Geometric Cauchy problems for surfaces associated to harmonic maps

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Credits: parts of this work are collaborations with Josef Dorfmeister, Martin Svensson and Peng Wang

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Geometric Cauchy problems
A classical problem
Björling’s problem for minimal surfaces:

Prescribed normal field on curve \(\rightarrow\) Unique minimal surface
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Schwarz formula:

\[f(z) = \Re\left\{ \alpha(z) - i \int_{x_0}^{z} N(w) \times \alpha'(w)dw \right\},\]

\(\alpha(z)\) and \(N(z)\) are the holomorphic extensions.
A classical problem

Björling’s problem for minimal surfaces:

Prescribed normal field on curve $\rightarrow$ Unique minimal surface

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$\alpha(z)$ and $N(z)$ are the holomorphic extensions.
Curve $\ldots$ generates surface of type $X$

e.g. space curve given by:

$$\kappa(s) = 1 - s^4, \quad \tau(s) = 0.$$ 

Find the (unique?) surface of (e.g.) constant Gauss curvature $K = 1$ containing this curve as:

1. a geodesic
2. a cuspidal edge singularity
3. or with some arbitrary prescribed surface normal

These are called geometric Cauchy problems
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As a geodesic curve (the CGC $K = 1$ solution)
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As a cuspidal edge singular curve
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e.g. Find the unique CGC $K = 1$ surface containing the curve with surface geometry given by:

$$\kappa_g(s) = 1, \quad \kappa_n(s) = 1, \quad \tau_g(s) = \sin(s)$$
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Special surfaces and harmonic maps
Special surfaces and harmonic maps

Many important classical surfaces correspond to *harmonic maps* from either $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$ into $G/K$.

**Examples:**
- Constant mean curvature (CMC) surfaces in space forms
- Constant Gauss curvature (CGC) surfaces in space forms
- Willmore surfaces
Example: Constant Gauss Curvature Surfaces

1. \( N : \mathbb{C} \rightarrow S^2 \) is harmonic iff

\[ N \times N_{zz} = 0, \]

iff

\[ f_z = iN \times N_z, \]

is integrable i.e. \((f_z)_{\bar{z}} = (f_{\bar{z}})_z\).

Moreover: \( f : \mathbb{C} \rightarrow \mathbb{R}^3 \) (with induced metric) is CGC, with \( K = 1 \).

2. \( N : \mathbb{R}^{1,1} \rightarrow S^2 \) is (Lorentzian)-harmonic iff

\[ N \times N_{xy} = 0, \]

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\[ f_x = N \times N_x, \quad f_y = -N \times N_y, \]

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Example: Constant Gauss Curvature Surfaces

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Loop group lift of a harmonic map into $G/K$

$$G = G^\mathbb{C}_\rho, \quad K = G_\sigma$$

Loop group $\Lambda G^\mathbb{C} := \{\gamma : S^1 \to G^\mathbb{C}\}$ . Twisted subgroup is the fixed point subgroup

$$\Lambda G^\mathbb{C}_{\hat{\sigma}}, \quad \text{for} \quad \hat{\sigma} x(\lambda) := \sigma(x(-\lambda)).$$

Real forms determined by the involutions:

$$\hat{\rho}_1 x(\lambda) := \rho(x(1/\bar{\lambda})), \quad \hat{\rho}_2 x(\lambda) := \rho(x(\bar{\lambda})).$$

Note:

$$\Lambda G = \Lambda G^\mathbb{C}_{\hat{\rho}_1}.$$
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**Riemannian case:** Harmonic maps $\mathbb{C} \supset U \to G/K$,

Characterized by $F : U \to \Lambda^* G_{\rho_1\hat{\sigma}} = \Lambda G_{\hat{\sigma}}$

$$F^{-1} dF = A_{-1} \lambda^{-1} dz + \alpha_0 + \overline{A_{-1}} \overline{\lambda} d\overline{z},$$

For any $\lambda_0 \in S^1$ the map

$$F|_{\lambda_0} : U \to G$$

projects to a harmonic map $f : U \to G/K$.

Call such $F$ an **admissible frame**.

**Lorentzian case:** $\mathbb{R}^{1,1} \supset V \to G/K$,

Characterized by $F : V \to \Lambda^* G_{\rho_2\hat{\sigma}}$

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$(x, y)$ null coord.s.
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Link with Soliton equations

Map into loop group

Flat loopalgebra-valued connection

Maurer-Cartan form

Adapted frame

Special submanifold

Special coordinates

Solution of soliton equation
Important loop group decompositions

Set $\Lambda^\pm G^C = \{ \gamma \in \Lambda G^C \mid \gamma = \sum_{n=0}^{\infty} a_n \lambda^\pm n \}$. 

