TESSELLATIONS OF HYPERBOLIC SPACE
NOTES FOR AN UNDERGRADUATE COLLOQUIUM, OCT 2013

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1. The axiom of parallels

As you have doubtless heard before, Euclid (∼300 B.C.)—in his "Elements" several millennia ago—tried to put geometry on a solid footing, using an axiomatic approach.

This amounted to a huge endeavour: the task is to find as few axioms as possible (self-explaining theorems about the object of your consideration) for the geometry and deduce all the known theorems about the geometry from those axioms.

In modern parlance, they look as follows:

(1) Each pair of points can be joined by one and only one straight line segment.
(2) Any straight line segment can be indefinitely extended in either direction.
(3) There is exactly one circle of any given radius with any given center.
(4) All right angles are congruent to one another.
(5) Through any point not lying on a straight line there passes one and only one straight line that does not intersect the given line.

Of such a set of axioms one needs to make sure in particular that

(1) the axioms do not lead to a contradiction ("consistency"),
(2) neither axiom can be deduced from the others ("independence").

This was handled quite satisfactory in Euclid's "Elements" in many ways—*but* already back then the fifth axiom, the axiom of parallels, seemed superfluous.

Key Question. Is the fifth axiom *independent* of the others?

The importance of Euclid's "Elements" and this somewhat nagging issue for the mathematical foundations spurred many attempts over the centuries to indeed prove that axiom from the remaining four axioms.

Eventually, (presumably) independent flashes of genius for five people: each of them showed—in their own way—that one can replace that fifth axiom by its "converse":

"*Through any point not lying on a straight line there pass at least two straight line that does not intersect the given line.*"

and still gets a consistent geometry! This was independently achieved by Bolyai, Lobachevsky and Gauss as well as Schweikart and (his nephew) Taurinus, all in the early 19th century.

Remark. Actually, one can instead replace "one and only one" also by "no" and get yet another (so-called *spherical*) geometry.

What is more, one can picture this kind of geometry within Euclidean spaces, albeit with some of the "rules" changed.

Some surprises:

- the angle sum in a triangle is $< \pi$; ⟦as opposed to $= \pi$ in Euclidean geometry⟧
- if two triangles are similar (i.e. have the same angles), then they are also congruent! ⟦as opposed to the Euclidean case where one has infinitely many non-congruent similar triangles⟧
- the *area* of a triangle can be read off from the *angles*;
- there are (non-empty) triangles with all angles being zero!
- the "hyperbolic Pythagoras": in a right-angled triangle with legs a, b and hypotenuse c one has

$$\cosh(a)\cosh(b) = \cosh(c)\,.$$

2. First glance at (plane) hyperbolic geometry

How can we picture such a strange geometry? (Many of you have probably seen instances of this.) There are several rather different models in which one can view it, we only mention three of them.

(H) the upper half plane model $\mathbb{H}$ (viewed as inside the complex plane),
(I) the disk model;
(L) the hyperboloid model (take the "upper" sheet ($z > 0$) of the two sheeted hyperboloid $x^2 + y^2 - z^2 = -1$).

The most intuitive of these, in a couple of respects, is perhaps the upper half plane model, in particular for those among you who have seen *Moebius transformations* in Complex Analysis. Recall that these are maps of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$, and most of them map the complex plane into itself, i.e. preserve it. We can consider in particular those Moebius transformations which preserve the *upper half plane*, i.e. $\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid y > 0\}$. Those transformations can be captured by transformations with *real* a, b, c, d.

Now an important fact is that these transformations not only preserve the upper half plane, but they even preserve the underlying "geometry" (distances ($\rightsquigarrow$ metric) and angles), hence are called *isometries*.

Fact. Any isometry of $\mathbb{H}^2$ is captured by some $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, in other words $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

Examples.

(1) The matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ encodes the map $z \mapsto \frac{1\cdot z+2}{0\cdot z+1} = z + 2$, i.e. it simply shifts each point in the (half) plane to the right by two units.
(2) The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ encodes the map $z \mapsto -\frac{1}{z}$, and can be seen geometrically as the inversion in the unit circle ($z \mapsto \frac{z}{|z|^2} = 1/\overline{z}$) followed by a reflection in the real line (complex conjugation) and a reflection in the origin ($z \mapsto -z$).

First glimpse of Number Theory. Now the two matrices above generate an *infinite* group[1] Γ, in fact a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (which in turn is of course a subgroup of $\mathrm{SL}_2(\mathbb{R})$,

[1] a *group* essentially means that we can compose any two such matrices and wind up with another matrix of the same sort, and that moreover each matrix has an "inverse"

characterised by the property that all its entries are *integers*). More precisely, one can show that Γ is not "far off" in the sense that three copies of it (mathematical notion: cosets) cover $\mathrm{SL}_2(\mathbb{Z})$. This Γ is an example of an "arithmetic" group; such groups play an important role in number theory, not least in the theory of modular forms[2] (which was e.g. central in the proof of Fermat's Last Theorem (Wiles, building on work of Frey, Serre, Ribet, Taniyama, in parts collaboration with Taylor)).

