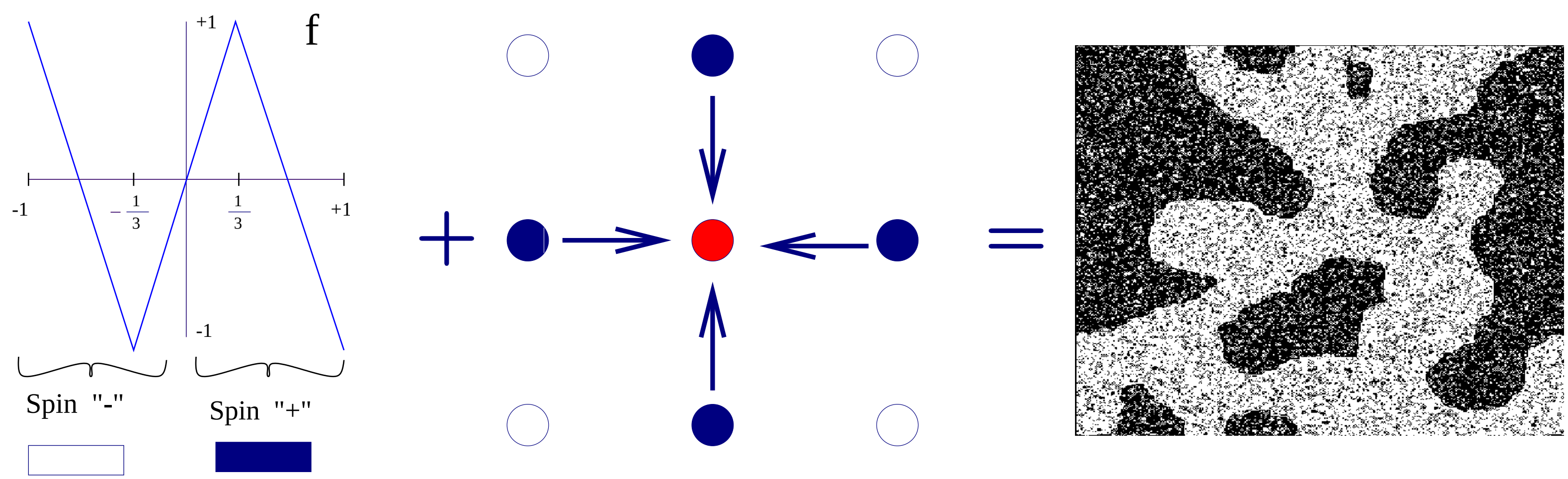
0 Content

1 Introduction

1.1 Phase transition in CMLs

Miller Huse model (⟶ J.Miller and D.A.Huse '93, PRE **48** 2528)

$$x_{n+1}^{(\nu_1,\nu_2)} = (1-\varepsilon) f\left(x_n^{(\nu_1,\nu_2)}\right) + \varepsilon/4 \sum_{\mu\in n.n.} f\left(x_n^{(\mu_1,\mu_2)}\right)$$

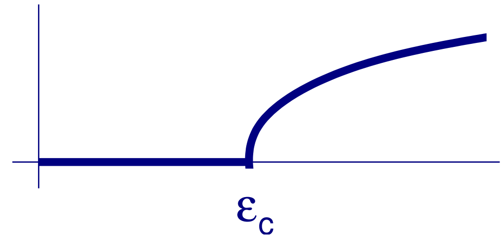


Ising-like phase transition for strong coupling $\varepsilon \geq \varepsilon_c$ in the limit of large system size

1.2 Critical behaviour

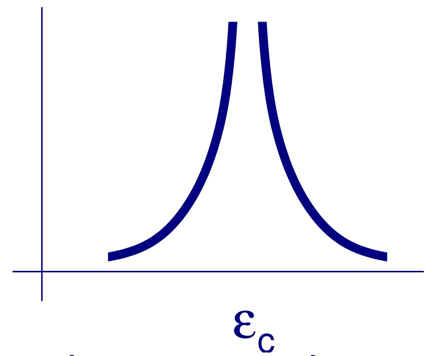
Magnetisation ($\sigma^{(\nu)} = \mathrm{sgn}(x^{(\nu)})$)

$$\bar{M} = \sum_\nu \sigma^{(\nu)}/L^2, \qquad m_L(\varepsilon) = \langle |\bar{M}| \rangle$$



