B-FIELDS, GERBES AND GENERALIZED GEOMETRY

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PART 2

(i) PREHOMOGENEOUS VECTOR SPACES

(ii) INVARIANT VOLUMES

(iii) GENERALIZED VARIATIONAL PROBLEMS

(iv) A 5-DIMENSIONAL EXAMPLE

The geometry of three-forms in six dimensions, J. Differential Geometry **55** (2000), 547 – 576.

Stable forms and special metrics, in "Global Differential Geometry: The Mathematical Legacy of Alfred Gray", M. Fernández and J. A. Wolf (eds.), Contemporary Mathematics **288**, American Mathematical Society, Providence (2001).

Generalized Calabi-Yau manifolds, Quart. J. Math. Oxford, **54** (2003) 281–308.

A Gerasimov and S Shatishvili

Towards integrability of topological strings I. Three-forms on Calabi-Yau manifolds **hep-th/0409238**

R Dijkgraaf, S Gukov, A Neitzke and C Vafa

Topological M-theory as Unification of Form Theories of Gravity **hep-th/0411073**

V Pestun and E Witten

The Hitchin functionals and the topological B-model at one loop **hep-th/0503083**

OPEN ORBITS

- Lie group G

- representation space V

- open orbit $U \subset V$

- relatively invariant polynomial $f : V \to \mathbf{R}$ $(f(gv) = \chi(g)f(v))$

PREHOMOGENEOUS VECTOR SPACES

- M Sato and T Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.

- Tatsuo Kimura, "*Introduction to prehomogeneous vector spaces*". Translations of Mathematical Monographs, **215**. American Mathematical Society, Providence, RI, 2003.

	Complex group	Representation	Stabilizer$_o$
1.	$H \times GL(m)$	$\rho \otimes \Lambda^1$	H
2.	$GL(n)$	$Sym^2\Lambda^1$	$O(n)$
3.	$GL(2m)$	Λ^2	$Sp((2)m)$
4.	$GL(2)$	$Sym^3\Lambda^1$	1
5.	$GL(6)$	Λ^3	$SL(3) \times SL(3)$
6.	$GL(7)$	Λ^3	G_2

	Complex group	Representation	Stabilizer$_o$
7.	$GL(8)$	Λ^3	$Ad(SL(3))$
8.	$SL(3) \times GL(2)$	$Sym^2\Lambda^1 \otimes \Lambda^1$	1
9.	$SL(6) \times GL(2)$	$\Lambda^2 \otimes \Lambda^1$	$SL(2)^3$
10.	$SL(5) \times GL(3)$	$\Lambda^2 \otimes \Lambda^1$	$PSL(2)$
11.	$SL(5) \times GL(4)$	$\Lambda^2 \otimes \Lambda^1$	1
12.	$SL(3)^2 \times GL(2)$	$\Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1$	$GL(1) \times GL(1)$

	Complex group	Representation	Stabilizer_o
13.	$Sp(2n) \times GL(2m)$	$\Lambda^1 \otimes \Lambda^1$	$Sp(2m) \times Sp(2n-m)$
14.	$GL(1) \times Sp(6)$	$\Lambda^1 \otimes \Lambda^3$	$SL(3)$
15.	$SO(n) \times GL(m)$	$\Lambda^1 \otimes \Lambda^1$	$SO(m) \times SO(n-m)$
16.	$GL(1) \times Spin(7)$	$\Lambda^1 \otimes S$	G_2
17.	$GL(2) \times Spin(7)$	$\Lambda^1 \otimes S$	$SL(3) \times SO(2)$
18.	$GL(3) \times Spin(7)$	$\Lambda^1 \otimes S$	$SL(2) \times SO(3)$

	Complex group	Representation	Stabilizer_o
19.	$GL(1) \times Spin(9)$	$\Lambda^1 \otimes S$	$Spin(7)$
20.	$GL(2) \times Spin(10)$	$\Lambda^1 \otimes S$	$SL(2) \times G_2$
21.	$GL(3) \times Spin(10)$	$\Lambda^1 \otimes S$	$SO(3) \times SL(2)$
22.	$GL(1) \times Spin(11)$	$\Lambda^1 \otimes S$	$SL(5)$
23.	$GL(1) \times Spin(12)$	$\Lambda^1 \otimes S$	$SL(6)$
24.	$GL(1) \times Spin(14)$	$\Lambda^1 \otimes S$	$G_2 \times G_2$

