Michaelmas 2012, NT III/IV, Problem Sheet 2.

1. (i) Factorize $8+9i$ into irreducibles in $\mathbb{Z}[i]$.
   (ii) Let $R=\mathbb{Z}[\sqrt{-5}]$. Factorize $11+\sqrt{-5}$ into irreducibles in R in two essentially different ways (i.e. the second factorization should use an irreducible which is not associate to any of the irreducibles used in the first). Deduce that R is not a unique factorization domain (UFD).
   (iii) Let $R=\mathbb{Z}[\sqrt{-13}]$. Show that $1+\sqrt{-13}$ is irreducible in R, but not prime and deduce that R is not a UFD.
   [*Hint:* using norms may be helpful.]

2. Suppose that $d<-2$. Show that 2 is irreducible in $\mathbb{Z}[\sqrt{d}]$. Find a value of $d<-2$ such that 2 is not prime in $\mathbb{Z}[\sqrt{d}]$.

3. Let $R=\mathbb{Z}[\sqrt{-26}]$. Show that each of the factors in the equation
$$3^3=(1+\sqrt{-26})(1-\sqrt{-26})$$
is irreducible (which of these are prime?), and deduce that R is not a UFD.

4. Find two units in $\mathbb{Z}[\sqrt{5}]$ which are greater than 1.

5.* Find all the solutions $(X,Y)\in\mathbb{Z}\times\mathbb{Z}$ to
   (i) $X^2+1=Y^7$ given that $\mathbb{Z}[i]$ is a UFD and to
   (ii) $X^2+8=Y^3$ given that $\mathbb{Z}[\sqrt{-2}]$ is a UFD.

6.* (i) Factorize 5, 19, 43 and $19\cdot 43=817$ as products of (one or more) irreducibles in $R=\mathbb{Z}[\sqrt{-2}]$.
   (ii) Using the fact that R is a UFD, find all the elements $\alpha\in R$ such that
$$\alpha\bar{\alpha}=817\,.$$
   (iii) Hence find all pairs of positive integers (a,b) such that $a^2+2b^2=817$.

7. Show that if H, I and J are (additive) subgroups of $(R,+)$ (R a ring) then
   (i) HJ and $H+J$ are subgroups of R;
   (ii) $H(I+J)=HI+HJ$;
   (iii) HI is an ideal if I is.
   (iv) $RI=I\Leftrightarrow I$ is an ideal.

8. For a ring R and elements a_j $(1\leqslant j\leqslant n)$ we introduce the notation $\langle a_1,\ldots,a_n\rangle_{\mathrm{gp}}:=\mathbb{Z}a_1+\cdots+\mathbb{Z}a_n$.
   Note that this is in general *different* from the ideal $(a_1,\ldots,a_n)_R$. [Why?]
   Show that if a, b, c and $d\in R$ then
   (i) $(a)_R(b)_R=(ab)_R$,
   (ii) $\langle a\rangle_{\mathrm{gp}}\langle b\rangle_{\mathrm{gp}}=\langle ab\rangle_{\mathrm{gp}}$,
   (iii) $\langle a,b\rangle_{\mathrm{gp}}\langle c,d\rangle_{\mathrm{gp}}=\langle ac,ad,bc,bd\rangle_{\mathrm{gp}}$.

9. Let α, β and γ lie in an integral domain R. Show that if $(\alpha,\beta)_R=(\gamma)_R$ then γ is a gcd of α and β in R.

10. Let $R=\mathbb{Z}[\sqrt{-21}]$. Express the ideal $(5,2+\sqrt{-21})_R(3,\sqrt{-21})_R$ in the form $(N,\alpha)_R$ where $N\in\mathbb{Z}$ and $\alpha\in R$.

11. Let $I=(1+\sqrt{-5},2)_R$ where $R=\mathbb{Z}[\sqrt{-5}]$.
   (i) Show that I^2 is a principal ideal but that I, itself, is not.
   (ii) Show that I is a maximal ideal. [Show $R/I\cong\mathbb{Z}_2$.]

12. Let $J=(1+\sqrt{-26},3)_R$ where $R=\mathbb{Z}[\sqrt{-26}]$.
   (i) Show that J^3 is a principal ideal but that J, itself, is not.
   (ii) Deduce that J^2, also, is not principal.
   (iii) Show that J is a maximal ideal. [Show $R/J\cong\mathbb{Z}_3$.]