Boundary of $\mathbb{H}^2$. One can view the real line (embedded in $\mathbb{C}$) as part of the boundary of $\mathbb{H}^2$, denoted $\partial\mathbb{H}^2$. Apart from the real line, there is one more point ("at infinity") that is on the boundary—altogether $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ (which can be identified with the real projective line in projective geometry).

Geodesics in $\mathbb{H}^2$. What are the shortest lines in $\mathbb{H}^2$ (according to the *hyperbolic metric*)? It turns out that these are the half-circles (and as a limiting case also the straight lines) orthogonal to the real line of $\partial\mathbb{H}^2$.

Orbits under group action. Now we can look at all the (infinitely many) translates of an element $z \in \mathbb{H}^2$ under Γ (the correct notion here is that of a "group action" of Γ on the upper half plane), which we will call the Γ–*orbit* of z.
As a 1-dimensional analogue, we can consider all the $\mathbb{Z}$–translates inside $\mathbb{R}$ and look for collections that contain a $\mathbb{Z}$-translate of each point on $\mathbb{R}$—clearly the interval $[0,1]$ suffices, but also, say, $[\pi, \pi+1]$; one could even use rather provocatively construed examples like $\{r \in \mathbb{Q} \mid 0 \leq r < 1\} \cup \{s \in \mathbb{R} \setminus \mathbb{Q} \mid 17.1 < s < 18.1\}$ but an obvious advantage of the former ones is that they are not scattered around; one has the rather intuitive (topological) notion of connectedness which will guide our preferences.

Fundamental domains. We want to find a collection of points in $\mathbb{H}^2$ which contain a Γ–translate of each element in $\mathbb{H}^2$. If such a collection is connected and is optimal in the sense that it contains only a *single* element of each Γ-orbit, then we call it a *fundamental domain* (for the action of Γ on $\mathbb{H}^2$).

Note that in our 1-dimensional example the interval $[0,1]$ is not quite a fundamental domain (for the action of $\mathbb{Z}$ on $\mathbb{R}$) because both 0 and 1 are in the same $\mathbb{Z}$-orbit; but a simple modification—dropping "1", say—gives us the half-open interval $[0,1)$ as a fundamental domain.

Now clearly each point in $\mathbb{H}^2$ has a Γ-translate in the half-strip $\{x+iy \mid -1 \leq x < 1,\ y > 0\}$: for $z_0 = x_0 + iy_0$ simply subtract integer multiples of 2 from z_0 so that it lands between -1 and 1 (in terms of formulas, $z_0 \mapsto z_0 - 2\lfloor \frac{x_0+1}{2} \rfloor$ does it, so we would apply the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-\lfloor \frac{x_0+1}{2} \rfloor} = \begin{pmatrix} 1 & -2\lfloor \frac{x_0+1}{2} \rfloor \\ 0 & 1 \end{pmatrix} \in \Gamma$ to replace z_0 by its translate inside that half-strip).

Furthermore, since the second matrix, corresponding to the map $z \mapsto -1/z$, maps elements from inside the unit circle to the outside, it seems plausible that the fundamental domain is covered by the figure below (this is to be thought as extended to the "point at infinity" where the two vertical boundary lines meet.

Indeed, it turns out that this is essentially the correct picture, except that again one needs to be a bit more careful at the boundary (only "half" the points are to be counted in.

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[2] roughly holomorphic functions which are invariant—up to some well controlled "multipliers"—under the infinitely many symmetries arising from elements in Γ

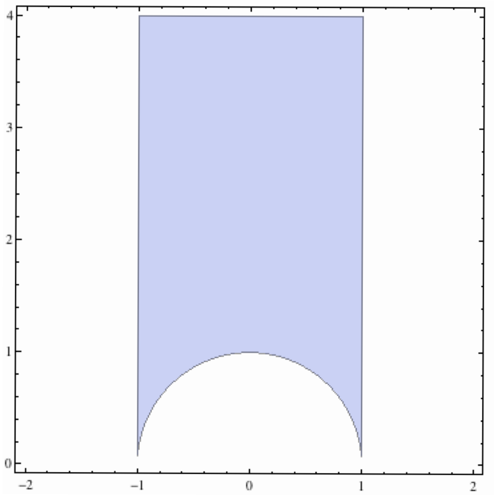
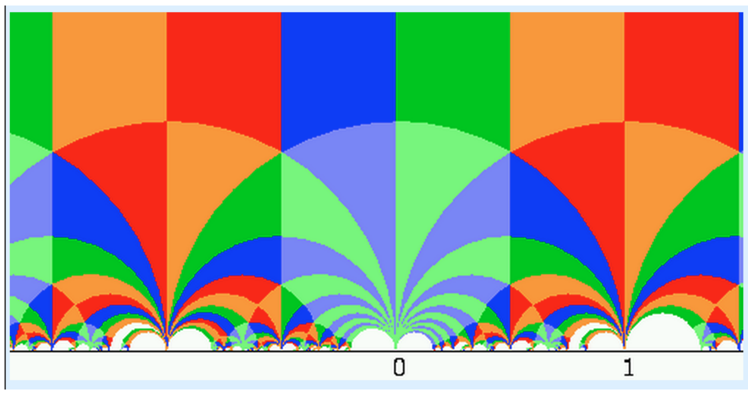


FIGURE 1. Fundamental domain for the group Γ

Note that this defines a triangle in $\mathbb{H}^2$ with all angles being zero, and with all vertices at the boundary. ("Ideal triangle".)

