Geometric Transitions, CY Integrable Systems, and Open GW Invariants

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* Geometric transitions + int. sys
  hep-th/0506196
  Diaconescu, Dijkgraaf, –, Hofman, Pantev
* Geometric transitions + MHS.
  hep-th/050619?
  Diaconescu, –, Grassi, Pantev
* Hitchin systems + twisted complexes,
  in prep
  Diaconescu, –, Pantev

Geometric transitions

$$X_\mu \rightsquigarrow X_0 \longleftarrow \tilde{X}$$

X_μ : family of (complex strs. on) CY's
X_0 : a singular CY in the family
$\tilde{X}$: its small resolution, still CY, contains some exceptional 2 cycles $\mathbb{P}^1 \approx S^2$.

Large N duality:

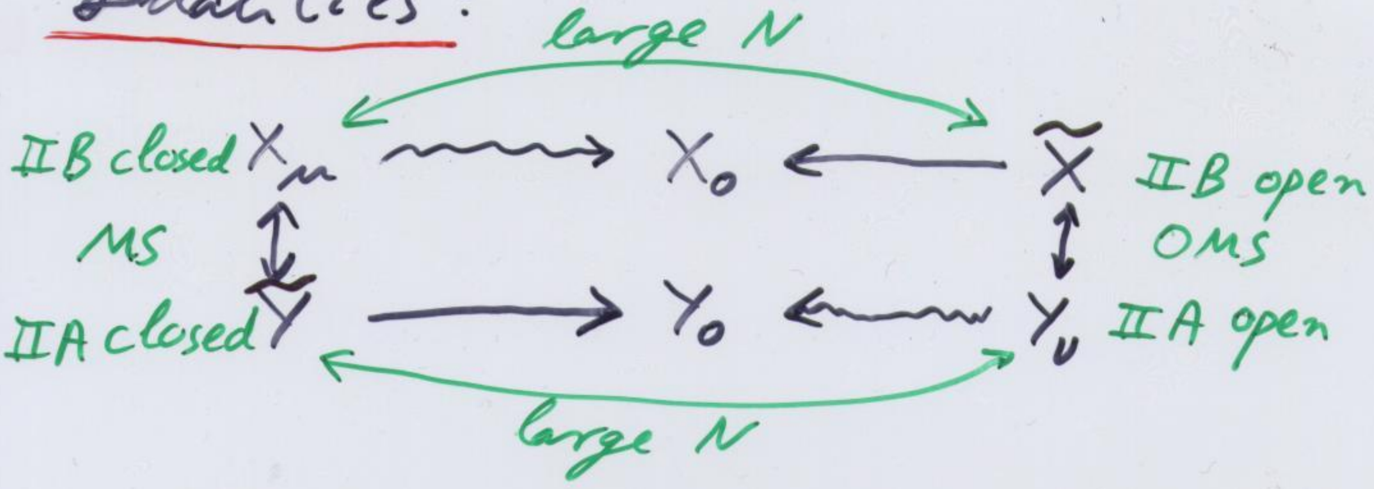
closed strings on X_μ $\Longleftrightarrow$ open strings on $\tilde{X}$

This involves:

$$\lim_{N \to \infty} \mathcal{M}_N ,$$

where $\mathcal{M}_N$ is a quantum moduli space of N branes on $\tilde{X}$.

Dualities:



LHS : Mirror symmetry

IIB on X_μ period integrals on cycles in X_μ	$\Leftrightarrow$	IIA on $\tilde{Y}$ closed string GW invariants on $\tilde{Y}$

RHS: Open Mirror Symmetry (MS w/ branes)

integrals on chains in $\tilde{X}$ bounding the vanishing cycles	$\Leftrightarrow$	Open string GW invariants, count holo maps from R.S. w/ boundary to Y_ν, boundary $\to L_\nu$

Mirror of the exceptional 2-cycles $\mathbb{P}^1 \approx S^2 \subset \tilde{X}$ are SLAG vanishing 3-cycles $L_\nu \subset Y_\nu$.

Large N duality interchanges left & right.
Sometimes, allows "calculation" of open GW invariants.

[D+V] relate B-model topological strings on local CYs, via large N duality, to <u>matrix models</u>.

