JEWELRY FROM TESSELLATIONS OF HYPERBOLIC SPACE

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1. The axiom of parallels

The perhaps most influential books on mathematics $\mathcal{D}$ were written by Euclid (∼300 B.C.) and were called the "Elements". For an online translation of these see [4].

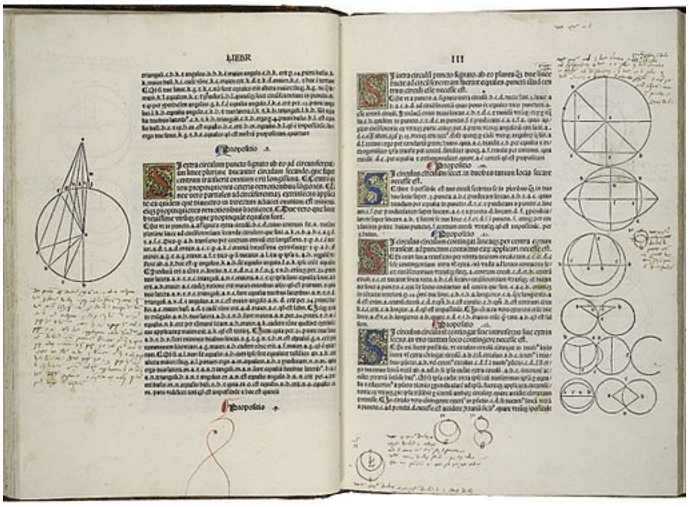

Figure 1. From an Italian translation of Euclid's Elements. Source: Wikipedia.

One of the tasks he set out on was to put geometry on a solid footing, using an axiomatic approach. This amounted to a huge endeavour: the task was to find as few axioms (=self-explaining facts about the object(s) of our consideration) for the geometry as possible, and to deduce all the known theorems about the geometry from those axioms.

In modern parlance, we can state the axioms to which he reduced the whole theory as follows ([4], p.7):

Euclid's axioms for geometry

(1) Each pair of points can be joined by one and only one straight line segment.
(2) Any straight line segment can be indefinitely extended in either direction.
(3) There is exactly one circle of any given radius with any given center.
(4) All right angles are congruent to one another.
(5) Through any point not lying on a straight line there passes one and only one straight line that does not intersect the given line.

Of such a set of axioms one needs to make sure in particular that

(1) the axioms do not lead to a contradiction ("consistency"), and
(2) neither axiom can be deduced from the others ("independence").

This axiomatic set-up was handled quite satisfactorily in Euclid's "Elements" in many ways. *But* already back then the fifth axiom, the axiom of parallels, was considered a bit of an enigma as it seemed superfluous, yet nobody was able to deduce it from the other axioms. From this the following question arose.

Key Question. Is the fifth axiom *independent* of the others?

The importance of Euclid's "Elements" and this somewhat nagging issue for the mathematical foundations (the possible redundance of the fifth axiom was presumably felt to be a blot on the perceived elegance of the axiomatic setting) spurred many attempts over the centuries to indeed prove that axiom from the remaining four ones.

Short Answer. Yes (the fifth axiom is independent of the others).

Surprisingly, the solution to this longstanding conundrum was found when (presumably independent) flashes of genius struck at least five people around the same time: each of them showed—in their own way—that one can replace that fifth axiom for example by the following

"*Through any point not lying on a straight line there are not precisely one straight lines that do not intersect the given line.*"

and one still gets a consistent geometry! This was independently achieved by Bolyai, Lobachevsky and Gauss as well as Schweikart and (his nephew) Taurinus, all in the early 19th century.

Remark. Actually, one can replace "one and only one" in **two** fundamentally different ways which roughly amount to

1) replacing it by "no" to get the so-called *spherical* geometry; or
2) replacing it by "more than one" to obtain the so-called *hyperbolic* geometry.

In the following we will be only interested in the second alternative

Hyperbolic alternative to axiom (5).

"(5') *Through any point not lying on a straight line there are at least two straight lines that do not intersect the given line.*"

Fortunately one can picture this kind of geometry using intuition from the spaces we are used to, the so-called *Euclidean* spaces, albeit with some of the "rules" changed.

Some surprises as to what happens in that new kind of geometry:

- the angle sum in a triangle is $< \pi$; ⟦as opposed to $= \pi$ in Euclidean geometry⟧
- if two triangles are similar (i.e. have the same angles), then they are also congruent! ⟦as opposed to the Euclidean case where one has infinitely many non-congruent similar triangles⟧
- the *area* of a triangle can be read off from the *angles*; ⟦as opposed to the Euclidean case where one has infinitely many triangles with the same angles but different area⟧
- there are (non-empty) triangles with all angles being zero!
- the "hyperbolic Pythagoras": in a right-angled triangle with side lengths a, b and hypotenuse c one has

$$\cosh(a)\cosh(b) = \cosh(c)\,.$$

2. First glance at (plane) hyperbolic geometry

How can we picture such a strange geometry? There are several rather different models in which one can view it, we only mention three of them.