Tessellations. Once a fundamental domain is found, its translates will determine a tessellation of the original space—simply take all its translates under the group (typically one allows for overlaps along the boundaries, so one is more casual about the actual "fundamental domain" thing.

So what does the tessellation look like?

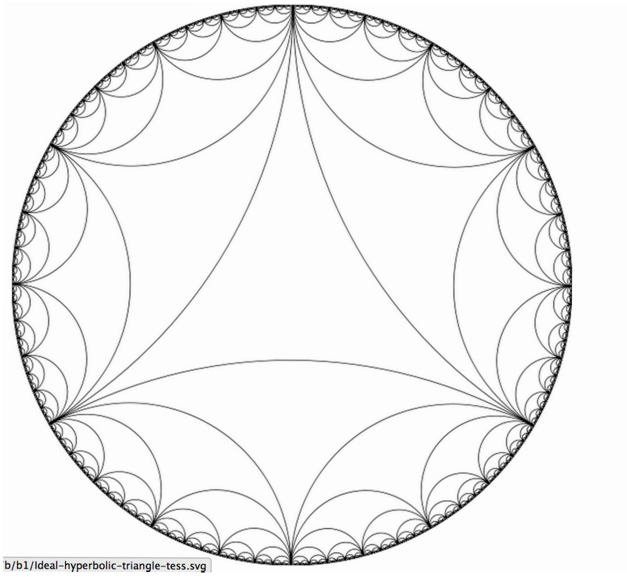


Tessellation of the hyperbolic plane: six tiles of different colour give a fundamental domain [Source: H. Verrill]

Moreover, one can view this picture in a different model of the hyperbolic plane, the disk model $\mathbb{D}^2$ (covering the unit disk inside the complex plane)—e.g. using the map

$$\begin{array}{ccc} \mathbb{H}^2 & \longrightarrow & \mathbb{D}^2 \\ z & \mapsto & \dfrac{z-i}{z+i} \end{array}$$

and the tessellation looks as follows [Source: Wikipedia]



This should ring a bell. . .

Foyer of the maths building, Durham University

A glance of hyperbolic 3-space. We get an analogous picture when we pass from 2D to 3D, i.e. from the hyperbolic plane the hyperbolic *space*. It is often depicted as the upper half space $\mathbb{H}^3$.

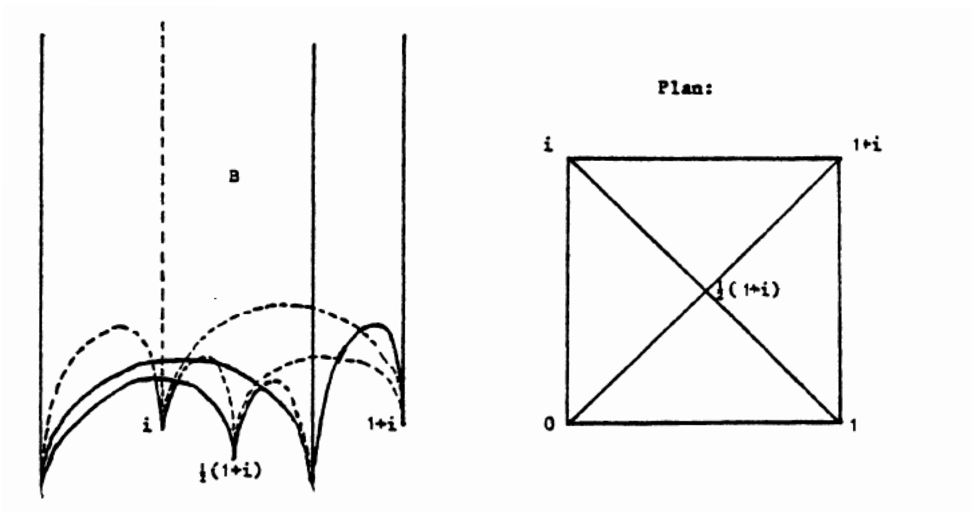
From 2D to 3D: the relevant analogues of the notions introduced above.

- Its boundary $\partial\mathbb{H}^3$ is given by the plane "underneath" together with a point at infinity (topologically a "1-point compactification of the plane, geometrically a 2-sphere.
- Its geodesics are again half-circles, this time orthogonal to the boundary *plane*—as a limiting case, if a half-circle passes through the point at infinity, it becomes a straight line in $\mathbb{H}^3$.
- The hyperplanes in $\mathbb{H}^3$ are half-spheres, again orthogonal to the boundary plane (limiting case half-spheres through ∞ which are planes orthogonal to that).
- Isometries now are encoded by elements in $\mathrm{SL}_2(\mathbb{C})$ (rather than $\mathrm{SL}_2(\mathbb{R})$ in the plane). Hence we are looking for interesting subgroups of that matrix group.