Finite size susceptibility

$$\chi_L(\varepsilon) = L^2(\langle \bar{M}^2 \rangle - \langle |\bar{M}| \rangle^2)$$



Scaling relations and critical exponents ($L \to \infty$)

- $m_\infty(\varepsilon) \sim (\varepsilon - \varepsilon_c)^{\beta}$
- $\chi_\infty(\varepsilon) \sim |\varepsilon - \varepsilon_c|^{-\gamma}$
- $\xi(\varepsilon) \sim |\varepsilon - \varepsilon_c|^{-\nu}$

Universality classes

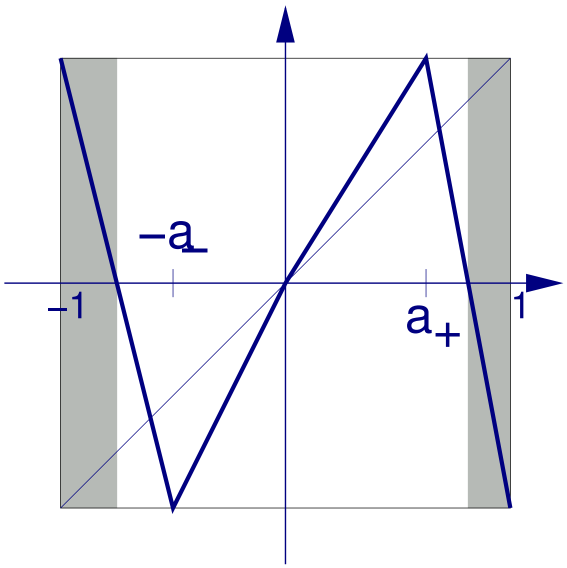
($\longrightarrow$ P.Marqc et al '97, PRE **55** 2606)

	2D Ising	Miller Huse	MH (async.)
β	1/8	0.111	0.126
γ	7/4	1.55	1.79
ν	1	0.887	1.02
β/ν	1/8	0.125	0.117
γ/ν	7/4	1.748	1.76
$(2\beta+\gamma)/\nu$	2	2.00	2.01

2 CML vs PCA

2.1 Symbolic dynamics for CMLs

Piecewise linear model with 2D n.n. symbolic coupling:



$$\sigma^{(\nu)} = \mathrm{sgn}\left(x^{(\nu)}\right), \qquad \nu = (\nu_1, \nu_2)$$

$$\Sigma^{(\nu)} = \sum_{\mu \in n.n.} \sigma^{(\mu)}$$

$$a_{\sigma^{(\nu)}} = a_{\sigma^{(\nu)}}\left(\Sigma^{(\nu)}\right)$$

Single site transition rates $\sigma^{(\nu)} \rightarrow \tau^{(\nu)}$

$$w^{(\nu)}\left(\sigma^{(\nu)} \rightarrow \tau^{(\nu)}\right) = \left(1 + \sigma^{(\nu)}\tau^{(\nu)} a_{\sigma^{(\nu)}}\left(\Sigma^{(\nu)}\right)\right)/2$$

$p_n(\underline{\sigma})$ probability for symbol state $\underline{\sigma} = \left(\sigma^{(\nu)}\right)$ at time n.

Master equation

$$p_{n+1}(\underline{\tau}) = p_n(\underline{\tau}) + \sum_{\underline{\sigma}} \left(W(\underline{\tau};\underline{\sigma})p_n(\underline{\sigma}) - W(\underline{\sigma};\underline{\tau})p_n(\underline{\tau})\right)$$

$$W(\underline{\tau};\underline{\sigma}) = \prod_{\nu} w^{(\nu)}\left(\sigma^{(\nu)} \rightarrow \tau^{(\nu)}\right)$$

Probabilistic cellular automaton (simultaneous updates).