	Complex group	Representation	Stabilizer$_o$
25.	$GL(1) \times G_2$	$\Lambda^1 \otimes \Lambda^2$	$SL(3)$
26.	$GL(2) \times G_2$	$\Lambda^1 \otimes \Lambda^2$	$GL(2)$
27.	$GL(1) \times E_6$	$\Lambda^1 \otimes \Lambda^1 (= 27)$	F_4
28.	$GL(2) \times E_6$	$\Lambda^1 \otimes \Lambda^1 (= 27)$	$SO(8)$
29.	$GL(1) \times E_7$	$\Lambda^1 \otimes \Lambda^6 (= 56)$	E_6
30.	$GL(1) \times Sp(2n) \times SO(3)$	$\Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1$	$Sp(2n-4) \times SO(2) \cdot U(2n-3)$

RIEMANNIAN HOLONOMY GROUPS

(M Berger, 1955)

$O(n) \qquad U(n) \qquad Sp(n) \qquad SU(n)$

$Sp(n) \cdot Sp(1) \qquad G_2 \qquad Spin(7)$

RIEMANNIAN HOLONOMY GROUPS

(M Berger, 1955)

$O(n)$(15) $U(n)$(15) $Sp(n)$(13) $SU(n)$(15)

$Sp(n) \cdot Sp(1)$(13) G_2(25) $Spin(7)$(16)

G transitive on $S^{n-1} \Rightarrow G \times \mathbf{R}^*$ has an open orbit on $\mathbf{R}^n$

NON-METRIC HOLONOMY GROUPS

(S Merkulov and L Schwachhöfer, 1999)

$(4), (5), (14), (15), (23), (27), (29)$

HOW DO WE TURN THIS INTO GEOMETRY?

OPEN ORBITS OF EXTERIOR FORMS

- (3): $GL(2m, \mathbf{R})$ on Λ^2 ($\deg f = m$ (Pfaffian))
- (5): $GL(6, \mathbf{R})$ on Λ^3 ($\deg f = 4$)
- (6): $GL(7, \mathbf{R})$ on Λ^3 ($\deg f = 7$)
- (7): $GL(8, \mathbf{R})$ on Λ^3 ($\deg f = 16$)
- (14): $Sp(6, \mathbf{R}) \times \mathbf{R}^*$ on Λ^3 ($\deg f = 4$)

- scalar matrices $\lambda I \in GL(n, \mathbf{R})$

- action on Λ^p is λ^p

- $\phi = |f|^{n/(p \deg f)}$

- $\phi : \Lambda^p \to \Lambda^n$

- invariant volume form

- $\rho \in U \subset \Lambda^p$ open orbit

- $\phi : U \to \Lambda^n$ invariant map, homogeneous of degree n/p

- derivative $D\phi : \Lambda^p \to \Lambda^n$

- $D\phi(\dot{\rho}) = \widehat{\rho} \wedge \dot{\rho}$

- $\widehat{\rho} \wedge \rho = (n/p)\phi$

complementary form $\widehat{\rho} \in \Lambda^{n-p}$

EXAMPLE (3): SYMPLECTIC

- $\rho = \omega \in \Lambda^2$
- $\phi(\omega) = \omega^m$
- $\widehat{\rho} = m\omega^{m-1}$

VARIATIONAL PROBLEM

- functional: $V = \int_M \phi(\rho)$
- restrict ρ to lie in a fixed de Rham cohomology class
- vary $\rho \mapsto \rho + d\varphi$ (open orbit if φ small)
- look for critical points

BASIC EQUATION

$$d\rho = d\widehat{\rho} = 0$$

- (3) symplectic manifold
- (5) Calabi-Yau threefold
- (6) Riemannian manifold with holonomy G_2

OPEN ORBITS OF SPINORS

- (23): $\mathbf{R}^* \times Spin(6,6)$ on S
- $\dim S = 32$, $\deg f = 4$, stabilizer $SU(3,3)$
- (24): $\mathbf{R}^* \times Spin(7,7)$ on S
- $\dim S = 64$, $\deg f = 8$, stabilizer $G_2 \times G_2$