(1) the upper half plane model $\mathbb{H}$ (viewed as inside the complex plane),
(2) the Poincaré disk model;
(3) the hyperboloid model (take the "upper" sheet ($z > 0$) of the two sheeted hyperboloid $x^2 + y^2 - z^2 = -1$).

The most intuitive of these is arguably the upper half plane model, in particular for people who have seen *Moebius transformations* in Complex Analysis. Recall that these are maps of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$, and most of them map the complex plane into itself, i.e. preserve it. We can consider in particular those Moebius transformations which preserve the *upper half plane* which is defined as $\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid y > 0\}$ (here x and y are real numbers and i satisfies $i^2 = -1$). Those transformations can be captured by transformations with *real* a, b, c, d.

Now an important fact is that these transformations not only preserve the upper half plane, but they even preserve the underlying "geometry" (i.e. distances ($\rightsquigarrow$ metric) and angles), hence are called *isometries.*

Fact. Any isometry of $\mathbb{H}^2$ is captured by some $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, in standard mathematical notation this is expressed as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

Examples.

(1) The matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ encodes the map $z \mapsto \frac{1 \cdot z+2}{0 \cdot z+1} = z + 2$, i.e. it simply shifts each point in the (half) plane to the right by two units.
(2) The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ encodes the map $z \mapsto -\frac{1}{z}$, and can be seen geometrically as the inversion in the unit circle ($z \mapsto \frac{z}{|z|^2} = 1/\overline{z}$) followed by a reflection in the real line (complex conjugation) and a reflection in the origin ($z \mapsto -z$).

An arithmetic group. Now $\mathrm{SL}_2(\mathbb{R})$ above is a group[1], and the two matrices above generate a subgroup of Γ (with infinitely many elements), in fact a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (which in turn is a subgroup of $\mathrm{SL}_2(\mathbb{R})$, and its elements are characterised by the property that all its entries are *integers*). More precisely, one can show that Γ is not "far off" in the sense that three copies of it (mathematical notion: cosets) cover $\mathrm{SL}_2(\mathbb{Z})$. This Γ is an example of an "arithmetic" group, as is $\mathrm{SL}_2(\mathbb{Z})$ itself; such groups play an important role in number theory.

Boundary of $\mathbb{H}^2$. One can view the real line (embedded in $\mathbb{C}$) as part of the boundary of $\mathbb{H}^2$, denoted $\partial\mathbb{H}^2$. Apart from the real line, there is one more point ("at infinity") that is on the boundary—altogether $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ (which can be identified with the real projective line in projective geometry). Think stereographic projection from the north pole of a circle to a line which is tangent to the south pole—in our picture the north pole plays the role of the point at infinity and the tangent to the south pole is identified with $\mathbb{R}$.

[1] a *group* essentially means that we can compose any two such matrices and wind up with another matrix of the same sort, and that moreover each matrix has an "inverse";

Geodesics in $\mathbb{H}^2$. What are the shortest lines in $\mathbb{H}^2$ (according to the so-called *hyperbolic metric* $\frac{dx\,dy}{y^2}$)? It turns out that these are the half-circles (and as a limiting case also the straight lines) orthogonal to the real line of $\partial\mathbb{H}^2$.

Orbits under group action. Now we can look at the set of all the (infinitely many) "translates" of an element $z \in \mathbb{H}^2$ under Γ (the correct notion here is that of a "group action" of Γ on the upper half plane). We will refer to this set as the Γ–*orbit* of z.
As a 1-dimensional analogue, we can consider all the $\mathbb{Z}$–translates of a real number (i.e. in $\mathbb{R}$) and look for collections that contain a $\mathbb{Z}$-translate of each point on $\mathbb{R}$—clearly the interval $[0, 1]$ suffices, as does, say, $[\pi, \pi + 1]$; one could even use rather provocatively construed examples like $\{r \in \mathbb{Q} \mid 0 \le r < 1\} \cup \{s \in \mathbb{R} \setminus \mathbb{Q} \mid 17.1 < s < 18.1\}$ but an obvious advantage of the former ones is that they are not scattered around; instead they form a "connected" set, and one has the rather intuitive (topological) notion of connectedness which will guide our choices of the selected points.