($\longrightarrow$ G.Gielis and R.S.MacKay '00, Nonl. **13** 867)

2.2 Detailed balance

$$0 = W(\underline{\tau};\underline{\sigma})p_*(\underline{\sigma}) - W(\underline{\sigma};\underline{\tau})p_*(\underline{\tau})$$

Kolmogorov criterion (⟶ A.Georges and L.P.Doussal '89, JSP **54** 1011)

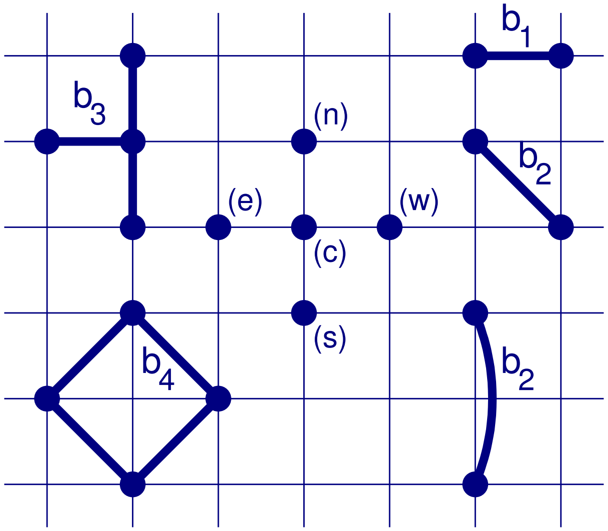
$$a_{\sigma^{(\nu)}}\left(\Sigma^{(\nu)}\right) = \tanh\left(J_0 + J_{nn}\sigma^{(\nu)}\Sigma^{(\nu)}\right)$$

Stationary density

$$p_*(\underline{\sigma}) = \prod_\nu \cosh\left(J_0 + J_{nn}\sigma^{(\nu)}\Sigma^{(\nu)}\right)/Z = \exp(-H(\underline{\sigma}))$$

interactions

$$b_k \sim J_{nn}^k$$



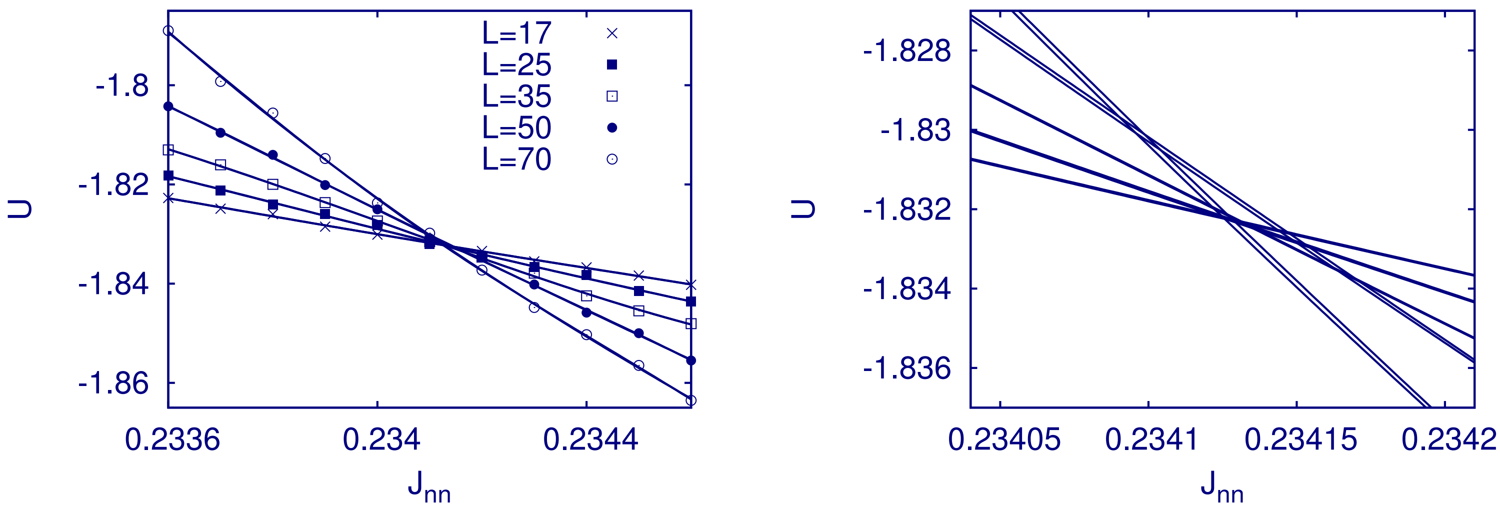
- Ising-type phase transition in the strong coupling limit $J_{nn} \geq J_c \approx 0.2203\ldots$.
- Detailed balance (reversibility): Hamiltonian with short range coupling (2D Ising universality)
- Violation of detailed balance: non Ising critical behaviour? (⟶ G.Grinstein et al '85, PRL **55** 2527)

