NON-DISCRETE COMPLEX HYPERBOLIC TRIANGLE GROUPS OF TYPE $(n, n, \infty; k)$

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Abstract. A complex hyperbolic triangle group is a group generated by three involutions fixing complex lines in complex hyperbolic space. Our purpose in this paper is to improve the result in [5] and to discuss discreteness of complex hyperbolic triangle groups of type $(n, n, \infty; k)$.

1. INTRODUCTION

A complex hyperbolic triangle is a triple (C_1, C_2, C_3) of complex lines in complex hyperbolic 2-space $\mathrm{H}^2_{\mathbf{C}}$. We assume that C_{k-1} and C_k either meet at the angle π/p_k for some integer $p_k \geq 2$ or else C_{k-1} and C_k are asymptotic, in which case they make an angle 0 and in this case we write $p_k = \infty$, where the indices are taken mod 3. Let Γ be a group of holomorphic isometries of $\mathrm{H}^2_{\mathbf{C}}$ generated by involutions i_1, i_2, i_3 fixing a complex lines C_1, C_2, C_3, respectively. We call Γ a complex hyperbolic triangle group of type (p_1, p_2, p_3). For each such triple (p_1, p_2, p_3) there is a one real parameter family of complex hyperbolic triangle groups. It is interesting to ask which values of this parameter correspond to discrete groups.

The study of complex hyperbolic triangle groups was begun in [3]. Since then there have been many developments (see [8], [9], [10], [11], [12], [13] and [14]). In a previous paper [5] we considered a complex hyperbolic triangle group of type (n, n, ∞) and gave intervals of non-discreteness for different values of n.

Our purpose here is to improve the result in [5] and to give examples of non-discrete complex hyperbolic triangle groups of type (n, n, ∞). Throughout this paper, Γ denotes a complex hyperbolic triangle group of type (n, n, ∞).

2. PRELIMINARIES

We recall some basic notions of complex hyperbolic geometry. The complex hyperbolic 2-space $\mathrm{H}^2_{\mathbf{C}}$ is defined as the complex projectivization of the set of negative vectors in $\mathbf{C}^{2,1}$ with the Hermitian form $< Z, W >= Z_0\bar{W}_0 + Z_1\bar{W}_1 - Z_2\bar{W}_2$, where $Z = (Z_0, Z_1, Z_2)$ and $W = (W_0, W_1, W_2)$ in $\mathbf{C}^{2,1}$. Let $\mathrm{PU}(2,1)$ be the projectivization of $\mathrm{SU}(2,1)$. The group of holomorphic isometries of $\mathrm{H}^2_{\mathbf{C}}$ is exactly $\mathrm{PU}(2,1)$. Just as in real hyperbolic geometry, nontrivial elements of $\mathrm{PU}(2,1)$ fall into three conjugacy classes depending on the number and the location of fixed points. Using the discriminant function

$$\rho(z) = |z|^4 - 8Re(z^3) + 18|z|^2 - 27,$$

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we can classify elements of $\mathrm{PU}(2,1)$ by traces of the corresponding matrices in $\mathrm{SU}(2,1)$. In [2, Theorem 6.2.4] Goldman states that an element in $\mathrm{SU}(2,1)$ is regular elliptic if and only if $\rho(\tau(A)) < 0$, where $\tau(A)$ is the trace of A.

The boundary $\partial \mathrm{H}^2_{\mathbf{C}}$ is homeomorphic to S^3 and one of representation we choose for this is $(\mathbf{C} \times \mathbf{R}) \cup \{\infty\}$, with points either ∞ or $(z,r)_H$ with $z \in \mathbf{C}$ and $r \in \mathbf{R}$. We call $(z,r)_H$ the $H-coordinates$. Let H denote this representation, that is, $(\mathbf{C} \times \mathbf{R}) \cup \{\infty\}$. We have the homeomorphism B taking S^3 to H given by the standard stereographic projection:

$$\begin{aligned} (z_1, z_2) &\mapsto \left(\frac{z_1}{1+z_2}, -\mathrm{Im}\left(\frac{1-z_2}{1+z_2} \right) \right)_H, \\ (0,-1) &\mapsto \infty. \end{aligned}$$

The Cygan metric δ is defined by

$$\delta((z,r)_H, (w,R)_H) = \left| |z-w|^2 + ir - iR + 2i\mathrm{Im}(z\overline{w}) \right|^{\frac{1}{2}}$$

for $(z,r)_H, (w,R)_H$ in $H - \{\infty\}$.