BASIC SCENARIO

- manifold M^n

- replace T by $T \oplus T^*$

- inner product of signature (n, n)

  $(X + \xi, X + \xi) = -i_X \xi$

- forms = spinors, Mukai pairing $\langle \rho_1, \rho_2 \rangle \in \Omega^n$

OPEN ORBITS OF SPINORS

(23): $\mathbf{R}^* \times Spin(6,6)$ on S

NJH: *Generalized Calabi-Yau manifolds*

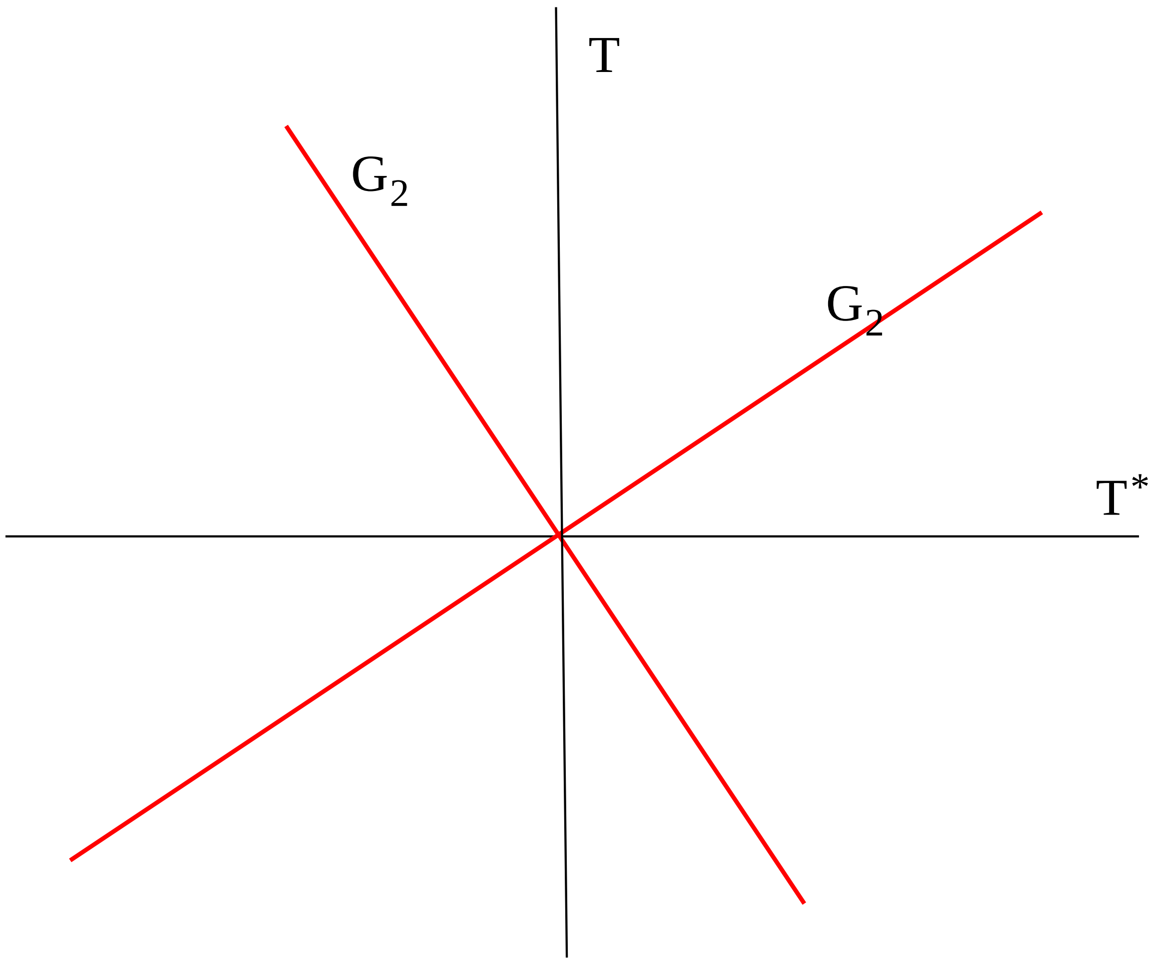
math.DG/0209099 (QJM **54** (2003) 281–308)