Typical picture:

$$\begin{array}{ccc} X_a = X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \mathbb{C}_x & \xrightarrow{\;W'\;} & \mathbb{C}_z \end{array}$$

$Z \subset \mathbb{C}^4 : -y^2 + uv + z^2 = 0$

$X \subset \mathbb{C}^4 : -y^2 + uv + W'(x)^2 = 0$

$W = W_a(x) = \sum_{i=0}^{n+1} a_i x^i =$ superpotential

X : CY3, singular at n points $\Big\|\; u = v = y = 0,\ W'(x) = 0.$

$\Omega_X = \frac{du\, dx\, dy}{u} = \dots$ holo 3form.

When $a = 0$, X is singular along a curve

$C \simeq \mathbb{C}_x : \{u = v = y = 0\}$.

X has a family of holomorphic spheres $= \mathbb{P}^1$

When $a \neq 0$, have transversally holomorphic family of (non-holomorphic) S^2's

Integrate Ω_X on these S^2's $\Rightarrow$ holo 1-form ω on $\mathbb{C}_x$.

$W(x) := \int_{x_0}^{x} \omega$: <u>classical superpotential</u>.

[DV] picture:

Combine superpotential deformations, X_a with smoothing deformations, X_μ:

$$X_{a,\mu} : -y^2 + uv + W'(x)^2 + f_\mu(x) = 0$$

$$f_\mu(x) = \sum_{i=0}^{n-1} \mu_i x^i$$

From matrix models, they get a quantized superpotential $W = W_{a,\mu}(\tilde{x})$

$\tilde{x}$: coordinate on the hyperelliptic curve

$$\tilde{C}_{a,\mu} : y^2 = W'(x)^2 + f_\mu(x)$$

Coefficients of W w.r.t. special coordinates ⇒ open string GW invariants on mirror Y_v.

Large N duality $\Rightarrow$ expansion in μ
(actually, in special coordinates equiv't to μ)

0-th order:

$$\lim_{\mu \to 0} \int_{\Gamma_\mu} \Omega_{X_\mu} = \int_{\Gamma} \Omega_X$$ (Clemens-Schmid)

Γ : 3-chain in X, $\partial\Gamma = \Sigma$ (exceptional $\mathbb{P}$'s)
Γ_0 : its image in X_0, a 3-cycle.
Γ_μ : its deformation to 3-cycle in X_μ.

Natural interpretation of W:
it is a "normal function", i.e.
a section of the family of
intermediate Jacobians $\mathcal{J}(X_\mu)$.

Want: behavior of $\mathcal{J}(X_\mu)$ near the transition, $\mu \to 0$.

a la [DV], we'll study this for
$\mathcal{J}(X_{a,\mu})$ near $a = \mu = 0$.

$$\begin{array}{ccccc} \widetilde{S} & c & \widetilde{m} & & \\ \downarrow & & \downarrow & & \\ S & \overset{a}{c} & m & \overset{\mu}{c} & L \end{array}$$

Our result: "proof" of genus 0 large N duality for (certain) B-model transitions.

* closed string ⟷ CY integrable system

"at $g=0$" ⟺ to 1st order,

CYIS ⟷ Hitchin I.S.

math

* open string ⟷ holo CS

↓

generalized matrix model

↓ (large N planar limit)

Same H.I.S.

phys

* **Main geometric prediction** of large-N duality: special geom. on L can be reconstructed from sheaf theory on CY's in $\widetilde{M}$.
* **More detailed:** L is foliated $\overline{\pitchfork}$ S, spec. geom. on leaf L_x through $x \in S$ is determined by a moduli problem in $D^b(\widetilde{X})$.
* **Linearization:** $\Gamma \subset SL_2(\mathbb{C})$ corresponds to the singularity type. V: rank 2, Γ-equiv't VB on C, $\det V = K_C$.
  Linearized $X := Tot(V)/\Gamma$.
  $\widetilde{X}$ = blowup contains ruled surface F.
* **Linearized foliation:** $\widetilde{M}$ is a VB of rank = rank(G) over S, L is a VB of rank = $(g-1)\cdot \dim G$ over S, $M = \widetilde{M}/W$ is singular along S.