Fundamental domains. We want to find a collection of points in $\mathbb{H}^2$ which contain a Γ–translate of each element in $\mathbb{H}^2$. If such a collection is connected and is optimal in the sense that it contains only a *single* element of each Γ-orbit, then we call it a *fundamental domain* (for the action of Γ on $\mathbb{H}^2$).

Note that in our 1-dimensional example the interval $[0, 1]$ is not quite a fundamental domain (for the action of $\mathbb{Z}$ on $\mathbb{R}$) because both 0 and 1 are in the same $\mathbb{Z}$-orbit; but a small modification—dropping "1", say—gives us the half-open interval $[0, 1)$ as a fundamental domain.

Now clearly each point in $\mathbb{H}^2$ has a Γ-translate in the half-strip $\{x+iy \mid -1 \le x < 1,\ y > 0\}$: for $z_0 = x_0 + iy_0$ simply subtract integer multiples of 2 from z_0 so that it lands between -1 and 1 (in terms of formulas, $z_0 \mapsto z_0 - 2\lfloor\frac{x_0+1}{2}\rfloor$ does it, so we would apply the matrix $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^{-\lfloor\frac{x_0+1}{2}\rfloor} = \begin{pmatrix}1 & -2\lfloor\frac{x_0+1}{2}\rfloor\\ 0 & 1\end{pmatrix} \in \Gamma$ to replace z_0 by its translate inside that half-strip).

Furthermore, since the second matrix, corresponding to the map $z \mapsto -1/z$, maps elements from inside the unit circle to the outside, it seems plausible that the fundamental domain is covered by the figure below (this is to be thought of as extended to the "point at infinity" where the two vertical boundary lines "meet").

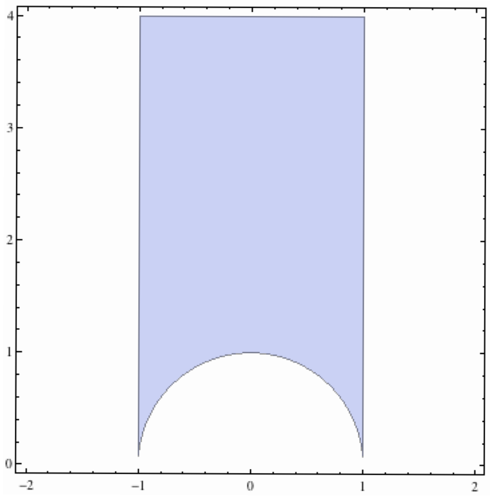

Figure 2. Fundamental domain for the group Γ (it is unbounded in the y-direction).

Indeed, it turns out that this is essentially the correct picture, except that again one needs to be a bit more careful at the boundary (only "half" the points are to be counted in).

Note that the fundamental domain just given defines a (limiting case of a) triangle in $\mathbb{H}^2$; all its angles are zero, and all its vertices are at the boundary $\partial\mathbb{H}^2$. It forms a so-called *ideal* triangle.

Tessellations. Once a fundamental domain for a given arithmetic group is found, its translates will determine a tessellation of the original space—simply take all its translates under the group (typically one allows for overlaps along the boundaries, so one is more casual about the actual "fundamental domain" thing).

So what does the tessellation look like?

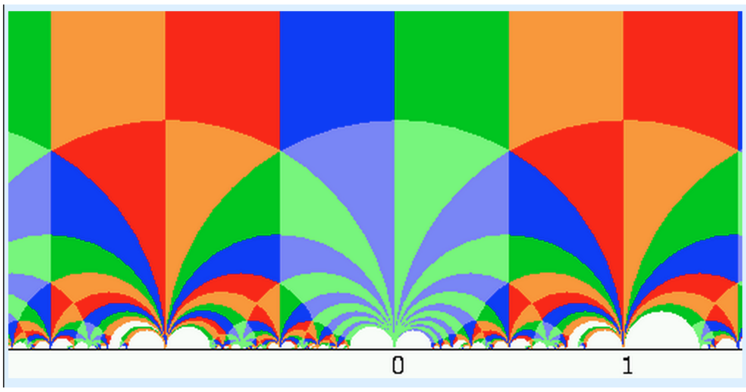

Figure 3. Tessellation of the hyperbolic plane: six tiles of different colour give a fundamental domain. Source: H. Verrill.

Moreover, one can view this picture in a different model of the hyperbolic plane, the disk model $\mathbb{D}^2$ (covering the unit disk inside the complex plane)—e.g. using the map

$$\begin{array}{ccc} \mathbb{H}^2 & \longrightarrow & \mathbb{D}^2 \\ z & \mapsto & \dfrac{z-i}{z+i} \end{array}$$

and the tessellation looks as follows

Figure 4. A hyperbolic tessellation of the plane. Each (curved) triangle has the same hyperbolic volume. Source: Wikipedia.