More details on this subject can be found in [2],[4] and [6].

3. COMPLEX HYPERBOLIC TRIANGLE GROUPS OF TYPE (n,n,∞)

In this section we show intervals of non-discreteness for different values of n.

By [14, Proposition 3.10.6], we can take three involutions i_j in C_j such that $\partial C_1 = \{(e^{i\phi}, 0)_H |\ \phi \in \mathbf{R}\}$, $\partial C_2 = \{(s,t)_H |\ t \in \mathbf{R}\}$, and $\partial C_3 = \{(se^{i\theta}, t)_H |\ t \in \mathbf{R}\}$, where $s = \cos(\pi/n)$. Thus we see that a family of complex hyperbolic triangle groups of type (n,n,∞) is parametrized (up to conjugacy) by $\cos\theta$.

For convenience we shall shorten compositions of involutions, for example $i_1 i_2 i_3 i_1$ will be written as i_{1231}. We have the forms of i_j as follows:

$$i_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$i_2 = \begin{bmatrix} 1 & -2s & -2s \\ -2s & 2s^2-1 & 2s^2 \\ 2s & -2s^2 & -2s^2-1 \end{bmatrix} \text{ and}$$

$$i_3 = \begin{bmatrix} 1 & -2se^{i\theta} & -2se^{i\theta} \\ -2se^{-i\theta} & 2s^2-1 & 2s^2 \\ 2se^{-i\theta} & -2s^2 & -2s^2-1 \end{bmatrix}.$$

It follows that

$$i_1 i_2 i_3 = i_{123} = \begin{bmatrix} -1 & 2s(e^{i\theta}-1) & 2s(e^{i\theta}-1) \\ 2s(e^{-i\theta}-1) & 4s^2(e^{i\theta}-1)+1 & 4s^2(e^{i\theta}-1) \\ 2s(e^{-i\theta}-1) & 4s^2(e^{i\theta}-1) & 4s^2(e^{i\theta}-1)-1 \end{bmatrix}.$$

In [11] Schwartz considered ideal triangle groups, that is complex hyperbolic triangle groups of type (∞,∞,∞) and proved that if the element i_{123} is regular elliptic, then it is not of finite order, hence the corresponding complex hyperbolic triangle group is not discrete. In [8] Parker explored groups of type (n,n,n) such that i_{123} is regular elliptic and showed that in this case there are some discrete groups. In the same manner as in the proof of Schwartz in [11, p.545], Wyss-Gallifent formulated Schwartz's statement for groups of type (n,n,∞) in [14, Lemma 3.4.0.19]. In [10]

Pratoussevitch made a refinement of the proof of Wyss-Gallifent. Here we quote the result due to Wyss-Gallifent and Pratoussevitch.

Theorem 1. *Let* $\Gamma =< i_1, i_2, i_3 >$ *be a complex hyperbolic triangle group of type* (n, n, ∞). *If the product* i_{123} *of the three generators is regular elliptic, then* Γ *is non-discrete.*

Using this theorem, we work out some conditions on $\cos\theta$ for Γ of type (n, n, ∞) to be non-discrete. A simple computation yields

$$\tau = trace(i_{123}) = 8s^2(e^{i\theta} - 1) - 1$$

and

$$\begin{aligned}\rho(\tau) \quad = \quad & 256s^2(1-\cos\theta)\left\{2 - 2s^2 + 13s^2(1-\cos\theta) - 16s^4(1-\cos\theta)(1+4\cos\theta)\right. \\ & \left. +64s^6(1-\cos\theta)\right\},\end{aligned}$$

where $s = \cos(\pi/n)$. Set $X = 1 - \cos\theta$ and

$$\rho(X) \quad = \quad 256s^2X\left\{64s^4X^2 + (64s^6 - 80s^4 + 13s^2)X + 2 - 2s^2\right\}.$$