We need:

1. The **Birkhoff decomposition**
   1.1
   
   \[ \Lambda^- G^C \cdot \Lambda^+ G^C \]

   is open and dense in the identity component of $\Lambda G^C$.

   1.2 For compact $G$:
   
   \[ \Lambda G^C_{\hat{\rho}_2} = \Lambda^- G^C_{\hat{\rho}_2} \cdot \Lambda^+ G^C_{\hat{\rho}_2} \]

   **Analogue:** $A = LU$ matrix factorization.

2. The **Iwasawa decomposition** (for compact $G$):
   
   \[ \Lambda G^C = \Lambda G \cdot \Lambda^+ G^C \]

   **Analogue:** $A = QR$ matrix factorization.
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   Analogue: $A = QR$ matrix factorization.
Riemannian-harmonic maps
Riemannian case (Dorfmeister/Pedit/Wu)

\[ F^{-1}dF = A_{-1} \lambda^{-1} dz + \alpha_0 + \overline{A_{-1}} \lambda d\bar{z}, \]

Birkhoff decompose: \( F(z) = F_-(z)F_+(z) \) (with normalization), then

\[ F_{-1}^{-1}dF_+ = B_{-1} \lambda^{-1} dz, \quad B_{-1} \text{ holo.}, B_{-1}(z) \in \mathfrak{g}_\mathbb{C}. \]

 Conversely: given a holomorphic 1-form with values in \( \text{Lie}(\Lambda G^{\mathbb{C}}_{\hat{\rho}_1 \hat{\sigma}}), \)

\[ \eta = \sum_{n=-1}^{\infty} B_n(z) \lambda^n dz, \]

1. solve \( \Phi^{-1}d\Phi = \eta \), with \( \Phi(z_0) = I \),
2. Iwasawa

\[ \Phi(z) = F(z)G_+(z) \]

Then \( F \) is an admissible frame.
Riemannian case (Dorfmeister/Pedit/Wu)

⇒
Given admissible frame $F: U \to \Lambda G_{\hat{\rho}_1\hat{\sigma}} = \Lambda G_{\hat{\sigma}}$

$$F^{-1}dF = A_{-1}\lambda^{-1}dz + \alpha_0 + \overline{A}_{-1}\lambda d\bar{z},$$

Birkhoff decompose: $F(z) = F_-(z)F_+(z)$ (with normalization), then

$$F_-^{-1}dF_- = B_{-1}\lambda^{-1}dz, \quad B_{-1} \text{ holo.}, \quad B_{-1}(z) \in g^\mathbb{C}.$$

⇐
Conversely: given a holomorphic 1-form with values in Lie($\Lambda G_{\hat{\rho}_1\hat{\sigma}}$),

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Then $F$ is an admissible frame.
Applications: e.g. CGC $K = 1$ (spherical) surfaces

$F : U \rightarrow \Lambda G$ admissible frame for the harmonic Gauss map.

The CGC surface can be obtained from $F$ by the Sym formula:

$$f = i\lambda \frac{\partial F}{\partial \lambda} F^{-1} \bigg|_{\lambda=1} =: S(F).$$
Numerical Implementation

e.g. DPW for spherical surfaces:

"holomorphic potential": \[ \eta = \sum_{i=-1}^{\infty} A_i \lambda^i dz \]
integrate: \[ \Phi^{-1} d\Phi = \eta. \]
Iwasawa: \[ \Phi = FH_. \]
Sym: \[ f = S(F). \]

Implementation: Can represent \[ \sum_{i=-n}^{n} A_i \lambda^i \] as a matrix:

\[
\begin{pmatrix}
A_0 & \cdots & A_n & 0 & \cdots & 0 \\
A_{-1} & A_0 & \cdots & A_n & 0 & \cdots & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & \cdots & A_{-n} & \cdots & A_0 & \cdots & A_n & \cdots & 0 \\
\vdots & & & & & \ddots & & \vdots \\
0 & \cdots & 0 & \cdots & 0 & A_{-n} & \cdots & A_0
\end{pmatrix}
\]

Loop group decompositions \( \leftrightarrow \) matrix decompositions
Examples

Simplest potentials:

\[ \eta = \left( \begin{array}{cc} 0 & a(z) \\ b(z) & 0 \end{array} \right) \lambda^{-1} dz. \]
Summary of DPW for spherical surfaces:

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All spherical surfaces can be constructed this way.

**Limitation:** Geometric information lost in the Iwasawa splitting, can *not* read off geometric information from \( \eta \).

**To exploit:** many choices of potential for a given surface.

Somewhat analogous method and statements hold for surfaces associated to *Lorentzian* harmonic maps (such as CGC \( K = -1 \)).
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Using DPW for Geometry

**Problem:** Find the potential \( \eta \) that produces the solution with some desired geometric properties.

**One approach** Use known potentials (e.g. rotational) to define more complicated solutions, e.g. potentials on \( n \)-punctured sphere with prescribed *end behaviour*.