3. Tessellations in hyperbolic space.

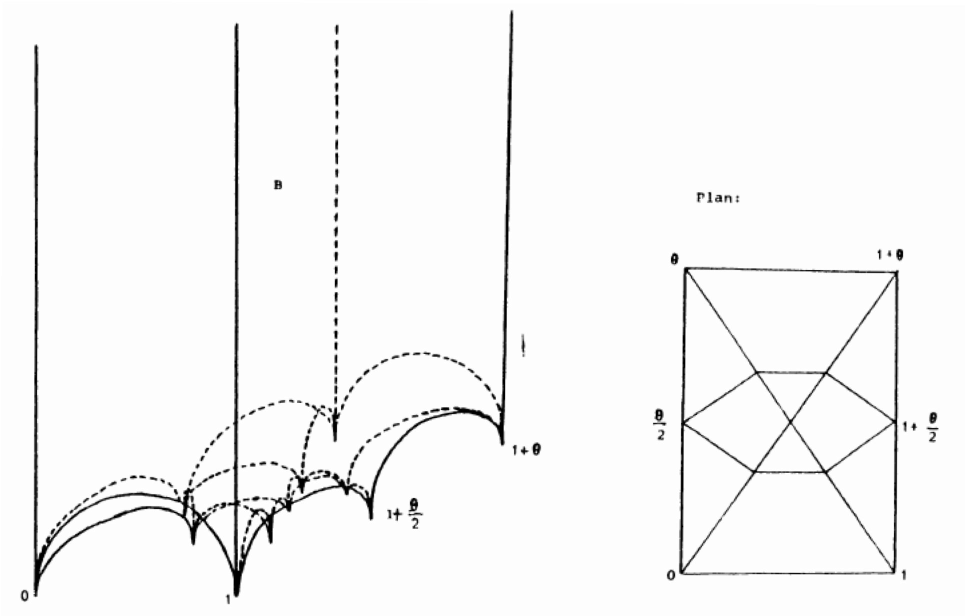
Let us consider a couple of examples.

Fundamental domains. 1. The perhaps simplest 3-dimensional example arises from the group $\mathrm{SL}_2(\mathbb{Z}[i])$ where $\mathbb{Z}[i] \subset \mathbb{C}$ denote the Gaussian integers. The actual fundamental domain is slightly more complicated, but if one passes to a subgroup (of index 4) of $\mathrm{SL}_2(\mathbb{Z}[i])$—this corresponds to gluing 4 copies of this fundamental domain together—then one obtains a nice octahedron with all eight vertices at the boundary. You can think of the octahedron as two square pyramids glued together along the base, and a fundamental domain for the full group $\mathrm{SL}_2(\mathbb{Z}[i])$ is given as half of such a square pyramid.



Tessellation of the hyperbolic plane using $\mathrm{SL}_2(\mathbb{Z}[i])$
[Source: J. Cremona, Comp. Math. 51 (1983)]

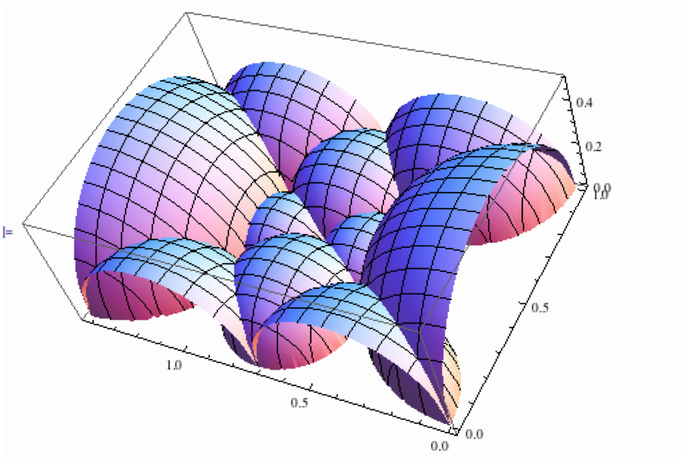
2. A second example is given by $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, where the picture is as follows.



Tessellation of the hyperbolic plane using $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$
[Source: J. Cremona, Comp. Math. 51 (1983)]