Solving the equation $\rho(X) = 0$ for X, we see that if $s \geq \sqrt{7/8}$, then there are two solutions a_n, b_n except 0, which lie between 0 and 1. But otherwise, there are no solutions except 0. Set $\alpha_n = 1 - a_n$ and $\beta_n = 1 - b_n$. We observe that if $s < \sqrt{7/8}$, then $\rho(X) \geq 0$ for $0 \leq X \leq 2$ and that if $s \geq \sqrt{7/8}$, then $\rho(X) < 0$ for $b_n < X < a_n$ and otherwise $\rho(X) \geq 0$. Since $\cos(\pi/9) > \sqrt{7/8} > \cos(\pi/8)$, we see that if $n < 9$, then the product i_{123} is not regular elliptic and that if $n \geq 9$, then it is regular elliptic for $\cos\theta \in (\alpha_n, \beta_n)$. Note that α_n and β_n are increasing functions of n. Denote by $E_{123}(n)$ the interval (α_n, β_n). It follows from Theorem 1 that if $n \geq 9$, then Γ is not discrete for $\cos\theta \in E_{123}(n)$.

Later we tabulate α_n and β_n together with another value γ_n, which is defined after Theorem 2.

Remark 1. If $s = 1$, then $\rho(\tau) < 0$ for θ with $61/64\ (= 0.9531...) < \cos\theta < 1$. This yields that $|A| > \tan^{-1}\sqrt{125/3}$, where A is the Cartan angular invariant. In [11] Schwartz showed that a group of type (∞, ∞, ∞) is discrete if and only if $|A| \leq \tan^{-1}\sqrt{125/3}$, which was conjectured by Goldman and Parker in [3].

Next we use a different way to find out some sufficient conditions on $\cos\theta$ for Γ to be non-discrete. Let g be an element of $\mathrm{PU}(2,1)$. We define the translation length $t_g(p)$ of g at $p \in H$ by $t_g(p) = \delta(g(p), p)$. In the case where a group contains a parabolic element, we know several criteria for a group to be non-discrete (see [6]). To state Theorem 2, we need the notion of isometric spheres. Let $h = (a_{mn})_{1\leq m,n\leq 3}$ be an element of $\mathrm{PU}(2,1)$ not fixing ∞. The isometric sphere of h is the sphere in the Cygan metric with center $h^{-1}(\infty)$ and radius $R_h = \sqrt{\frac{2}{|a_{22}-a_{23}+a_{32}-a_{33}|}}$ (see [3], [4] and [6]).

Here we recall the complex hyperbolic version of Shimizu's lemma due to Parker [7].

Theorem 2. *Let G be a discrete subgroup of* $\mathrm{PU}(2,1)$ *that contains the Heisenberg translation g with the form*

$$g = \begin{bmatrix} 1 & \tau & \tau \\ -\overline{\tau} & 1-(|\tau|^2-it)/2 & -(|\tau|^2-it)/2 \\ \overline{\tau} & (|\tau|^2-it)/2 & 1+(|\tau|^2-it)/2 \end{bmatrix}.$$

The transformation g fixes ∞ and maps the point with H-coordinates $(\zeta, v)_H$ to the point with H-coordinates $(\zeta+\tau, v+t+2\mathrm{Im}(\tau\bar{\zeta}))_H$. Let h be any element of G not fixing ∞ and with isometric sphere of radius R_h. Then

$$R_h^2 \le t_g(h^{-1}(\infty))t_g(h(\infty)) + 4|\tau|^2.$$

To improve the result in [5], we take $g = i_{23}$ and $h = i_{1231}$ in Theorem 2. We have