Drawback: there are not that many known potentials.
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Another idea: prescribed geometry along a curve

The **geometric Cauchy problem:**

- Specify *sufficient geometric data* along a curve for a unique solution
- Find formulas for DPW-type potentials in terms of this data.
Solving the GCP for harmonic maps

Recall:
Riemannian harmonic:

\[ F \leftarrow \Phi \quad \text{via} \quad \Phi = FH_+ \quad \text{Iwasawa} \]

Many choices of potentials, hence of \( \Phi \).

**Essential idea:** Find potentials such that the Iwasawa/Birkhoff decomposition is *trivial* along the curve, i.e. such that

\[ F|_\gamma = \Phi|_\gamma. \]

**Main point:** \( F \) contains the *geometric information*, while \( \Phi \) are the “*Weierstrass data*”.
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Solving the GCP

Outline:

- Choose coordinates \( z = x + iy \) so that the curve is \( y = 0 \).
- Prescribe sufficient information to construct the loop group frame \( F_0(x) \) along \( y = 0 \), from \( \gamma \) and \( N \).
- Write \( \alpha = F^{-1} dF = \left( A_{-1} \lambda^{-1} + \alpha_0 + \overline{A_{-1}} \lambda \right) dx \).
- Let \( \eta \) be the holomorphic extension of \( \alpha \).
- Apply DPW to \( \eta \): solve \( \Phi^{-1} d\Phi = \eta \), Iwasawa split \( \Phi = FH_+ \), then \( F \) is an admissible frame.
- Along \( y = 0 \) we have \( F(x, 0) = \Phi(x, 0) = F_0(x) \) by construction.
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▶ Choose coordinates $z = x + iy$ so that the curve is $y = 0$.

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▶ Write $\alpha = F^{-1}dF = (A_{-1}\lambda^{-1} + \alpha_0 + \overline{A_{-1}}\lambda)\,dx$.

▶ Let $\eta$ be the holomorphic extension of $\alpha$.

▶ Apply DPW to $\eta$: solve $\Phi^{-1}d\Phi = \eta$, Iwasawa split $\Phi = FH_+$, then $F$ is an admissible frame.

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Theorem

Give real analytic functions

\[ \kappa_g(s), \quad \kappa_n(s), \quad \tau_g(s), \]

The unique spherical surface containing a curve along \{y = 0\} with the prescribed geodesic and normal curvature and geodesic torsion is obtained from the DPW potential

\[
\eta = \left[ \left[ \frac{\tau_g(z) - i}{2} e_1 - \frac{\kappa_n(z)}{2} e_2 \right] \frac{1}{\lambda} + \kappa_g(z) e_3 + \left[ \frac{\tau_g(z) + i}{2} e_1 - \frac{\kappa_n(z)}{2} e_2 \right] \lambda \right] dz.
\]

(All functions extended holomorphically, Here \( e_i \) are an o.n. basis for \( g \).)
Singular geometric Cauchy problem

Similarly, given real analytic

\[ \kappa(s), \quad \tau(s), \]

with \( \kappa \neq 0 \), holomorphically extend and then:

\[ \hat{\eta} = \left( \frac{\tau(z) - i}{2} \lambda^{-1} e_1 + \kappa(z)e_3 + \frac{\tau(z) + i}{2} \lambda e_1 \right) \, dz, \]

generates the singular curve solution.
Lorentzian-harmonic maps
"DPW"' for Lorentzian harmonic maps (Krichever, M. Toda)

⇒: Given $F : V \rightarrow \Lambda G^\mathbb{C}_{\hat{\rho}_2\hat{\sigma}}$,

$$F^{-1}dF = A_1\lambda dx + \alpha_0 + A_{-1}\lambda^{-1}dy,$$

Birkhoff: $F(x, y) = X_+(x, y)G_-(x, y) = Y_-(x, y)G_+(x, y)$ (with normalizations), then

$$X_+^{-1}dX_+ = B_1(x)\lambda dx,$$
$$Y_-^{-1}dY_- = C_{-1}(y)\lambda^{-1}dy.$$

⇐ Conversely: given 1-forms $(\chi, \psi)$ on $\mathbb{R}$ with values in $\text{Lie}(\Lambda G^\mathbb{C}_{\hat{\rho}_2\hat{\sigma}})$,

$$\chi = \sum_{n=-\infty}^{1} B_n(x)\lambda^n dx, \quad \psi = \sum_{n=-1}^{\infty} C_n(y)\lambda^n dy,$$

1. Solve $X^{-1}dX = \chi$, and $Y^{-1}dY = \psi$,
2. Birkhoff decompose

$$X^{-1}(x)Y(y) = H_-(x, y)H_+(x, y)$$

Then $F := XH_-$ is an admissible frame.
"DPW" for Lorentzian harmonic maps (Krichever, M. Toda)

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Then \( F := XH_- \) is an admissible frame.
The GCP for Lorentz-harmonic maps

"DPW" construction:

\[ F = XH_\leftarrow (X, Y) \quad \text{via} \quad X^{-1}Y = H_-H_+ \quad \text{Birkhoff} \]

Many choices of potentials, hence of \((X, Y)\).