A glance of hyperbolic 3-space. We get an analogous picture when we pass from 2D to 3D, i.e. from the hyperbolic plane to the hyperbolic *space*. It is often depicted as the 'upper half' of the usual (Euclidean) space $\mathbb{R}^3$, consisting of those points $(x, y, z) \in \mathbb{R}^3$ for which $z > 0$, and it is denoted $\mathbb{H}^3$.

From 2D to 3D: the relevant analogues of the notions introduced above.

- Its boundary $\partial\mathbb{H}^3$ is given by the plane "underneath" (consisting of the points $(x, y, z) \in \mathbb{R}^3$ for which $z = 0$) together with a point at infinity. This gives topologically a "1-point compactification" of the plane, and geometrically we obtain a 2-sphere).
- Its geodesics are again certain half-circles; in this 3-dimensional case they have to be orthogonal to the boundary *plane.* As a limiting case one gets that, if a half-circle passes through the point at infinity, it becomes a straight line in $\mathbb{H}^3$. Again, think stereographic projection from the north pole, this time of a *sphere*, to the *plane* which is tangent to the south pole—in our picture the north pole plays the role of the point at infinity and the tangent plane to the south pole is identified with the plane 'underneath' the upper half plane.
- The hyperplanes in $\mathbb{H}^3$ are half-spheres, again orthogonal to the boundary plane (a limiting case being half-spheres through ∞ which are planes that intersect this boundary plane at a right angle).
- Isometries now are encoded by elements in $\mathrm{SL}_2(\mathbb{C})$ (rather than $\mathrm{SL}_2(\mathbb{R})$ which provide all the isometries for the hyperbolic plane). Hence we are looking for interesting subgroups of that matrix group.

A beautiful introductory text, including historical bits, was written by Milnor [8].

3. Tessellations in hyperbolic space.

Let us consider a couple of examples.

Fundamental domains. 1. The perhaps simplest 3-dimensional example arises from the group $\mathrm{SL}_2(\mathbb{Z}[i])$ where $\mathbb{Z}[i] \subset \mathbb{C}$ denote the Gaussian integers. The actual fundamental domain is slightly more complicated, but if one passes to a subgroup (of index 4) of $\mathrm{SL}_2(\mathbb{Z}[i])$—this corresponds to gluing 4 copies of this fundamental domain together—then one obtains a nice octahedron with all eight vertices at the boundary. You can think of the octahedron as two square pyramids glued together along the base, and a fundamental domain for the full group $\mathrm{SL}_2(\mathbb{Z}[i])$ is given as half of such a square pyramid.

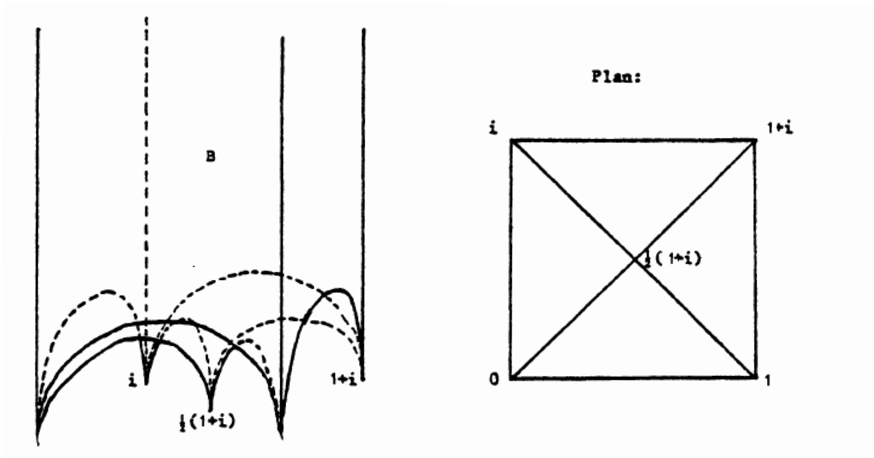

Figure 5. Tessellation of hyperbolic 3-space using $\mathrm{SL}_2(\mathbb{Z}[i])$. Source: J. Cremona [3].