$$i_{23} = \begin{bmatrix} 1 & 2s(1-e^{i\theta}) & 2s(1-e^{i\theta}) \\ -2s(1-e^{-i\theta}) & 1+4s^2(e^{i\theta}-1) & 4s^2(e^{i\theta}-1) \\ 2s(1-e^{-i\theta}) & -4s^2(e^{i\theta}-1) & 1-4s^2(e^{i\theta}-1) \end{bmatrix} \text{ and}$$
$$i_{1231} = \begin{bmatrix} 1 & -2s(1-e^{i\theta}) & 2s(1-e^{i\theta}) \\ 2s(1-e^{-i\theta}) & 1+4s^2(e^{i\theta}-1) & -4s^2(e^{i\theta}-1) \\ 2s(1-e^{-i\theta}) & 4s^2(e^{i\theta}-1) & 1-4s^2(e^{i\theta}-1) \end{bmatrix}.$$

It is seen that i_{23} is a Heisenberg translation with fixed point ∞ and that i_{1231} has isometric sphere of radius

$$R_{i_{1231}} = \sqrt{\frac{1}{8s^2\{2(1\cos\theta)\}^{\frac{1}{2}}}}.$$

We have

$$\begin{aligned} t_{i_{23}}(i_{1231}^{-1}(\infty))t_{i_{23}}(i_{1231}(\infty)) &= |8s^2(1-e^{i\theta}) + 2i\sin\theta| \\ &= \{128s^4(1-\cos\theta) - (32s^2-4)(1-\cos^2\theta)\}^{\frac{1}{2}}. \end{aligned}$$

Theorem 2 implies that Γ is not discrete, if

$$(*) \quad \frac{1}{8s^2\{2(1-\cos\theta)\}^{1/2}} > \left\{128s^4(1-\cos\theta) - (32s^2-4)(1-\cos^2\theta)\right\}^{1/2} + 32s^2(1-\cos\theta).$$

Set $X = 1-\cos\theta$ and

$$Y = F_s(X) = \left\{128s^4X - (32s^2-4)X(2-X)\right\}^{1/2} + 32s^2X - \frac{1}{8s^2(2X)^{1/2}}.$$

Considering the graph of the function $Y = F_s(X)$, we observe that there is some $r_n \in (0,1)$ such that $F_s(X) < 0$ for $0 < X < r_n$ and $F_s(X) \ge 0$ for $r_n \le X \le 2$. Also we have $r_{n+1} < r_n$. Put $\gamma_n = 1 - r_n$. It follows that the above inequality $(*)$ is true only for $\cos\theta$ with $\gamma_n < \cos\theta < 1$, where γ_n is an increasing function of n. Thus Γ is not discrete for $\cos\theta \in (\gamma_n, 1)$. We denote the interval $(\gamma_n, 1)$ by $P(n)$.

We show the beginning of the list of approximations of α_n, β_n and γ_n.

TABLE 1. Approximations of α_n, β_n and γ_n.

n	α_n	β_n	γ_n
3	——	——	0.8923
4	——	——	0.9691
5	——	——	0.9819
6	——	——	0.9862
7	——	——	0.9882
8	——	——	0.9893
9	0.9312	0.9319	0.9900
10	0.9367	0.9423	0.9905
11	0.9403	0.9510	0.9908
12	0.9427	0.9580	0.9910
13	0.9445	0.9637	0.9913
14	0.9458	0.9684	0.9914
15	0.9469	0.9722	0.9915
16	0.9477	0.9754	0.9916
17	0.9484	0.9781	0.9917
18	0.9489	0.9804	0.9918
19	0.9494	0.9823	0.9918
20	0.9498	0.9840	0.9919
21	0.9501	0.9854	0.9919
22	0.9504	0.9867	0.9919
23	0.9506	0.9878	0.9920
24	0.9509	0.9887	0.9920
25	0.9510	0.9896	0.9920
26	0.9512	0.9904	0.9920
27	0.9814	0.9911	0.9920
28	0.9515	0.9917	0.9921
29	0.9516	0.9922	0.9921
30	0.9517	0.9927	0.9921
50	0.9527	0.9973	0.9922
200	0.9531	0.9998	0.9922

Remark 2. We observe that $\cos\theta = 0.9922$ satisfies the inequality

$$\frac{1}{8\{2(1-\cos\theta)\}^{1/2}} > \left\{128(1-\cos\theta) - 28(1-\cos^2\theta)\right\}^{\frac{1}{2}} + 32(1-\cos\theta),$$

which is obtained by substituting 1 for s in the inequality $(*)$. Therefore $\beta_n > \gamma_n$ for $n \geq 29$, that is $E_{123}(n) \cap P(n) \neq \emptyset$.