**Analogous to Riemannian case:** Find potentials such that the Birkhoff decomposition is trivial along the curve, i.e. such that

\[ F|_\gamma = X|_\gamma = Y|_\gamma. \]
The GCP for Lorentz-harmonic maps

"DPW" construction:

\[ F = XH_\perp \leftrightarrow (X, Y) \quad \text{via} \quad X^{-1}Y = H_-H_+ \quad \text{Birkhoff} \]

Many choices of potentials, hence of \((X, Y)\).

**Analogous to Riemannian case:** Find potentials such that the Birkhoff decomposition is trivial along the curve, i.e. such that

\[ F|_\gamma = X|_\gamma = Y|_\gamma. \]
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Solving the GCP (non-characteristic curve)

Required admissible frame:

\[ F^{-1}dF = A_1 \lambda dx + \alpha_0 + A_{-1} \lambda^{-1} dy, \]

Potential pairs of form:

\[ \chi = X^{-1}dX = \sum_{n=-\infty}^{1} B_n(x) \lambda^n dx, \]
\[ \psi = Y^{-1}dY = \sum_{n=-1}^{\infty} C_n(y) \lambda^n dy, \]

Related by \( F := XH_\_ \), where

\[ X^{-1}(x) Y(y) = H_\-(x, y) H_+(x, y) \]

- Choose null coordinates s.t. initial curve given by \( y = x \).
- Set \( u = (x + y)/2, \ v = (x - y)/2 \), then initial curve is \( v = 0 \), and \( dy = dx = du \) along the curve.
- Construct \( F_0(u) = F(u, 0) \), so
  \[ \alpha_0 = F_0^{-1}dF_0 = A_1 \lambda du + \alpha_0 + A_{-1} \lambda^{-1} du. \]
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- Set \( \chi = \psi = \alpha_0 \).
Pseudospherical surfaces (Lorentzian harmonic)

- Analogous results to spherical surfaces
- Main difference: solution not unique for characteristic curves

Convenient way to generate examples:

Given curvature functions $\kappa$ and $\tau$ there is a unique CGC $K = -1$ surface containing this curve as a cuspidal edge (degenerate where $\kappa = 0$ or $\tau = \pm 1$).

$$\kappa(s) = 1 - s^4, \quad \tau(s) = 0$$
Examples

\[ \kappa(s) = 2 - s^2 \]
\[ \tau = 0 \]

\[ \kappa(s) = s^2 \]
\[ \tau = 1/2 \]
Examples that are not weakly regular

Viviani figure 8 space curve \( \gamma(t) = 0.3 \left( 1 + \cos t, \sin t, 2 \sin \frac{t}{2} \right) \).

- \( \tau = \pm 1 \) twice each on the curve.
- Solution to SG-equation not defined at these points
- The Lorentzian harmonic map is defined
Examples that are not weakly regular
Willmore surfaces
Willmore Surfaces

Elliptic PDE
- Gauss map *Riemannian-harmonic* (like spherical surfaces)
- Uniqueness: need more than just the surface normal.
It is sufficient to prescribe the *dual surface* $\hat{Y}$ in addition to $Y$ and the conformal Gauss map along the curve.
Equivariant Willmore Surfaces
Summary

- We discussed surface classes with harmonic Gauss maps.
- All solutions can be constructed from holomorphic Weierstrass-type data (Riemannian) or d’Alembert-type data (Lorentzian) called potentials.
- The challenge is to explicitly write down the potential for a given geometric problem.
- We can solve this given geometric Cauchy data along a curve.
D. Brander and J. Dorfmeister
The Björling problem for non-minimal constant mean curvature surfaces.

D. Brander and M. Svensson
The geometric Cauchy problem for surfaces with Lorentzian harmonic Gauss maps.

D. Brander
Pseudospherical surfaces with singularities.

D. Brander
Spherical Surfaces.

D. Brander and P. Wang
On the Björling problem for Willmore surfaces.

D. Brander
Singularities of Spacelike Constant Mean Curvature Surfaces in Lorentz-Minkowski Space.

D. Brander and M. Svensson
Timelike constant mean curvature surfaces with singularities.