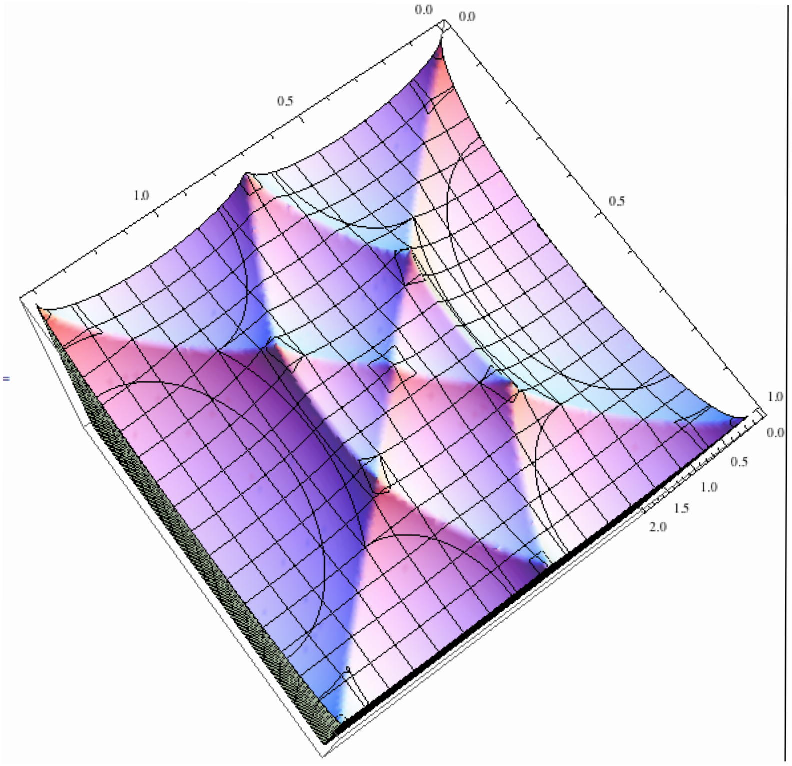
How should we interpret these pictures?
The left hand one shows (parts of) 14 hyperplanes in $\mathbb{H}^3$, four of them being "straight", while 10 of them are given in terms of half-spheres. The interior of this figure gives a polyhedron which is essentially a fundamental domain arising from the action of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.
The right hand picture shows a projection of this polyhedron from the point "at infinity" to the plane "below".

A schematic 3D-picture of the half-spheres bounding the polyhedron looks as follows:



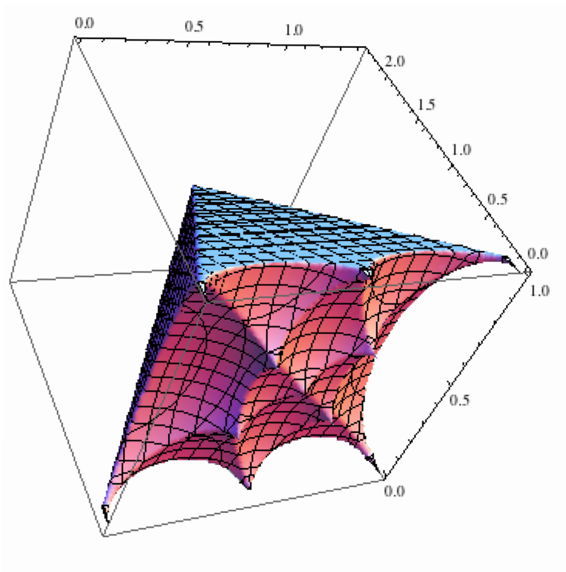
Ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ [uses Mathematica]

Here's how it looks from below:

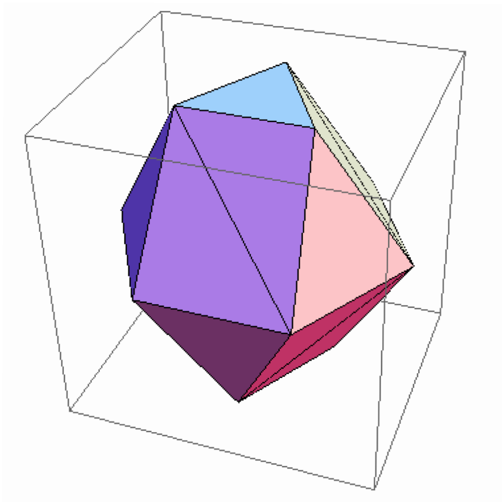


Ideal fundamental domain arising from $SL_2(\mathbb{Z}[\sqrt{-2}])$ [uses Mathematica]

If we push the vertex "up at infinity" down to a finite point, we can see a compact approximation of the polyhedron (with the same combinatorial data)

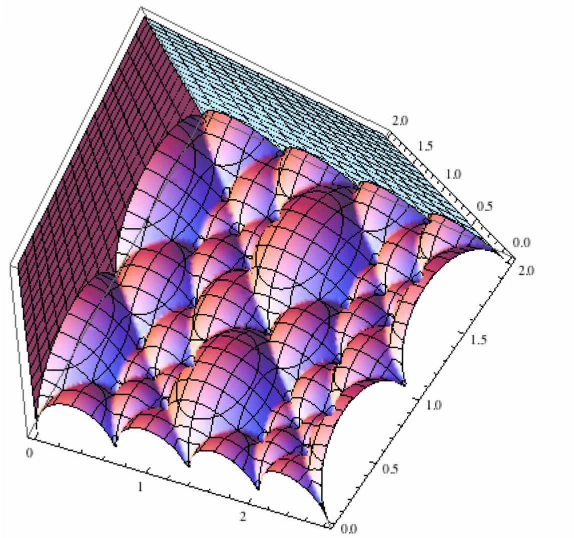


Approximate ideal fundamental domain arising from $SL_2(\mathbb{Z}[\sqrt{-2}])$ [uses Mathematica] and we can try to recognise it as a more familiar polyhedron, at least after straightening out the faces. Indeed we find its Euclidean counterpart as follows—a cuboctahedon (please ignore the diagonals in the picture drawn below).