Remark 3. In [15], Xie and Jiang discussed discreteness of groups containing a regular elliptic element. We can apply their result to our case, but we could not improve our results here.

4. COMPLEX HYPERBOLIC TRIANGLE GROUPS OF TYPE $(n, n, \infty; k)$

Let $\Gamma =< i_1, i_2, i_3 >$ be a complex hyperbolic triangle group of type (n, n, ∞). If the trace of the element i_{1213} is equal to $1 + 2\cos\frac{2\pi}{k}$, where k is a positive integer ≥ 3, then Γ is said to be of type $(n, n, \infty; k)$. If $trace(i_{1213}) = 3$, then Γ is said to be of type $(n, n, \infty; \infty)$.

In this section we show examples of non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$. We have

$$trace(i_{1213}) = 3 - 16s^2\cos\theta + 16s^4$$

and

$$trace(i_{2123}) = 20s^2 - 16s^2\cos\theta - 1.$$

Denote the intervals consisting of the parameter $\cos\theta$ for which i_{1213} and i_{2123} are regular elliptic by $E_{1213}(n)$ and $E_{2123}(n)$, respectively. Then $E_{1213}(n) = (s^2, 1)$ and $E_{2123}(n) = (\frac{5}{4} - \frac{1}{4s^2}, 1)$. We see that s^2 and $\frac{5}{4} - \frac{1}{4s^2}$ are increasing functions of n. The following lemma shows the relations among $\alpha_n, \beta_n, \gamma_n, s^2$ and $\frac{5}{4} - \frac{1}{4s^2}$.

Lemma 1.
(1) $s^2 \leq \frac{5}{4} - \frac{1}{4s^2}$ *for* $n \geq 3$.
(2) $s^2 < \alpha_n < \beta_n < \frac{5}{4} - \frac{1}{4s^2}$ *for* $9 \leq n \leq 13$.
(3) $\alpha_n < s^2 < \beta_n < \frac{5}{4} - \frac{1}{4s^2}$ *for* $14 \leq n \leq 28$.
(4) $\alpha_n < s^2 < \gamma_n < \beta_n$ *for* $n \geq 29$.

Proof.
(1) is immediate.
Putting $\cos\theta = s^2$ into $\rho(trace(i_{123}))$ gives

$$256s^2(1 - s^2)^2(2 + 13s^2 - 16s^4).$$

This is negative when

$$s^2 > \frac{13 + 3\sqrt{33}}{32}.$$

We find that

$$\cos^2\frac{\pi}{13} < \frac{13 + 3\sqrt{33}}{32} < \cos^2\frac{\pi}{14}.$$

Thus $s^2 < \alpha_n < \beta_n$ for $9 \leq n \leq 13$. The inequality above together with Remark 1 yields that $\alpha_n < s^2 < \beta_n$ for $n \geq 14$.

Putting $\cos\theta = \frac{5}{4} - \frac{1}{4s^2}$ into $\rho(trace(i_{123}))$, we have

$$16(1 - s^2)^2(64s^4 - 96s^2 + 37) > 0.$$

Therefore $\beta_n < \frac{5}{4} - \frac{1}{4s^2}$. Thus (2) and (3) are proved. By Remark 2, we have (4).