3 Finite size scaling

3.1 Equilibrium model

Binder cumulant (determination of J_c)

$$U_L(J_{nn}) = \frac{\langle \bar{M}^4 \rangle - 3\langle \bar{M}^2 \rangle^2}{\langle \bar{M}^2 \rangle^2} = U(L^{1/\nu}(J_{nn} - J_c)) \ (= f(\xi/L))$$



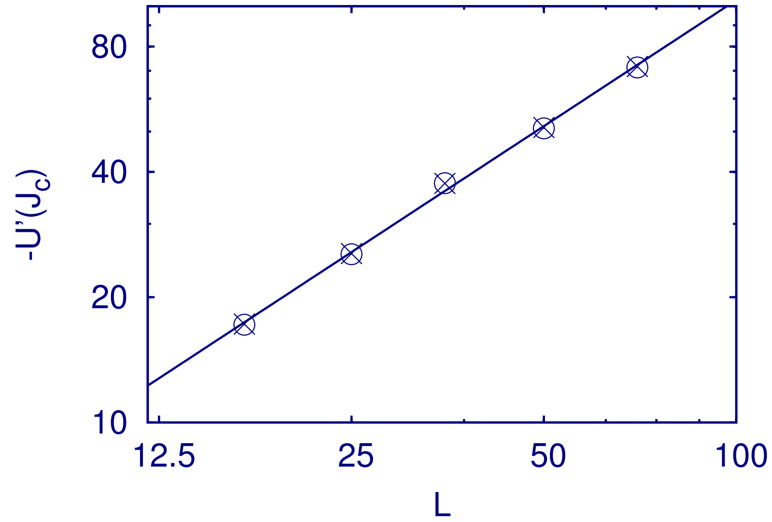
critical coupling $J_c = 0.23413 \pm 0.00003$ (for $J_0 = 1.25$).

Critical exponents

$$\partial U/\partial J_{nn}|_{J_c} \sim L^{1/\nu}$$

$$\nu = 0.994 \pm 0.015$$

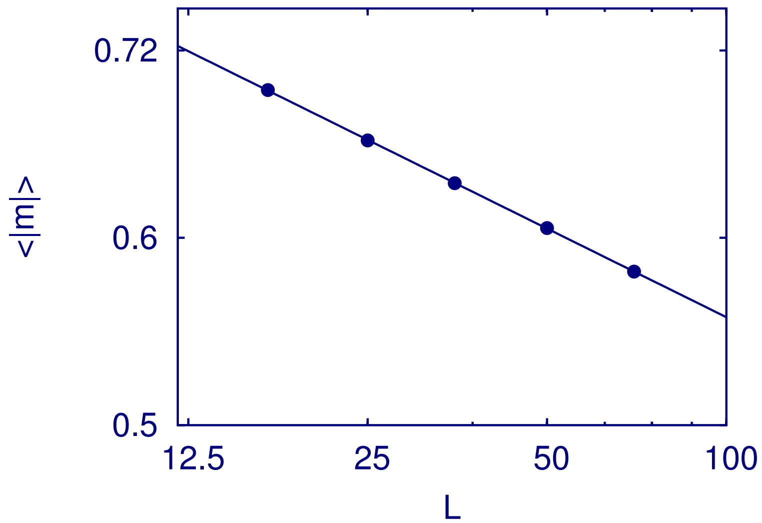
(2D Ising: 1)



