

ALGEBRA II Problems: Week 18 (Finitely generated abelian groups)

Epiphany Term 2014

1. Write down the torsion coefficients of

$$(a) \mathbb{Z}_{15} \times \mathbb{Z}_2 \times \mathbb{Z}_{20}; \quad (b) \mathbb{Z}_{10} \times \mathbb{Z}_{36} \times \mathbb{Z}_{14} \times \mathbb{Z}_{21}.$$

2. Let G be an abelian group of order 100. Show that G must contain an element of order 10. What are the torsion coefficients of G if no element of G has order greater than 10?
3. Classify the abelian groups of order 32, 60 and 144.
4. If the order of a finite abelian group is not divisible by a square, show that the group must be cyclic.
5. Let G be a finite abelian group and write $A(q) = A_G(q)$ for the number of elements x of G which satisfy $x^q = e$. Find the torsion coefficients of G when $A(3) = 81$, $A(9) = 243$, $A(5) = 25$, $A(25) = 625$ and $x^{225} = e$ for all $x \in G$.
6. Find the rank and the torsion coefficients of the abelian group determined by generators w, x, y, z and relations $3w + 5x - 3y = 0$, $4w + 2x - 2z = 0$.
7. Find the rank and the torsion coefficients of the abelian group determined by generators v, w, x, y, z and relations:

$$\begin{aligned} v - 7w + 14y - 21z &= 0; \\ 5v - 7w - 2x + 10y - 15z &= 0; \\ 3v - 3w - 2x + 6y - 9z &= 0; \\ v - w + 2y - 3z &= 0. \end{aligned}$$

8. How many elements of order (a) 3, (b) 9, (c) 4, (d) 12 does $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{36}$ contain?
9. Let G be a finite abelian group of order 360 which does not contain any elements of order 12 or 18. Find the torsion coefficients of G . How many elements of order 6 does G contain?