2. A second example is given by $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, where the picture is as follows.

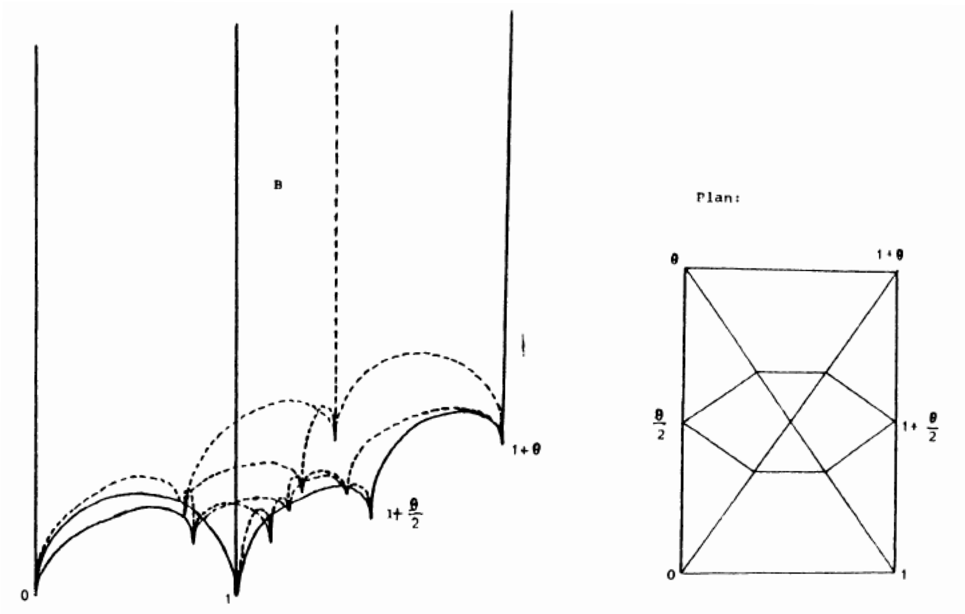

Figure 6. Tessellation of the hyperbolic plane using $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$. Source: J. Cremona [3].

How should we interpret these pictures (we note that similar projection pictures of a number of cases had been given in [6], too)?
The left hand one shows (parts of) 14 hyperplanes in $\mathbb{H}^3$, four of them being "straight", while 10 of them are given in terms of half-spheres. The **interior** of this figure gives a polyhedron which is essentially a fundamental domain arising from the action of (a suitable large subgroup of) $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.
The right hand picture shows a projection of this polyhedron from the point "at infinity" to the plane "below".

A schematic 3D-picture—Figures 7–11 below were produced by the author using Mathematica—of the half-spheres bounding the polyhedron looks as follows (the polyhedron itself consists of the points **above** those half-spheres):

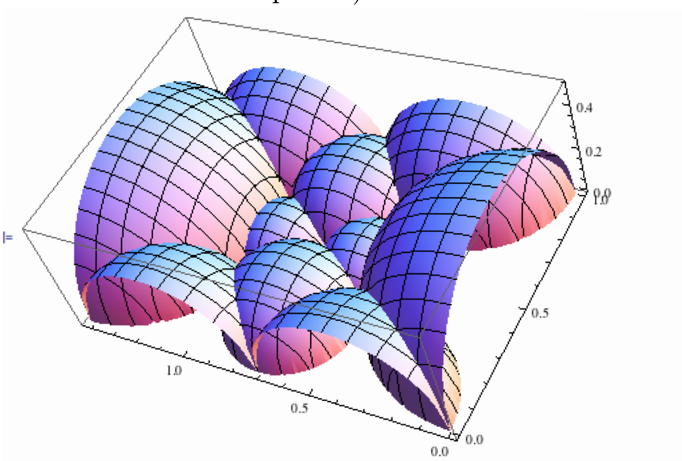

Figure 7. Ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

Here's how the (closure of the) actual fundamental domain looks from below:

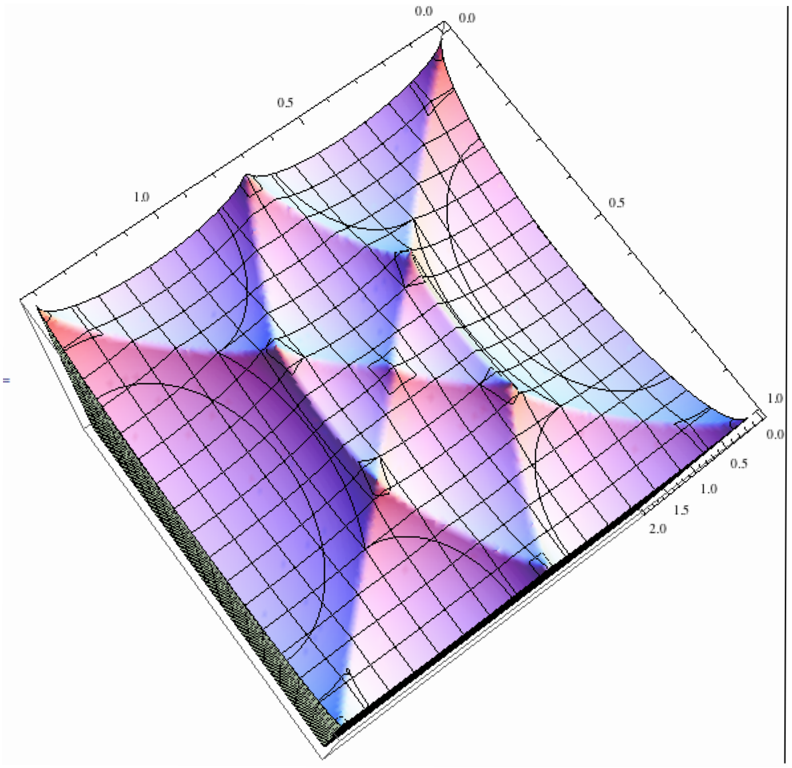

Figure 8. Ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

If we push the vertex "up at infinity" down to a finite point, we can see a compact approximation of the polyhedron (with the same combinatorial data)

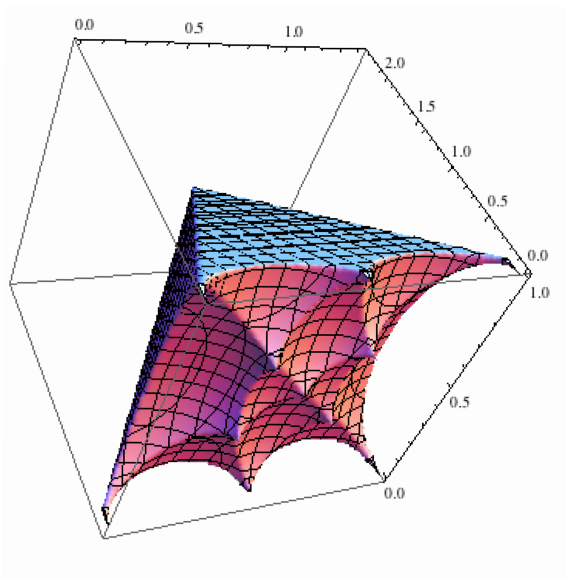

Figure 9. Approximate ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

and we can try to recognise it as a more familiar polyhedron, at least after straightening out the faces. Indeed, we can pass to the Klein model where geodesics correspond to straight lines, and we find its Euclidean counterpart as follows—a cuboctahedron.

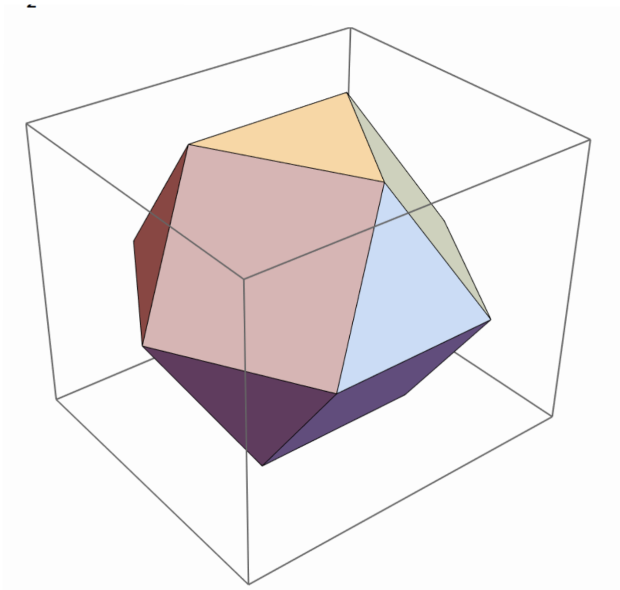

Figure 10. "Straightened" version of ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

3.1. **From fundamental domain to tessellation.** On the one hand, it is reasonably straightforward to picture parts of the *tessellation* of $\mathbb{H}^3$ arising from this fundamental domain via simple translates using the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with a an integer or, say, an integer multiple of $\sqrt{-2}$ (in the following picture we glue four copies of the fundamental domain together).

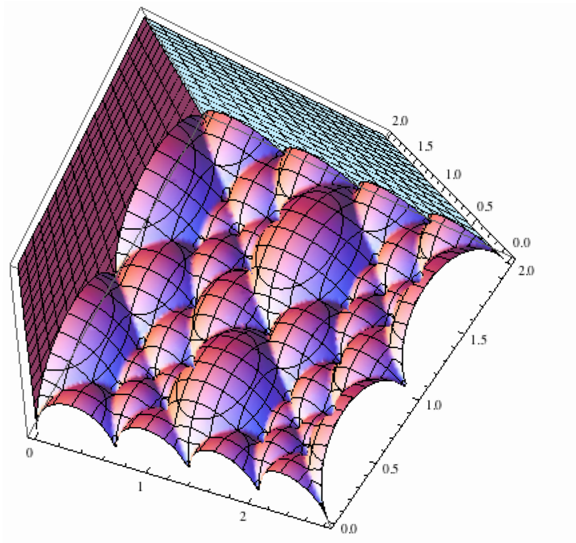

Figure 11. Four copies of ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

On the other hand, it is considerably harder to picture the image under e.g. the "inversion" $z \mapsto -1/z$, and moreover some of the faces become quite small.