$$\langle|\bar{M}|\rangle(J_c) \sim L^{-\beta/\nu}$$

$$\beta/\nu = 0.1245 \pm 0.0005$$

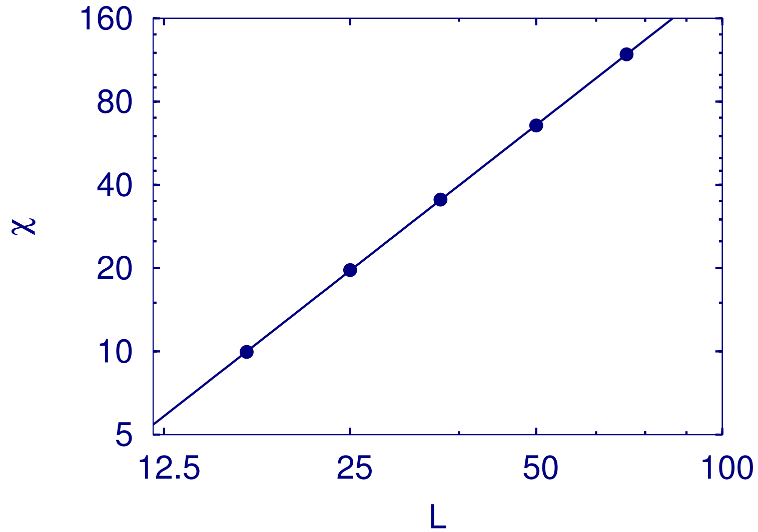
(2D Ising: $1/8$)



$$\chi(J_c) \sim L^{\gamma/\nu}$$

$$\gamma/\nu = 1.748 \pm 0.0003$$

(2D Ising: $7/4$)



3.2 Toom PCA

$$\sigma_{n+1}^{(\nu)} = \begin{cases} \text{sgn}\left(\Sigma_n^{(\nu)}\right) & \text{with prob. } (1+\varepsilon)/2 \\ -\text{sgn}\left(\Sigma_n^{(\nu)}\right) & \text{with prob. } (1-\varepsilon)/2 \end{cases}$$

2D "north-east" coupling

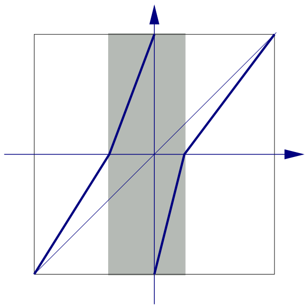
$$\Sigma^{(\nu)} = \sigma^{(\nu_1,\nu_2)} + \sigma^{(\nu_1+1,\nu_2)} + \sigma^{(\nu_1,\nu_2+1)}$$

- phase transition at $\varepsilon_c \approx 0.8222$ (⟶ D.Makowiec '99, PRE **60** 3787)
-