"Straightened" version of ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ [uses Mathematica]

3.1. **From fundamental domain to tessellation.** On the one hand, it is reasonably straightforward to picture parts of the *tessellation* of $\mathbb{H}^3$ arising from this fundamental domain via simple translates using the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with a an integer or, say, an integer multiple of $\sqrt{-2}$ (in the following picture we glue four copies of the fundamental domain together),



Four copies of ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ [uses Mathematica]

On the other hand, it is much harder to picture the image under e.g. the "inversion" $z \mapsto -1/z$.

On the next page there is a rather sophisticated picture/impression of how it looks like in a related case (here the tessellation is via dodecahedra).

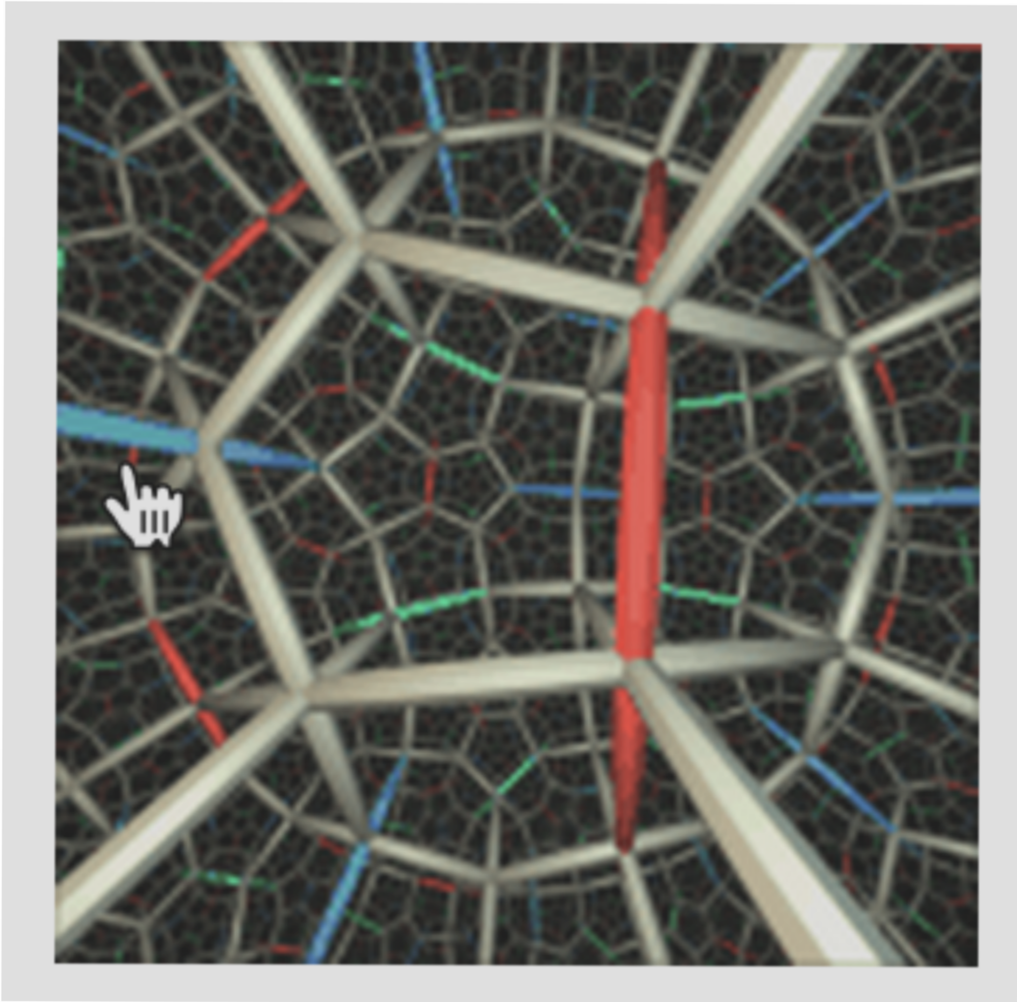
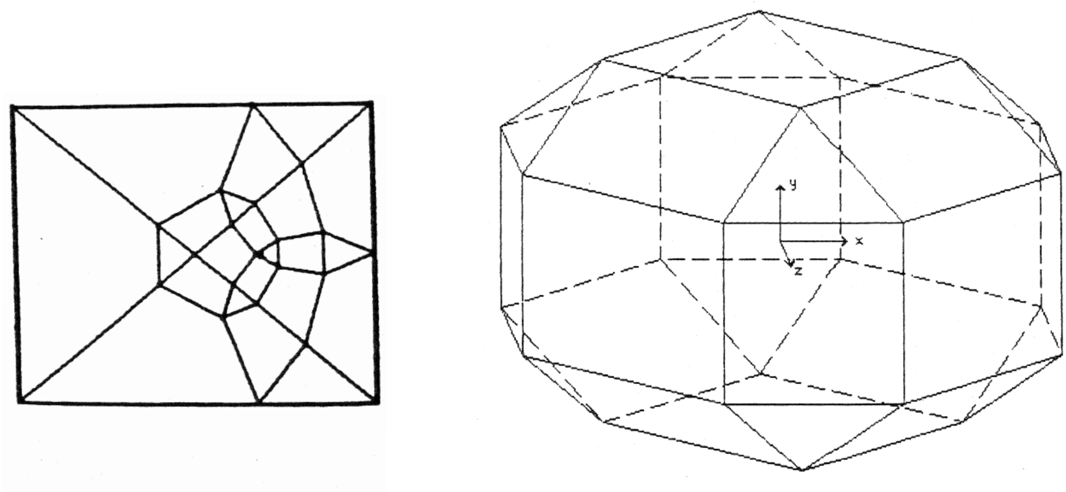