A simple compact example: (cf. [KMP])

$X_{a,\mu} = Q \cap R$

Q = quadric $\subset \mathbb{P}^5$
R = quartic $\subset \mathbb{P}^5$

$a = \mu = 0 \Leftrightarrow \text{rank}(Q) = 3 \Leftrightarrow \text{Sing}(Q) = \mathbb{P}^2 \subset \mathbb{P}^5$

$\Rightarrow \text{Sing}(X_{a,\mu})$ = plane quartic $\Rightarrow$ X has 1-param family of $\mathbb{P}^1$'s
$g=3\ C \subset \mathbb{P}^2$

$\mu = 0 \Leftrightarrow \text{rank}(Q) \leq 4 \Leftrightarrow \text{Sing}(Q) \supset \mathbb{P}^1 \subset \mathbb{P}^5$

$\Rightarrow \text{Sing}(X_{a,\mu}) \supset$ 4 points $\Leftrightarrow a \in H^0(C, K_C)$.

rank(Q) = 5,6 $\Rightarrow$ X (generically) n.s.

rank(Q)	# param's Q	$h^{2,1}$
3	14	83
4	17	86
6	20	89

$S \overset{a}{\subset} m \overset{\mu}{\subset} L$

83 86 89

a-parameters = 86 - 83 = 17 - 14 = 3
μ-parameters = 89 - 86 = 20 - 17 = 3.

Integrable system:

$$\begin{array}{ccc} T & \longrightarrow & X \\ & & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

everything is algebraic (or: analytic),

T: complex tori.

(X, σ): holo. symplectic variety.

$\pi : X \to B$ holomorphic Lagrangian fibration.

$\sigma|_T \equiv 0$

$\dim T = \dim B = \frac{1}{2} \dim X$.

Example 1:

X compact Riemann surface

Hodge: $H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$

$\Rightarrow$ Jacobian $J(X) = H^1(X, \mathcal{O}) / (H^{1,0} + H^1(X, \mathbb{Z}))$

is an algebraic torus.

Example 2:

X compact Kahler 3fold

Hodge: $H^3(X, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$

Intermediate Jacobian:

$$J(X) = H^3(X, \mathbb{C}) / (H^{3,0} + H^{2,1} + H^3(X, \mathbb{Z}))$$

is a polarized, non-algebraic torus.

E.g. X = Calabi-Yau 3fold,

(i.e. $\Omega^3_X \approx \mathcal{O}$, $H^1(X, \mathbb{C}) = 0$)

$\Rightarrow$ signature of period is $(1, h^{21})$.

Q: When is a family of complex tori Lagrangian?

$B \underset{\text{open}}{\subset} V^*$, $V \approx \mathbb{C}^g$.

$\pi : X \to B$ family of complex tori, with <u>period</u> <u>map</u>:

$$p : B \longrightarrow (\mathrm{Sym}^2 V)_{n.d}$$

<u>The cubic condition</u> [D, Markman]

TFAE:

① $\exists$ complex symplectic σ on X s.t.
$\pi : (X, \sigma) \to B$ is Lagrangian
σ induces identity: $T_{X/B} \longrightarrow \pi^* T^*_B$, with $T_{X/B} \cong \pi^* V$ and $\pi^* T^*_B \cong \pi^* V$

② $p : B \to \mathrm{Sym}^2 V$ is (locally in B) the Hessian of a holomorphic function on B: "prepotential"

③ $dp_b \in \mathrm{Hom}(T_B, \mathrm{Sym}^2 V) \approx V \otimes \mathrm{Sym}^2 V$
actually lives in: $\mathrm{Sym}^3 V \subset V \otimes \mathrm{Sym}^2 V$.

Calabi-Yau integrable system:

X : CY3, $\Omega^3_X = 0$

$\mathcal{M}$ = moduli space = {complex structures on X}/isom.

$$T_{[X]}\mathcal{M} = H^1(T_X) \approx H^1(\Omega^2_X) = H^{2,1}$$

(Bogomolov, Tian, Todorov : unobstructed)

$\widetilde{\mathcal{M}} \to \mathcal{M}$: natural $\mathbb{C}^*$-bundle

(choose : holo. volume form ω)

$\mathcal{J} \to \mathcal{M}$: universal int. Jacobian

$\widetilde{\mathcal{J}}' \to \widetilde{\mathcal{M}}'$: pullback.

$$\begin{array}{ccc} \widetilde{\mathcal{J}}' & \to & \mathcal{J} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}' & \to & \mathcal{M} \end{array}$$

[D+M, '94] : $\widetilde{\mathcal{J}} \to \widetilde{\mathcal{M}}$ is an analytically integrable system.