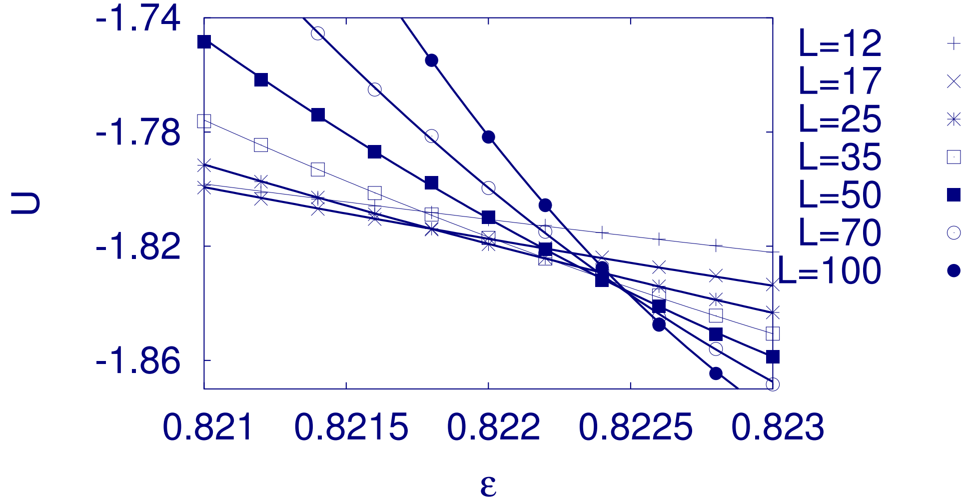
ν	β	γ
0.85	0.12	1.59

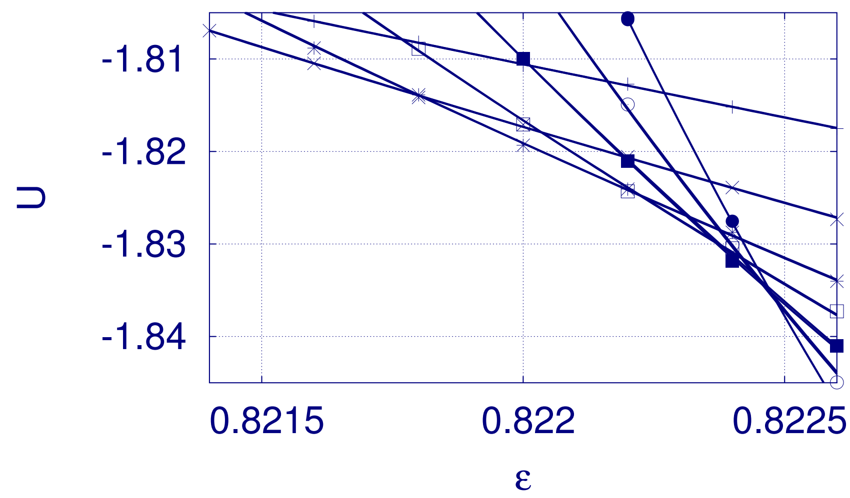
CML

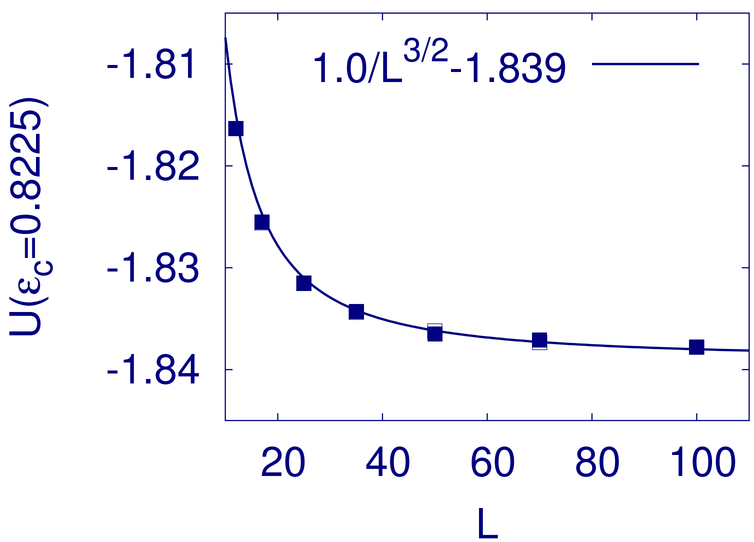
$$a_{\sigma^{(\nu)}}\left(\Sigma^{(\nu)}\right) = \left(1 - \varepsilon\sigma^{(\nu)}\text{sgn}\left(\Sigma^{(\nu)}\right)\right)/2$$



Binder cumulant







$$U_L(\varepsilon) = U(L^{1/\nu}(\varepsilon - \varepsilon_c)) + c/L^{\alpha}$$

$$\varepsilon_c = 0.82250 \pm 0.00001$$

$$\nu = 0.979 \pm 0.014$$

$$\alpha = 1.5$$

Critical exponents

	$(\varepsilon_c = 0.8225)$	$(\varepsilon_c = 0.8222)$	TCA	2D Ising
β	0.122	0.128	0.12	1/8
γ	1.68	1.618	1.59	7/4
ν	0.979 ± 0.014	0.917 ± 0.007	0.85 ± 0.02	1
β/ν	0.1243 ± 0.0002	0.140 ± 0.002	0.139	1/8
γ/ν	1.717 ± 0.009	1.764 ± 0.014	1.857	7/4
$(2\beta + \gamma)/\nu$	1.967	2.044	2.135	2