4. Further arithmetic examples.

There are a dozen further arithmetic example, elaborated upon in [5], all of which arise from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ (or at least a closely related such matrix group—the mathematically precise notion is for it to be "commensurable"[2]) for some small d.

4.1. **The case $d = 6$.** It turns out that the case $d = 6$ is one of the rare cases where we can find a fundamental domain which is a single convex polytope in its own right (rather than a union of such). More precisely, one obtains a rhombicuboctahedron, which we depict as a 2-dimensional projection from the point at infinity on the left and in its Euclidean avatar on the right:

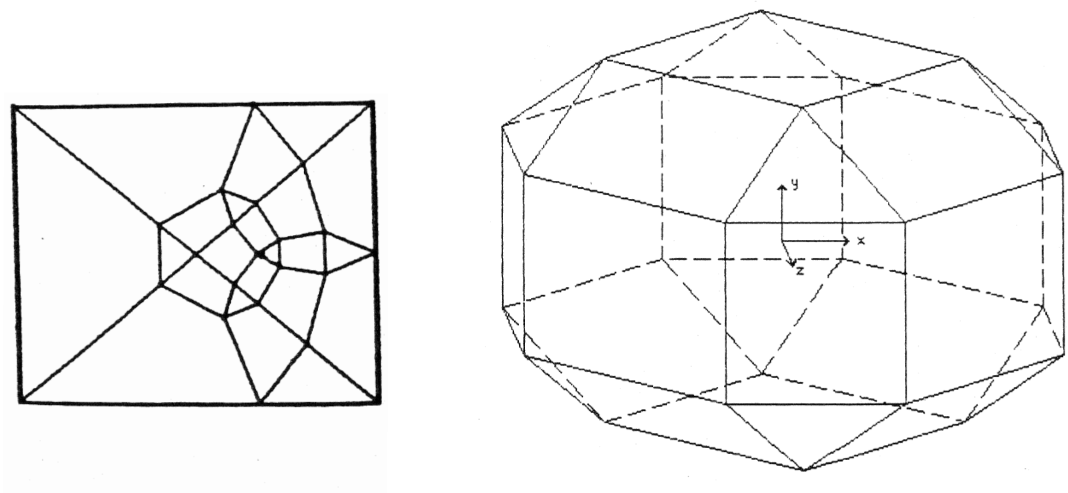

Figure 12. A polytope tessellating hyperbolic 3-space, projected from the point at infinity to the plane at the boundary given by $\partial\mathbb{H}^3 \setminus \{\infty\}$ (left), and a "straightened" version of that polytope in Euclidean space (right).

Visualisation challenge: Can you "see" that the left hand one really is (combinatorially, i.e. not taking into account distances) a projection of the right hand one from any one of its vertices?

Occurrence in "real life". This polyhedron is very reminiscent of (the convex hull of) "Rubik's snake" which is depicted below.

Figure 13. A picture of Rubik's snake, a rather flexible mathematical toy.

[2]two groups are called commensurable if their intersection is of finite index in each

Upshot. So we could say *hyperbolically*: we can tessellate hyperbolic 3-space with copies of (the convex hull of) Rubik's snake...

5. Hyperbolic polytopes for decoration and jewellery

We can apply the same procedure to groups closely related to $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ for many integers $d > 0$, and it turns out that in a good number of cases one finds interesting looking yet rather skewed polytopes. Several students in Durham working on summer research projects have been toying around with these over the years and found ways to depict them (M. Spencer) and to exhibit their symmetry better by 'spherifying' them with suitable affine transformations (J. Inoue).