* fibers $\mathcal{J}(X)$ are Lagrangian
* the image of any Abel-Jacobi map is isotropic.

The cubic = **Yukawa's**:

$$\otimes^3 H^1(T_X) \to H^3(\Lambda^3 T_X) = H^3(\Omega_X^{-3}) \xrightarrow{\cdot\omega^2} H^3(\Omega^3_X) \xrightarrow{\int} \mathbb{C}$$

$\mathcal{X} \to B$ family of CY3's

$\forall b \in B$, $C_b = C_b^+ - C_b^-$: a 1-cycle in X_b, homologous to 0.

$\Rightarrow$ **Abel-Jacobi map** = "normal function"

$$AJ : B \to J(\mathcal{X}/B)$$
$$b \mapsto \int_{\Gamma_b} \in H^3(X_b, \mathbb{C}) / \ldots\ldots = J(X_b)$$

where Γ_b is a 3-chain in X_b, $\partial\Gamma_b = C_b$.

* Independent of choices.

Various extensions:

* C not null-homologous: replace $J(X)$ by Deligne cohomology group.

* Special case $\mathcal{X} = X \times B$:

$$AJ : \mathcal{H}ilb^0(X) \longrightarrow J(X)$$

* $\exists$ "transversally holomorphic" version: C is a real surface (non holomorphic) but it "varies holomorphically".

Other examples (from algebraic geom.)

S: complex symplectic surface

$C \subset S$: a holo. curve

$\Rightarrow$ short exact sequence

(*) $0 \to T_C \to T_S|_C \to N_{C/S} \to 0$

S symplectic $\Rightarrow N_C = \omega_C$ = canonical bundle

The SES (*) determines an extension class in:

$$\begin{aligned} Ext^1(N_{C/S}, T_C) &= H^1(N_{C/S}^{-1} \otimes T_C) \\ &= H^1(T_C^{\otimes 2}) \\ &= H^0(\omega_C^{\otimes 3})^* \longrightarrow Sym^3 H^0(C, \omega_C)^* \end{aligned}$$

$\Rightarrow$ A.I.S.

Base $= H^0(C, \omega_C) = H^0(C, N_{C/S}) \sim$ deformations of C in S

Fiber over C is $\mathcal{J}(C)$.

E.g. S = K3 (or T^4): Mukai's I.S.

Related to: symplectic structure on moduli spaces of vector bundles or coherent sheaves on K3.

Another example:

B = curve (= compact R.S.)

$S := T^*B$, holomorphically symplectic

$$\begin{array}{ccl} T^*B & \supset C & = \text{"spectral curve"} \\ \searrow & \swarrow & \\ & B & \left(\begin{array}{l}n\text{-sheeted} \\ \text{branched cover}\end{array}\right) \end{array}$$

$\Rightarrow$ **Hitchin's I.S.**

Base $\equiv \{C\} = H^0(S, \mathcal{O}(C)) \approx \bigoplus_{i=1}^{n} H^0(B, K_B^{\otimes i})$

Fiber over $C \equiv \mathcal{J}(C)$.

Total space = {Higgs bundles (V, φ) on B}

V: rank n vector bundle on B

$\varphi : V \to V \otimes \omega_B$: Higgs field

Variants:

* meromorphic Higgs bundles $\rightsquigarrow$ Markman's Poisson I.S.
  ($\varphi : V \to V \otimes \omega_B(D)$ for fixed D)
* Replace the LB by a $\mathbb{C}^*$-bundle $\Rightarrow$ Sklyanin's.
* Replace the LB by an elliptic fibration $\Rightarrow$ moduli spaces of bundles on elliptic fibr'n.
* Replace vector bundles by principal G-bundles

. . . . . .

Hitchin system: (for group G)

B : a curve

G: a reductive group

Total space: $\{$ Higgs bundles (V, φ)
V: G-bundle on B
$\varphi \in \Gamma(B, \mathrm{ad}\, V \otimes K_B)$ $\}$

Base $= \{ C \to B$ spectral cover $\}$
$= \oplus_{i=1}^{r} \Gamma(B, K_B^{\otimes d_i})$

$\{d_i\}$ = degrees of invariant polynomials for G.