Let Γ be of type $(n, n, \infty; k)$, where $n > 3$. Considering $trace(i_{1213})$, we have

$$\frac{8s^4 + 1 - \cos\frac{2\pi}{k}}{8s^2} = \cos\theta < 1,$$

which leads to

$$8s^4 - 8s^2 + 1 = 8\cos^4\frac{\pi}{n} - 8\cos^2\frac{\pi}{n} + 1 = \cos\frac{4\pi}{n} < \cos\frac{2\pi}{k}.$$

Hence $k \geq [\frac{n}{2}] + 1$ for $n > 3$. Thus we have only to consider the cases where $k \geq [\frac{n}{2}] + 1$.

To find non-discrete groups of type $(n, n, \infty; k)$, we ask which groups have their parameters $\cos\theta$ in $E_{123}(n)$ or $P(n)$. We only treat special cases, because we can do the remainder in the same manner.

First we find groups whose parameters lie in $P(n)$. For $n \leq 12$ there is no group of type $(n, n, \infty; k)$ for which parameter is located in $P(n)$. For $n \geq 13$, we find some groups of type $(n, n, \infty; k)$, which are not discrete by Theorem 2. As an example, we consider the case where $n = 21$. From Table 1, it is seen that

$$\gamma_{21} = 0.9919 < \cos^2\frac{\pi}{21} + \frac{1 - \cos\frac{2\pi}{k}}{8\cos^2\frac{\pi}{21}} < 1$$

for $11 \leq k \leq 13$, that is, the groups of types $(21, 21, \infty; 11), (21, 21, \infty; 12)$ and $(21, 21, \infty; 13)$ have their parameters in $P(21)$. Therefore, these three groups are not discrete.

Next consider groups whose parameters lie in $E_{123}(n)$. For $n \leq 8$, $E_{123}(n) = \emptyset$. There is no group of type $(9, 9, \infty; k)$ whose parameter is in $E_{123}(9)$. For $10 \leq n \leq 13$, there are a finite number of groups of type $(n, n, \infty; k)$ with parameters in $E_{123}(n)$. As an example, we treat the case where $n = 13$. In this case $E_{123}(13) \subset E_{1213}(13)$. It follows from Table 1 that

$$\alpha_{13} = 0.9445 < \cos^2\frac{\pi}{13} + \frac{1 - \cos\frac{2\pi}{k}}{8\cos^2\frac{\pi}{13}} < 0.9637 = \beta_{13}$$

for $12 \leq k \leq 38$. Therefore, groups of type $(13, 13, \infty; k)$ for $12 \leq k \leq 38$ are not discrete. By Lemma 1, $E_{123}(n) \cap E_{1213}(n) \neq \emptyset$ and $\alpha_n < \cos^2\frac{\pi}{n} < \beta_n$ for $n \geq 14$. Hence there are infinitely many groups of type $(n, n, \infty; k)$ with their parameters in $E_{123}(n)$, which are not discrete. We deal with the case where $n = 17$ as an example. From Table 1,

$$\cos^2\frac{\pi}{17} + \frac{1 - \cos\frac{2\pi}{k}}{8\cos^2\frac{\pi}{17}} < 0.9781 = \beta_{17}$$

for $k \geq 15$. Therefore the groups of type $(17, 17, \infty; k)$ for $k \geq 15$ are not discrete.

Finally we consider the element i_{2123}. Assume that i_{2123} is a regular elliptic element. Then $trace(i_{2123})$ is written as

$$trace(i_{2123}) = 20s^2 - 16s^2\cos\theta - 1 = 1 + 2\cos\phi\pi,$$

which yields that

$$\cos\phi\pi = 10s^2 - 8s^2\cos\theta - 1,$$

where ϕ is a real number. Substituting

$$\frac{8s^4 + 1 - \cos\frac{2\pi}{k}}{8s^2}$$

for $\cos\theta$, we have

$$\cos\phi\pi = -8s^4 + 10s^2 - 2 + \cos\frac{2\pi}{k} = -\cos\frac{4\pi}{n} + \cos\frac{2\pi}{n} + \cos\frac{2\pi}{k}.$$