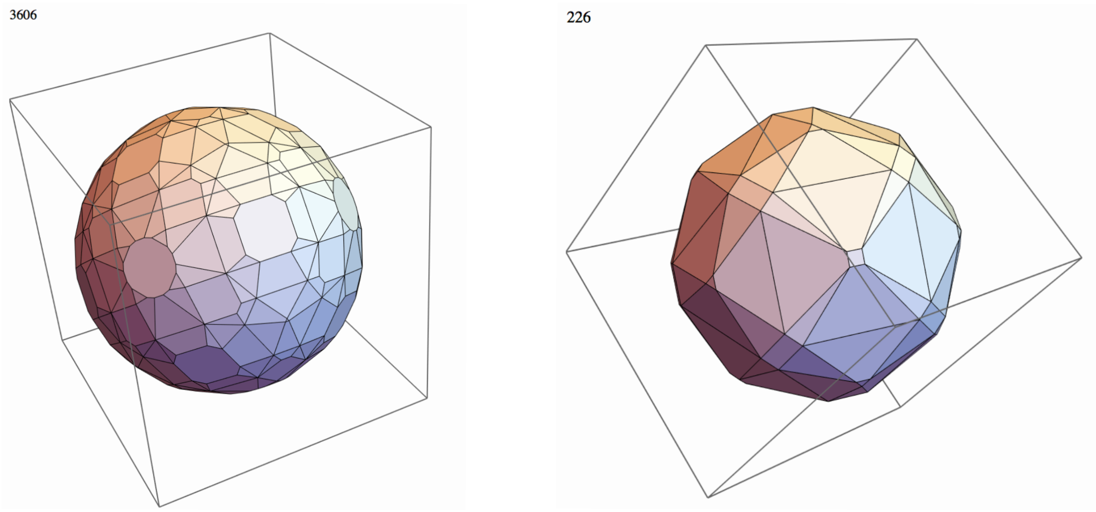

Figure 14. Pictures of polytopes arising from an ideal tessellation of hyperbolic 3-space from a group commensurable with $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ with $d = 3606$ and $d = 226$, respectively.

These pictures triggered a desire to realise the polytopes as models, and other students were able to produce the first such models via 3D-printing (E. Woodhouse, J. Inoue). There are plenty of computer aided design (CAD) programs like OpenSCAD and Rhino3D which allow to manipulate the data which encode the vertices, edges and faces of the models. Moreover, one has the option to produce files from the ensuing models that can be uploaded to the web page of a 3D printing service (like 'Shapeways') who in turn print and ship the results—in a good variety of materials—to their customers. Our first such trial runs produced wireframe models of those polytopes in plastics, and in steel materials like Bronze Steel or Gold Steel; examples of the latter are the following rather decorative models (arising from $d = 3606$ and $d = 226$, respectively).

Figure 15. Pictures of 3D-printed wireframe models of the above in Polished Bronze Steel and Polished Gold Steel, respectively.

While Shapeways can also print in cast metals, one needs to scale things down in size quite a bit in order to make the output financially affordable—in fact, they also offer the option to print in plated metals. Furthermore, if one wants to use them as jewellery like necklaces or earrings, it is sometimes convenient to adapt them in shape (e.g. by flattening them along a suitably chosen direction), even though some of the symmetries will be dropped in the process.

We reproduce a few pictures to give an idea of the results.

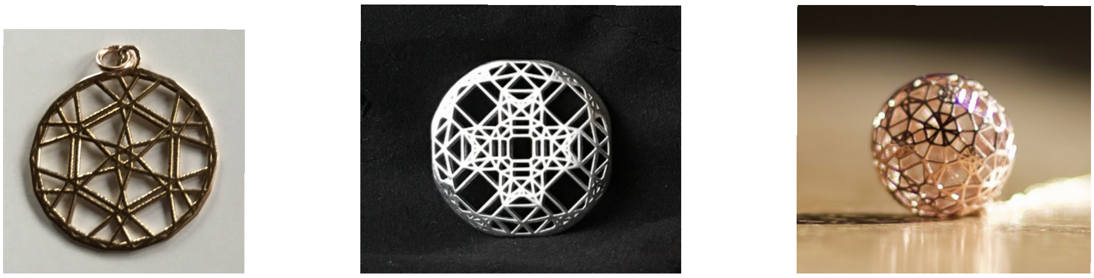

Figure 15. Pictures of polytopes in Polished Bronze, Rhodium plated and Rose Gold plated, respectively, arising from an ideal tessellation of hyperbolic 3-space for $d = 34$, $d = 11782$ and $d = 1409$, respectively.

Some of these can actually be ordered from Shapeways via the following clickable link:
`https://www.shapeways.com/shops/herbert-gangl-maths-gems`

It is surprising how nicely poised many of the ensuing polytopes emerge, as there does not seem to be a compelling reason a priori that the vertices of such a polytope (given as the simultaneous integer solutions of a linear and a quadratic Diophantine equation) should have such a rich (hidden) symmetry at all. It is a pleasing empirical observation that many polytopes arising in this way appear to be combinatorially different—this is in notable contrast to a different tessellation procedure for closely related groups given by Yasaki [9] where only nine combinatorially different polytopes appear to arise. In fact, the maximal number of vertices for our polytopes might even grow indefinitely with increasing d (the current record: for $d = 20009$ we find a polytope with 2496 vertices).

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