Fiber over $[C]$ is a Prym variety

Prym (C/B),

roughly $J(C)/J(B)$.

Relevant cases for A_1 singularities:

$G = \begin{cases} SL_2(\mathbb{C}) \\ \mathbb{P}GL_2 = SL_2/(\pm) \end{cases}$

Spectral curves: $\boxed{W^2 = \beta}$

β: quadratic differential
w: multi valued differential

Our setup:

$X_{0,0}$: CY3 with curve C of singularities (say, of type G, e.g. simplest: A_1.)

$X_{a,0}$: CY3's with finite number n of singularities which can be resolved: $\tilde{X}_{a,0} \to X_{a,0}$.

$X_{a,\mu}$: smoothing of $X_{a,0}$.

$$\mathcal{S} \subset \tilde{\mathcal{M}} \to \mathcal{M} \subset \mathcal{L}$$

e.g.: 83 86 89

We want to understand CYIS ($\mathcal{L}$) near $a = \mu = 0$.

Claim: to first order,

$$\text{CYIS}(\mathcal{L}) \approx \text{CYIS}(\mathcal{S}) \times \text{Hitchin}(C, G).$$

In fact: $\exists$ family of IS's parametrized by $t \in \mathbb{C}$, s.t. for $t \neq 0$ get CYIS($\mathcal{L}$), for $t = 0$ get CYIS($\mathcal{S}$) $\times$ Hitchin(C, G).

2D analogue: [D, Ein, Lazarsfeld]

S = K3 surface
$\cup$
C = curve, $D \in |nC|$

Mukai's I.S. for line bundles on $D \subset S$ degenerates to:
Hitchin's I.S. for C, group $G = SL(n, \mathbb{C})$.

nilpotent cone in Mukai = sheaves supported on original C

is an affine twist of:
nilpotent cone in Hitchin $= \{(V, \varphi) \mid \varphi$ is nilpo$\}$

Data for the affine twist $\Leftrightarrow$ extension class encoded in n-th order neighborhood of C in S.

Idea: $\exists$ degeneration of S to $N_{C/S} = T^*C$ induces the degeneration of Mukai to Hitchin.

The deformation to normal cone for CY 3's:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X_{0,0} \\ \cup & & \cup \\ F & \longrightarrow & C \end{array} \quad \text{ruled surface}$$

$$N_{F\backslash X} = \mathrm{Tot}(K_F)$$

$$H^0(C, K_C) \approx H^1(F, K_F) \to H^{2,1}(X).$$

$$H^{1,0}(C) \xrightarrow{\sim} H^{1,0}(F) \longrightarrow H^{2,0}(X)$$

$$\begin{array}{ccccc} \tilde{S} & \subset & \tilde{\mathcal{M}} & & \\ \| & & \downarrow & & \\ S & \subset & \mathcal{M} & \subset & L \end{array}$$

$$\begin{array}{l} N_{S\backslash\tilde{\mathcal{M}}} = H^0(C, K_C) \otimes (\text{weights of } G) \\ \downarrow \\ N_{S\backslash L} = \oplus_{i=2}^{n} H^0(C, K_C^{\otimes i}) = \text{Hitchin base} \end{array}$$

The map is non-linear.

Hitchin base $\leftrightarrow$ spectral covers $\tilde{C} \xrightarrow{n} C$

Image of $N_{S\backslash\tilde{\mathcal{M}}}$ $\leftrightarrow$ completely reducible covers, $\tilde{C} = \cup_{i=1}^{n} C_i$.

Outline: geometric proof

- Identify Hitchin base:

$$B = \mathrm{Maps}\left(C, (\underline{t} \otimes K_C)/W\right)$$

- $\underline{t}/W$ parametrizes deformations of the surface $\mathbb{C}^2/\Gamma$, $\Gamma \subset SL_2(\mathbb{C})$. This is $\mathbb{C}^*$-equivariant.

- So get family $\mathcal{X} \to B$ of open CYs, each fibered over C with:

  fibers = deformations of $\mathbb{C}^2/\Gamma$.

- B also parametrizes the W-Galois cameral covers: $\widetilde{C}_b \to C$, and for each rep ρ of G, corresponding spectral covers: $\overline{C}_{b,\rho} \to C$.

$$\begin{array}{c} \widetilde{C} \\ \downarrow \\ B \times C \end{array}$$

- Everything pulls back from $(\underline{t} \otimes K_C)/W$ and (locally in C) from $\underline{t}/W$.