By Lemma 1 and Table 1, $E_{1213}(n) \supset E_{2123}(n) \supset P(n)$ and $E_{2123}(n) \cap E_{123}(n) = \emptyset$ for $n = 5, 7, 9, 11, 12$ and 14. In each group of type $(5, 5, \infty; 3)$,$(7, 7, \infty; 4)$, $(9, 9, \infty; 5)$, $(11, 11, \infty; 6)$, $(12, 12, \infty; 7)$ or $(14, 14, \infty; 8)$, i_{2123} is regular elliptic. It follows from [1, Theorem 7] that for $(n, k) = (5, 3)$, $(7, 4)$, $(9, 5)$, $(11, 6)$, $(12, 7)$ and $(14, 8)$, there are no rational numbers $\phi's$ satisfying

$$\cos\phi\pi = -\cos\frac{4\pi}{n} + \cos\frac{2\pi}{n} + \cos\frac{2\pi}{k},$$

that is,

$$\begin{aligned}
&\cos\frac{\pi}{5} + \cos\frac{2\pi}{5} - \cos\phi\pi = \frac{1}{2},\\
&\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} - \cos\phi\pi = 0,\\
&\cos\frac{2\pi}{9} - \cos\frac{4\pi}{9} + \cos\frac{2\pi}{5} - \cos\phi\pi = 0,\\
&-\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\phi\pi = \frac{1}{2},\\
&\cos\frac{\pi}{6} + \cos\frac{2\pi}{7} - \cos\phi\pi = \frac{1}{2},\\
&\cos\frac{3\pi}{7} - \cos\frac{\pi}{4} + \cos\phi\pi = \frac{1}{2},
\end{aligned}$$

Therefore, i_{2123} is of infinite order in these cases. Hence the groups above are not discrete.

Thus we have

Theorem 3. *Let $\Gamma =< i_1, i_2, i_3 >$ be a complex hyperbolic triangle group of type $(n, n, \infty; k)$. Let $k \geq [n/2] + 1$. The following groups are non-discrete.*

1) $(5, 5, \infty; 3)$.
2) $(7, 7, \infty; 4)$.
3) $(9, 9, \infty; 5)$.
4) $(10, 10, \infty; 9)$.
5) $(11, 11, \infty; 6), (11, 11, \infty; 10), (11, 11, \infty; 11)$.
6) $(12, 12, \infty; 7)$, *and* $(12, 12, \infty; k)$, *where* $11 \leq k \leq 16$.
7) $(13, 13, \infty; 7)$, *and* $(13, 13, \infty; k)$, *where* $12 \leq k \leq 38$.
8) $(14, 14, \infty; 8)$, *and* $(14, 14, \infty; k)$, *where* $k \geq 12$.
9) $(15, 15, \infty; 8)$, *and* $(15, 15, \infty; k)$, *where* $k \geq 13$.
10) $(16, 16, \infty; 9)$, *and* $(16, 16, \infty; k)$, *where* $k \geq 14$.
11) $(17, 17, \infty; 9)$, *and* $(17, 17, \infty; k)$, *where* $k \geq 15$.
12) $(18, 18, \infty; 10)$, *and* $(18, 18, \infty; k)$, *where* $k \geq 16$.
13) $(19, 19, \infty; 10), (19, 19, \infty; 11)$, *and* $(19, 19, \infty; k)$, *where* $k \geq 17$.
14) $(20, 20, \infty; 11), (20, 20, \infty; 12)$, *and* $(20, 20, \infty; k)$, *where* $k \geq 18$.
15) $(21, 21, \infty; 11), (21, 21, \infty; 12), (21, 21, \infty; 13)$, *and* $(21, 21, \infty; k)$ *where* $k \geq 19$.
16) $(22, 22, \infty; 12), (22, 22, \infty; 13), (22, 22, \infty; 14)$, *and* $(22, 22, \infty; k)$, *where* $k \geq 19$.
17) $(23, 23, \infty; 12), (23, 23, \infty; 13), (23, 23, \infty; 14), (23, 23, \infty; 15)$, *and* $(23, 23, \infty; k)$, *where* $k \geq 20$.
18) $(24, 24.\infty; 13), (24, 24, \infty; 14), (24, 24, \infty; 15), (24, 24, \infty; 16)$, *and* $(24, 24, \infty; k)$, *where* $k \geq 21$.
19) $(25, 25, \infty; 13), ..., (25, 25, \infty; 17)$, *and* $(25, 25, \infty; k)$, *where* $k \geq 22$.
20) $(26, 26, \infty; 14), ..., (26, 26, \infty; 19)$, *and* $(26, 26, \infty; k)$, *where* $k \geq 23$.