FIGURE 2. Tessellation of hyperbolic 3-space, from "Not Knot" (video on YouTube, highly recommended!) by C. Gunn et al.

4. Further arithmetic examples.

We can look at further arithmetic examples, typically arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ (or at least a closely related such "commensurable" matrix group). It turns out that in the case $d = 6$ we can find a fundamental domain which is a rhombicuboctahedron, given in the abstract of this colloquium talk:



A polytope tessellating hyperbolic 3-space, projected to the Euclidean plane from one of its vertices (left), and a "straightened" version of that polytope in Euclidean space (right).

Visualisation challenge: Can you "see" that the left hand one really is a projection of the right hand one?

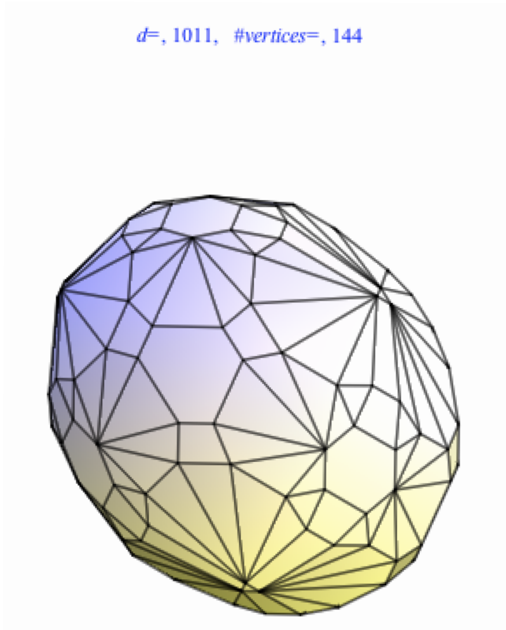
Occurrence in "real life". This polyhedron is very reminiscent of "Rubik's snake" which is depicted below.

A picture of Rubik's snake, a rather flexible mathematical toy.

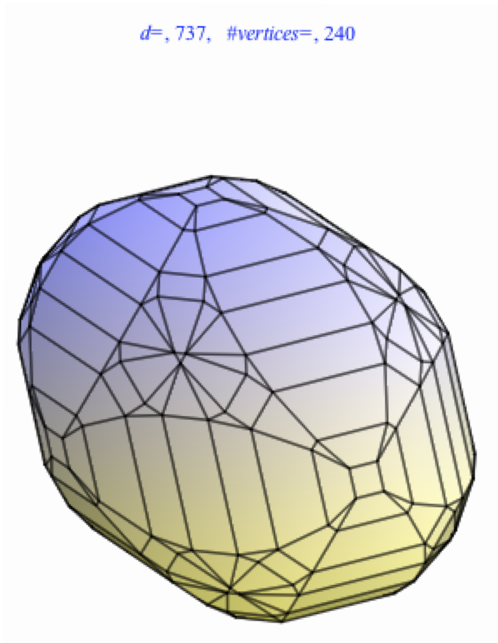
Upshot. So we could say *hyperbolically*: we can tessellate hyperbolic 3-space with (curved versions of) copies of Rubik's snake...

Symmetric polyhedra with many vertices. If one considers many such cases (last year I have let the departmental cluster run [thanks to Peter Craig for the initiation] for $d < 20000$), one occasionally finds a beautiful hyperbolic polytope with many vertices.

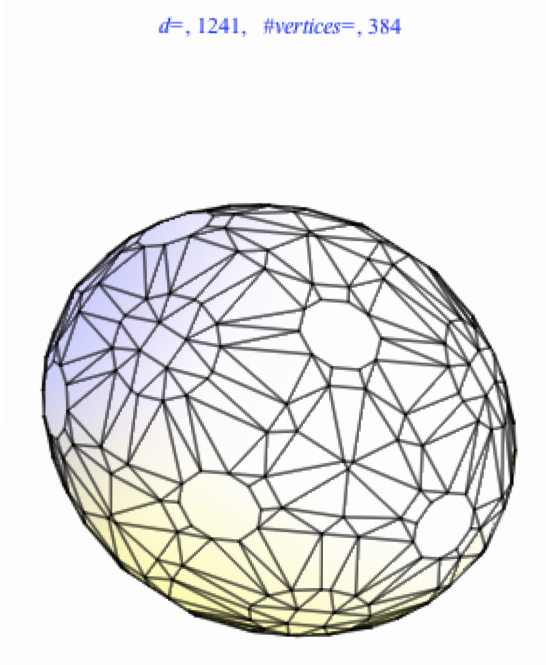
Here is a selection, all of them produced with the help of Maple, following an original visualisation idea found by Matthew Spencer (former Durham student).



