Michaelmas 2012, NT III/IV, Problem Sheet 5.

1. In an obvious generalization of the notion of an "algebraic integer", for a field L containing a ring R, we say that α is **integral over** R if it satisfies a monic polynomial equation with coefficients in R.

   Furthermore, R is said to be **integrally closed** if every element in the quotient field of R which is integral over R, is already contained in R. (For example, $\mathbb{Z}$ and $\mathbb{Z}[i]$ are integrally closed in $\mathbb{Q}$ and $\mathbb{Q}(i)$, respectively, while $\mathbb{Z}[\sqrt{-3}]$ is not integrally closed in $\mathbb{Q}(\sqrt{-3})$—why?)

   With these definitions show that every UFD is integrally closed.

2. Find how many solutions (a, b) there are (a) with a, $b \in \mathbb{Z}$ and (b) with a, $b \in \mathbb{N}$ to the following equations. (c) In each case give (just) one of the solutions, if there is one.

   (i) $a^2 + b^2 = 2^3 \times 7^4 \times 37^5 \times 41$,
   (ii) $a^2 + b^2 = 27 \times 41 \times 43$,
   (iii) $a^2 + 25b^2 = 4 \times 29 \times 113^4$,
   (iv)[1] $a^2 + 7b^2 = 8 \times 23 \times 43$,
   (v)[2] $a^2 - ab + b^2 = 3 \times 7 \times 61$,
   (vi)[3] $a^2 - 5b^2 = 11$,
   (vii) $a^2 - 2b^2 = 21$,
   (viii) $a^2 + 2b^2 = 3^{14} \times 43^{10}$,
   (ix) $a^2 + 11b^2 = 3^{12} \times 5^{16}$,
   (x)[1] $a^2 + 7b^2 = 4 \times 23^r \times 43^s$, $r, s \in \mathbb{Z}^{>0}$.

   [([1]) Note that for any solution $(a + b\sqrt{-7})/2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$.
   ([2]) Note that $a^2 - ab + b^2 = (a + \omega b)(a + \overline{\omega} b)$, where $\omega = \frac{-1+\sqrt{-3}}{2}$.
   ([3]) It may be useful to find a solution (indeed several) to $a^2 - 5b^2 = 1$.]

3. Let p and q be distinct odd prime integers. Given that there is at least one solution, find how many solutions there are to
$$a^2 + 2b^2 = p^{11} q^{13}$$
   with a and b in $\mathbb{Z}$.

4. Let $p(\neq 11)$ be an odd prime integer which is not prime in $\mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right]$. How many solutions are there to
$$X^2 + 11Y^2 = 4p^{23},$$
   where X and Y are positive integers *not divisible by* p?

5. Find all the integer solutions to $X^2 + 11 = Y^3$. [Make sure you work in a UFD.]

6. Show that $\mathbb{Z}[\sqrt{-7}]$ contains elements α and β such that $\alpha\widetilde{\alpha} = 11$ and $\beta\widetilde{\beta} = 23$. For given positive integers s and t, how many integer solutions are there to
$$X^2 + 7Y^2 = 3^4 11^s 23^t \ ?$$
   [You may assume that $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$ is a UFD.]