Farmer and Trespasser

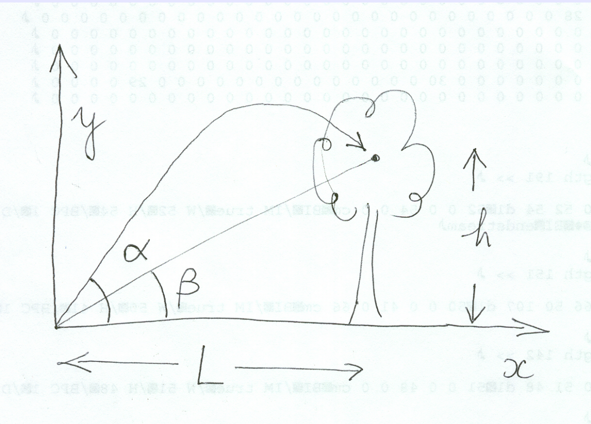
"Get orf moi laaaand!"

Durham University

problem

Farmer Palmer, lying stealthily on the (flat) ground, shoots at a trespasser who has climbed to the top of a tree of height h at distance L away.
If bullets leave his gun with speed v then find an equation determining the required angle of projection α.
Show that $\tan\alpha = v^2/gL$ for maximum range, and that then the bullets' initial velocity bisects the angle between the vertical and the straight line joining gun and trespasser.

diagram



derivation

integrating N2 gives

$$x(t) = vt\cos\alpha \qquad y(t) = vt\sin\alpha - gt^2/2$$

when putting

$$x(T) = L \quad \text{and} \quad y(T) = h$$

and eliminating time of flight T — with the use of

$$\tan = \sin / \cos \quad \text{and} \quad \sin^2 + \cos^2 = 1$$

— leads to the **key equation**.

key equation

$$h = -\frac{gL^2}{2v^2}(1 + \tan^2\alpha) + L\tan\alpha$$

— which is quadratic for $\tan\alpha$.

maximum range

in range there are two real solutions for $\tan\alpha$ and out of range the solutions are complex — so at **maximum range** they coincide — when

$$\tan\alpha = -\frac{L}{2(-gL^2/2v^2)} = \frac{v^2}{gL}$$

— remembering that when $ax^2 + bx + c = 0$ has coincident roots they are given by $x = -b/2a$.

angles

let β be the angle between the ground and the straight line joining gun and trespasser — then $\tan\beta = h/L$, which the **key equation** gives as

$$\tan\beta = \frac{h}{L} = -\frac{gL}{2v^2}(1 + \tan^2\alpha) + \tan\alpha$$

and where at **maximum range** we have

$$gL/v^2 = 1/\tan\alpha$$

— hence

$$\tan\beta = \frac{\tan^2\alpha - 1}{2\tan\alpha} = -\cot 2\alpha$$

and so $\beta = 2\alpha - \pi/2$ or

$$\pi/2 - \alpha = \alpha - \beta$$

as required

... and finally

why does angle α at **maximum range** depend on horizontal distance L and not on vertical height h?

references

http://en.wikipedia.org/wiki/Farmer_Palmer

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