* Hitchin fibers are generalized Pryms, modelled on $H^1(C, \tilde{C} \times^W \Lambda_{roots})$.

* Int. Jacs are complex tori modelled on $H^3(X, \mathbb{Z})$ (or: H_3)

* Both cohomologies can be computed by Leray, boils down to two local systems over B, both pullback from $\underline{t}/W$: $\underline{t} \longrightarrow \underline{t}/W$ vs. W-orbits in $(\Lambda \times \underline{t})/W \longrightarrow \underline{t}/W$.

Holomorphic CS + twisted Higgs complexes

* wrap N topological B-branes (+ N antibranes) on exceptional curves of $\widetilde{X}_{\widetilde{m}}$

$$Q^+ = \Sigma_{a=1}^{N} \mathcal{O}_{F_{p_a}} \qquad Q^- = \Sigma_{b=1}^{N} \mathcal{O}_{F_{q_b}}$$

complex: $Q = Q^+ \oplus Q^-[-1]$
gives boundary topological B-model.

* Offshell string states:

$$\boxed{A = \oplus_{k=0}^{3} \oplus_{m,n \in \mathbb{Z}} \Omega_{\widetilde{X}}^{0,k}(E_m^\vee \otimes E_n)}$$

where $\mathcal{E} = E_m \hookrightarrow E_{m+1} \to \ldots$ is a locally free resolution of Q.

* $\widetilde{X}$ is total space of a LB over $F \Rightarrow$ convert bundles on $\widetilde{X}$ to Higgs bundles on F: $\boxed{A = \Omega_F \otimes \mathrm{End}(Q)}$,

$$\boxed{\begin{aligned}\Omega_F &= \oplus_{p=0}^{2} \oplus_{q=0}^{1} \Omega_F^{0,p} \otimes \Lambda^q N_{F/\widetilde{X}} \\ &= \oplus_{p=0}^{2} \oplus_{q=0}^{1} \Omega_F^{2q,p}\end{aligned}}$$

(dimensional reduction)

* Holomorphic CS action:

$$\phi = \phi^{0,1} + \phi^{2,0}$$

for ghost number $p+q=1$ fields:

$$S_{CS} = \int_F \mathrm{Tr}\left(\tfrac{1}{2}\phi\bar{\partial}\phi + \tfrac{1}{3}\phi^3\right)$$
$$= \int_F \mathrm{Tr}\left(\phi^{2,0} \wedge F^{0,2}\right)$$

$F^{0,2}$ = (0,2) part of curvature of deformed connection $A + \phi^{0,1}$

* Extends to open strings specified by a complex E, via construction of twisted complexes / Bondal-Kapranov, Lazaroiu

- DG category of VBs on F with $\Lambda^{0,\cdot}$-valued maps
- shift extension
- twisted complexes (MC)

$$\Psi = \sum \psi^{q,p}_{n,m} \qquad \psi^{q,p}_{n,m} \in \Omega_c^{q,p}(E_m^\vee \otimes E_n)$$

action:

$$\int_S \mathrm{Tr}\left(\psi^{10}_{01} \bar\partial \psi^{00}_{10} + \psi^{10}_{12} \bar\partial \psi^{00}_{21} + \psi^{11}_{02} \psi^{00}_{21} \psi^{00}_{10} \right)$$

$\Rightarrow$ EOM:

$$\bar\partial_{10} \psi^{00}_{10} = 0 \qquad \psi^{00}_{21} \psi^{00}_{10} = 0$$

$$\bar\partial_{21} \psi^{00}_{21} = 0 \qquad \psi^{10}_{01} \psi^{00}_{10} = 0$$

$$\bar\partial_{01} \psi^{10}_{01} + \psi^{11}_{02} \psi^{00}_{21} = 0 \qquad \psi^{00}_{10} \psi^{10}_{01} - \psi^{10}_{12} \psi^{00}_{21} = 0$$

$$\bar\partial_{21} \psi^{10}_{21} + \psi^{00}_{10} \psi^{11}_{02} = 0 \qquad \psi^{00}_{21} \psi^{10}_{12} = 0$$