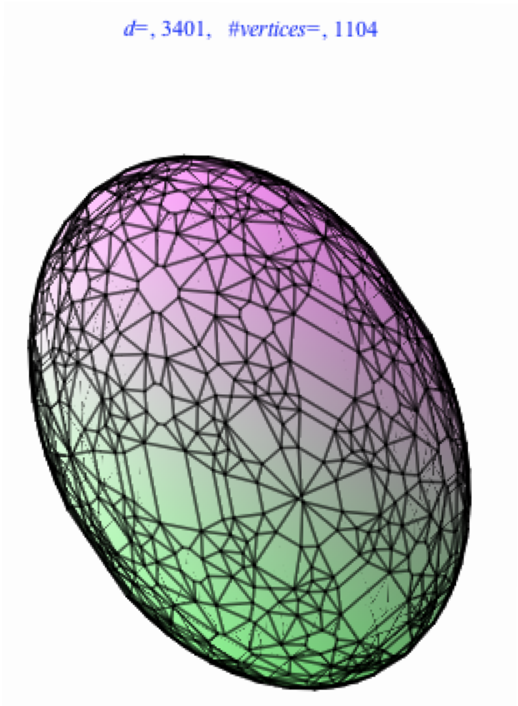
A hyperbolic polytope (144 vertices) in an ideal tessellation for $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-1011}])$.



A hyperbolic polytope (240 vertices) in an ideal tessellation for $\mathrm{SL}_2(\mathbb{Z}[\frac{1+\sqrt{-737}}{2}])$.



A hyperbolic polytope (144 vertices) in an ideal tessellation for $\mathrm{SL}_2(\mathbb{Z}[\frac{1+\sqrt{-1241}}{2}])$.



A hyperbolic polytope (144 vertices) in an ideal tessellation for $\mathrm{SL}_2(\mathbb{Z}[\frac{1+\sqrt{-3401}}{2}])$.

Past project work. In a summer project 2012, Matthew Spencer tried to implement an algorithm to produce these fundamental domains, and in a summer project 2013, Emily Woodhouse brought some of these polyhedra "to life" by preparing them into a 3D-printable form. In particular it was not obvious how to pass from the rather flat shape to a round form while keeping the symmetries. She will present tangible results in one of the forthcoming Undergraduate Colloquia.

Outreach. Yasaki found an independent method to produce a natural tessellation of the same spaces, with at most eight different types of polytopes—combining the two might give rise to a kind of "polytope puzzle".

Outlook. There is still a lot to be explored about those polyhedra (possibly even for project work). Getting a grip on questions related to those tessellations—possibly even in higher dimensions—can tell us quite a bit about questions relating especially number theory to different areas such as topology and geometry, with topics like

- cohomology of modular groups, in particular of so-called *Bianchi groups*;
- producing elements in algebraic K-theory;
- relations to special values of L-functions and, somewhat related,
- relations to Mahler measures.

Also, in joint work with Gunnells, Hanke, Dutour Sikirič, Schürmann and Yasaki we seem to have stumbled just last month over a new way to produce "ideal class numbers" (a number theoretic notion) as an Euler–Poincaré characteristic (a topological notion)...

References

[1] A. Beardon, *The Geometry of discrete groups,* GTM Springer **91**, 1983.
[2] J.W. Cannon, W.J. Floyd, R. Kenyon, W.R. Parry, *Hyperbolic geometry*, in: Flavors of Geometry, MSRI Publications, Volume 31, 1997.
[3] J. Cremona, *Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields*, Comp. Math. 51 (1983), 275–324.
[4] The Not Knot, amazing video on YouTube, `http://www.youtube.com/watch?v=AGLPbSMxSUM` (Part 1) and `http://www.youtube.com/watch?v=MKwAS5omW_w` (Part 2).
[5] H. Verrill, Fundamental domain drawer; `https://www.math.lsu.edu/~verrill/fundomain/`.

Summary. So what have we learned in particular today? Let wrap it up:

"To a line in the plane and a point outside
there's a single parallel"...
If this axiom is dropped we get a new geometry
that is consistent just as well;

where the cosh of leg 1 times the cosh of leg 2
is the cosh of the hypotenuse,
where the angle sum in a 3-gon is
less than π, this may sound quite abstruse.

"To a line in the plane and a point outside
there's a single parallel"...
If this axiom is dropped we get a new geometry
that is consistent just as well;

if two 3-gons have angles that are all the same,
then they're congruent, actually;
if two points are conjoined by a shortest line,
then that's curved most aesthetically.

||: "To a line in the plane and a point outside
there's a single parallel"...
If this axiom is dropped we get a new geometry
that is consistent just as well! :|| (H.G.)