(24): $\mathbf{R}^* \times Spin(7,7)$ on S

F Witt: *Generalized G_2-manifolds*

math.DG/0411642

GENERALIZED G2 GEOMETRY



GENERALIZED G2 STRUCTURE

- Riemannian connections $\nabla^{\pm}$ with *skew torsion* $\pm H \in \Omega^3$

- unit spinors $\epsilon^{\pm}$, covariant constant $\nabla^{\pm}\epsilon^{\pm} = 0$

- function Φ : $(d\Phi \pm H) \cdot \epsilon^{\pm} = 0$

- M compact $\Rightarrow$ ordinary G_2 structure, $H = 0$, $\Phi = const.$

- $T \oplus T^* = V \oplus V^\perp$.

- reflection $R(v) = v$ on V, $R(v) = -v$ on $V^\perp$

- Lift $R \in O(n,n)$ to $\tilde{R} \in Pin(n,n)$

- $\tilde{R} : \Omega^* \mapsto \Omega^*$ *generalized Hodge star operator* $*$

- $\widehat{\rho} = *\rho$

ANOTHER OPEN ORBIT OF SPINORS

- (20): $GL(2) \times Spin(5,5)$ on $\mathbf{R}^2 \otimes S$
- $\dim S = 16$, $\deg f = 4$
- stabilizer $SL(2) \times G_2^{(2)}$
- **Special geometric structure on 5-manifolds**

- $SL(2) \times G_2^{(2)} \mapsto SO(1,2) \times SO(4,3) \subset SO(5,5)$
-
-
-
-

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-

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- $T \oplus T^* = V \oplus V^{\perp}$, V trivial, rank 3

- $S = \mathbf{R}^2 \otimes \mathbf{R}^8 = S(V) \otimes S(V^{\perp})$

-

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- $SL(2) \times G_2^{(2)} \mapsto SO(1,2) \times SO(4,3) \subset SO(5,5)$

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- $S = \mathbf{R}^2 \otimes \mathbf{R}^8 = S(V) \otimes S(V^\perp)$

- $\mathbf{R}^2 \otimes S = \mathbf{R}^2 \otimes \mathbf{R}^2 \otimes S(V^\perp)$

-

- $SL(2) \times G_2^{(2)} \mapsto SO(1,2) \times SO(4,3) \subset SO(5,5)$
- $T \oplus T^* = V \oplus V^{\perp}$, V trivial, rank 3
- $S = \mathbf{R}^2 \otimes \mathbf{R}^8 = S(V) \otimes S(V^{\perp})$
- $\mathbf{R}^2 \otimes S = \mathbf{R}^2 \otimes \mathbf{R}^2 \otimes S(V^{\perp})$
- $SL(2, \mathbf{R}) \times G_2^{(2)}$ stabilizes $\epsilon \otimes \varphi$

GEOMETRIC PROBLEM

- 5-manifold M

- pair of closed forms $\rho \in \Omega^{ev} \otimes \mathbf{R}^2$

- What is the interpretation of $d\rho = 0 = d\hat{\rho}$?

THE INVARIANT FUNCTIONAL

- spinor (Mukai) pairing $\Omega^{ev} \otimes \Omega^{od} \to \Lambda^5 T^*$

  $\langle \varphi, \psi \rangle = \varphi_0 \psi_5 - \varphi_2 \psi_3 + \varphi_4 \psi_1$

- Define $P(\varphi, \varphi) \in (T \oplus T^*) \otimes \Lambda^5 T^*$ by

  $(P(\varphi, \varphi), X + \xi) = \langle (X + \xi) \cdot \varphi, \varphi \rangle$

- $(P(\varphi, \varphi), P(\psi, \psi)) \in (\Lambda^5 T^*)^2$

- *Volume*: $\phi = (P(\varphi, \varphi), P(\psi, \psi))^{1/2}$

- $\rho = (\rho_1, \rho_2) \in S \otimes \mathbf{R}^2$
-
-
-

- $\rho = (\rho_1, \rho_2) \in S \otimes \mathbf{R}^2$
- $P(\rho_1 + z\rho_2, \rho_1 + z\rho_2) = \phi \otimes [v_+ + 2zh + z^2 v_-]$
-
-