21) $(27, 27, \infty; 14), ..., (27, 27, \infty; 21)$, *and* $(27, 27, \infty; k)$, *where* $k \geq 24$.
22) $(28, 28, \infty; 15), ..., (28, 28, \infty; 23)$, *and* $(28, 28, \infty; k)$, *where* $k \geq 25$.
23) $(29, 29, \infty; k)$ *for any* k (≥ 15).
24) $(n, n, \infty; k)$ *for any* n (> 29) *and* k $(\geq [n/2] + 1)$.

Remark 3. In our forthcoming paper we show that the following 10 groups are discrete:
$(3,3,\infty; 4), (3, 3, \infty; 6), (3, 3, \infty; \infty)$;
$(4,4,\infty; 3), (4, 4, \infty; 4), (4, 4, \infty; 6), (4, 4, \infty; \infty)$;
$(6,6,\infty; 4), (6, 6, \infty; 6), (6, 6, \infty; \infty)$.
But we do not know if groups of type $(n, n, \infty; k)$ without reference are discrete.

References

[1] J.H. Conway and A.J. Jones, Trigonometric diophantine equations (On vanishing sums of roots of unity), *Acta Arithmeica 30 (1976), 229-240.*
[2] W.M. Goldman, Complex Hyperbolic Geometry, *Oxford University Press (1999).*
[3] W.M. Goldman and J. R. Parker, Complex hyperbolic ideal triangle groups, *J. Reine Angew. Math. 425 (1992), 71-86.*
[4] S. Kamiya, On discrete subgroups of $\mathrm{PU}(1, 2; \mathbf{C})$ with Heisenberg translations, *J. London Math. Soc. 62 (2000), no.3, 627-642.*
[5] S. Kamiya, Remarks on complex hyperbolic triangle groups, *Complex analysis and its applictions, 219-223, OCAMI Stud., 2, Osaka Munic. Univ. Press, Osaka , 2007.*
[6] S. Kamiya and J. R. Parker, Discrete subgroups of PU(2,1) with screw parabolic elements, *Math. Proc. Cambridge Phil. Soc. 144 (2008), 443-455.*
[7] J. R. Parker, Uniform discreteness and Heisenberg translation, *Math. Z. 225 (1997), 485-505.*
[8] J. R. Parker, Unfaithful complex hyperbolic triangle groups I: Involutions, *Pacific J. Math. 238, no.1 (2008), 145-169.*
[9] A. Pratoussevitch, Traces in complex hyperbolic triangle groups *Geomatriae Dedicata 111 (2005), 159-185.*
[10] A. Pratoussevitch, Non-discrete complex hyperbolic triangle groups of type (m, m, ∞), *(to appear)*
[11] R. E. Schwartz, Ideal triangle groups, dented tori, and numerical analysis, *Ann. of Math. 153 (2001), 533-598.*
[12] R. E. Schwartz, Degenerating the complex hyperbolic ideal triangle groups, *Acta Math, 186 (2001), 105-154*
[13] R. E. Schwartz, Complex hyperbolic triangle groups, *Proceedings of ICM, vol. II (2002), 339-349.*
[14] J.Wyss-Gallifent, Complex hyperbolic triangle groups, *Ph.D. thesis, Univ. of Maryland, 2000.*
[15] B.H.Xie and Y.P.Jiang, Discreteness of subgroups of $\mathrm{PU}(2, 1)$ with regular elliptic elements, *Linear Algebra Appl. 428 (2008), 732-737.*

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