- $\rho = (\rho_1, \rho_2) \in S \otimes \mathbf{R}^2$

- $P(\rho_1 + z\rho_2, \rho_1 + z\rho_2) = \phi \otimes [v_+ + 2zh + z^2 v_-]$

- v_+, h, v_- span V

-

- $\rho = (\rho_1, \rho_2) \in S \otimes \mathbf{R}^2$

- $P(\rho_1 + z\rho_2, \rho_1 + z\rho_2) = \phi \otimes [v_+ + 2zh + z^2 v_-]$

- v_+, h, v_- span V

- $v \in V$, $v \cdot \rho = a(v)\widehat{\rho}$, $a(v) \in \mathfrak{sl}(2, \mathbf{R})$

- $d\rho = 0 = d\widehat{\rho}$
-
-
-

- $d\rho = 0 = d\widehat{\rho}$
- $d(v \cdot \rho) = 0, d\rho = 0$
-
-

- $d\rho = 0 = d\widehat{\rho}$
- $d(v \cdot \rho) = 0, d\rho = 0$
- $2[A, B] \cdot \alpha = d((A \cdot B - B \cdot A) \cdot \alpha) + 2A \cdot d(B \cdot \alpha) -$
  $-2B \cdot d(A \cdot \alpha) + (A \cdot B - B \cdot A) \cdot d\alpha$ (Courant bracket)
-

- $d\rho = 0 = d\hat{\rho}$

- $d(v \cdot \rho) = 0, d\rho = 0$

- $2[A, B] \cdot \alpha = d((A \cdot B - B \cdot A) \cdot \alpha) + 2A \cdot d(B \cdot \alpha) -$
  $-2B \cdot d(A \cdot \alpha) + (A \cdot B - B \cdot A) \cdot d\alpha$ (Courant bracket)

- $\Rightarrow [v, w] = 0$ if $v, w \in V$

- 3-dimensional space of sections of $T \oplus T^*$
- Courant-commuting
- $v = X + \xi$, X is volume-preserving
- (v, v) is constant, signature $(2, 1)$

GENERIC NORMAL FORM

$$\begin{aligned}\rho_1 &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 \\ \rho_2 &= 1 + dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5\end{aligned}$$

$$V = \left\{ \frac{\partial}{\partial x_5} + dx_1, \quad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2} + dx_2 \right\}$$

EVOLUTION EQUATIONS FOR G_2

- metric on forms in a fixed cohomology class, $\dot{\rho} = d\sigma$

  $(\dot{\rho}, \dot{\rho}) = \int_M \dot{\rho} \wedge \sigma$

- "gradient flow" of $V(\rho)$

$$\frac{\partial \rho}{\partial t} = d\widehat{\rho}$$

- $\Rightarrow$ holonomy $Spin(7)$ on $\mathbf{R} \times M^7$

EVOLUTION EQUATIONS FOR GENERALIZED G_2

- metric on forms in a fixed cohomology class, $\dot{\rho} = d\sigma$

  $(\dot{\rho}, \dot{\rho}) = \int_M \langle \dot{\rho}, \sigma \rangle$

- "gradient flow" of $V(\rho)$

$$\frac{\partial \rho}{\partial t} = d\hat{\rho}$$

- $\Rightarrow$ generalized $Spin(7)$ on $\mathbf{R} \times M^7$

GENERALIZED $Spin(7)$

- $Spin(7) \times Spin(7)$ structure
- two cases: ρ odd or even
- $\nabla^+, \nabla^-, H, \epsilon^+, \epsilon^-, \Phi$

F Witt: *Generalized G_2-manifolds* **math.DG/0411642**

J Gauntlett, D Martelli & D Waldram, *Superstrings with intrinsic torsion* Phys. Rev. D (3) **69** (2004), no. 8, 086002, 27 pp.

EVOLUTION EQUATIONS FOR $Spin(5,5)$?

Courant-Nahm equations: $v_i = X_i + \xi_i$

$$\frac{\partial v_1}{\partial t} = [v_2, v_3], \quad \frac{\partial v_2}{\partial t} = -[v_3, v_1], \quad \frac{\partial v_3}{\partial t} = -[v_1, v_2]$$